Teaching Notes

On the bounds for the perimeter of an ellipse

Let \( L(a, b) \) denote the perimeter of an ellipse with semi-axes \( a, b \). It is well known that \( \pi (a + b) \leq L(a, b) \leq 2\pi \sqrt{\frac{a^2 + b^2}{2}} \), with equality if, and only if, \( b = a \). There are several proofs of this remarkable fact in the literature; see, for example, [1], [2] and [3]. The aim of this note is to compile a proof suitable for high school students as much as possible. We start with the standard parametrisation \( x = a \cos t, y = b \sin t \) which gives rise to

\[
L(a, b) = 4 \int_0^{\frac{\pi}{2}} \sqrt{a^2 \cos^2 t + b^2 \sin^2 t} \, dt.
\]

Using the substitution \( u = \frac{\pi}{2} - t \) we get \( L(a, b) = L(b, a) \), hence

\[
L(a, b) = 2 \int_0^{\frac{\pi}{2}} \left( \sqrt{a^2 \cos^2 t + b^2 \sin^2 t} + \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} \right) \, dt.
\]

Since \((a^2 \cos^2 t + b^2 \sin^2 t) + (a^2 \sin^2 t + b^2 \cos^2 t) = a^2 + b^2\) and \(A + B \leq \sqrt{\frac{A^2 + B^2}{2}}\) for all non-negative real \(A, B\) with equality if, and only if, \(B = A\), we have

\[
\sqrt{a^2 \cos^2 t + b^2 \sin^2 t} + \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} \leq 2 \sqrt{\frac{a^2 + b^2}{2}}
\]

for each \(t\), with equality for each \(t\) if, and only if, \(b = a\). Therefore

\[
L(a, b) = 2 \int_0^{\frac{\pi}{2}} \left( \sqrt{a^2 \cos^2 t + b^2 \sin^2 t} + \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} \right) \, dt \leq 2\pi \sqrt{\frac{a^2 + b^2}{2}}
\]

with equality if, and only if, \(b = a\).

Also since \((\cos^2 t + \sin^2 t)^2 = 1\) we get \(\cos^4 t + \sin^4 t = 1 - 2 \cos^2 t \sin^2 t\), whence

\[
(a^2 \cos^2 t + b^2 \sin^2 t)(a^2 \sin^2 t + b^2 \cos^2 t)
= a^2 b^2 (1 - 2 \cos^2 t \sin^2 t) + (a^2 + b^2) \cos^2 t \sin^2 t
= a^2 b^2 + (a^2 - b^2)^2 \cos^2 t \sin^2 t,
\]

and hence

\[
(a^2 \cos^2 t + b^2 \sin^2 t + \sqrt{a^2 \sin^2 t + b^2 \cos^2 t})^2
= a^2 + b^2 + 2\sqrt{a^2 b^2 + (a^2 - b^2)^2} \cos^2 t \sin^2 t \geq (a + b)^2
\]

for each \(t\), with equality for each \(t\) if, and only if, \(b = a\), giving
L(a, b) = \int_0^{\frac{\pi}{2}} \left( a^2 \cos^2 t + b^2 \sin^2 t + \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} \right) dt \geq \pi (a + b)

with equality if, and only if, b = a. Thus

\pi (a + b) \leq L(a, b) \leq 2\pi \sqrt{\frac{a^2 + b^2}{2}},

with equality if, and only if, b = a as required.

References

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Cauchy-Schwarz via collisions

Consider a line of n railway trucks with masses \( m_1, m_2, \ldots, m_n \) moving on a smooth straight track with velocities \( v_1 > v_2 > \ldots > v_n \) (with negative velocities allowed) and spaced so that they successively couple together in the order Truck \( n - 1 \) to Truck \( n \), Truck \( n - 2 \) to Trucks \( n - 1 \) and \( n \), Truck \( n - 3 \) to Trucks \( n - 2 \) and \( n - 1 \) and \( n \), etc. If, when all \( n \) trucks are coupled together, their common velocity is \( V \), conservation of momentum shows that \( V = \frac{\sum m_i v_i}{\sum m_i} \). But kinetic energy cannot be gained in the collisions, so

\[
\frac{1}{2} \sum m_i v_i^2 \geq \frac{1}{2} \left( \sum m_i \right) V^2 = \frac{1}{2} \left( \sum m_i v_i \right)^2
\]

and thus

\[
\left( \sum m_i v_i \right)^2 \leq \left( \sum m_i \right) \left( \sum m_i v_i^2 \right).
\]

Moreover, physical intuition suggests that no kinetic energy will be lost only in the special case in which \( v_1 = v_2 = \ldots = v_n \) where there are no collisions and the total kinetic energy is the same whether the trucks are viewed individually or en masse.