

A new description of the Bowen–Margulis measure

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(Received 22 July 1987 and revised 5 May 1988)

Abstract. The Bowen–Margulis measure on the unit tangent bundle of the universal covering of a compact manifold of negative curvature is determined by its restriction to the leaves of the strong unstable foliation. We describe this restriction to any strong unstable manifold W as a spherical measure with respect to a natural distance on W .

Let M be a compact connected Riemannian manifold of negative curvature $-\infty < -b^2 \leq K \leq -a^2 < 0$ and fundamental group Γ . The geodesic flow g^t acts on the unit tangent bundle SM of the universal covering \tilde{M} of M . SM admits foliations W^{ss} , W^s , W^{su} , W^u which are invariant under g^t and the action of Γ on SM . The leaves of W^{ss} (resp. W^s , W^{su} , W^u) are called the *strong stable* (resp. *stable*, *strong unstable*, *unstable*) manifolds of SM (see [6]). We write $A \subset W^i$ if $A \subset SM$ is contained in a leaf of W^i ($i = ss, s, u, su$).

The *Bowen–Margulis measure* $\tilde{\mu}$ on SM is the lift to SM of the unique g^t -invariant Borel-probability measure on SM of maximal entropy ([2], [6]). $\tilde{\mu}$ has natural restrictions to measures $\tilde{\mu}^i$ on the leaves of W^i ($i = ss, s, u, su$) and is determined by $\tilde{\mu}^{su}$.

The purpose of this paper is to show that for every $v \in SM$ the measure $\tilde{\mu}^{su}$ on the leaf $W^{su}(v)$ of W^{su} containing v is a spherical measure with respect to a natural distance on $W^{su}(v)$. In order to define this distance we have to fix some notations:

For $v \in SM$ let φ_v be the geodesic line in \tilde{M} with initial direction $\varphi'_v(0) = v$. φ_v determines a point $\varphi_v(-\infty) = \xi$ of the *ideal boundary* $\partial\tilde{M}$ of \tilde{M} . $W^u(v)$ then consists of all unit tangent vectors of geodesic lines γ in \tilde{M} which satisfy $\gamma(-\infty) = \xi$. In particular the restriction to $W^u(v)$ of the canonical projection $P: SM \rightarrow \tilde{M}$ is a diffeomorphism of $W^u(v)$ onto \tilde{M} .

$v \in SM$ determines a *Busemann function* θ_v at ξ which is normalized by $\theta_v \varphi_v(0) = 0$. For $t \in \mathbb{R} \cup \{\infty\}$ denote by $\pi_{t,v}: \tilde{M} \cup (\partial\tilde{M} - \xi) \rightarrow \theta_v^{-1}(t)$ the projection along the geodesics which are asymptotic to ξ . Then for every $y \in \partial\tilde{M} - \xi$ the curve $\gamma: t \rightarrow \pi_{t,v}(y)$ is the unique unit-speed geodesic in \tilde{M} with $\gamma'(0) \in W^{su}(v)$ and $\gamma(\infty) = y$.

The projection $\pi: SM \rightarrow \partial\tilde{M}$, $w \rightarrow \varphi_w(\infty)$ maps $W^{su}(v)$ homeomorphically onto $\partial\tilde{M} - \xi$ and $\pi(w) = \pi_{\infty,w} \circ P(w)$ for all $w \in SM$. If $w \in W^{su}(v)$ then $\varphi_w(-\infty) = \varphi_v(-\infty)$ and $\theta_w = \theta_v$, hence $\pi_{t,w} = \pi_{t,v}$ for all $t \in \mathbb{R}$.

In the sequel we will suppress the index v of the various objects depending on $v \in SM$ whenever v is arbitrarily fixed.

The level sets $\theta^{-1}(t)$ of $\theta(t \in \mathbb{R})$ are C^1 -manifolds, the horospheres through ξ . The restriction of the Riemannian metric to $\theta^{-1}(t)$ induces a distance $d_{t,v} = d_t$ on $\theta^{-1}(t)$. Fix $R > 0$ and define, for $x, y \in \partial\tilde{M} - \xi$, $f(x, y) = \sup \{t \in \mathbb{R} \mid d_t(\pi_t(x), \pi_t(y)) \leq R\}$. The function $\eta = \eta_{v,R} : (\partial\tilde{M} - \xi) \times (\partial\tilde{M} - \xi) \rightarrow \mathbb{R}_+$, $(x, y) \rightarrow e^{-f(x,y)}$ is symmetric and $\eta(x, y) = 0$ if and only if $x = y$.

Using the upper curvature bound $-a^2$ on \tilde{M} , η can be estimated as follows:

LEMMA 1. *If $x, y \in \partial\tilde{M} - \xi$ and $d_t(\pi_t(x), \pi_t(y)) = \varepsilon \leq R$, then*

$$\eta(x, y) \geq e^{-t} \left(\frac{\varepsilon}{R}\right)^{1/a}.$$

Proof. Let $\tau = a^{-1}(\log R/\varepsilon)$; then $K \leq -a^2$ implies by the estimates in [4] that $d_{t+\tau}(\pi_{t+\tau}(x), \pi_{t+\tau}(y)) \geq R$. Thus $\eta(x, y) \geq e^{-t}(\varepsilon/R)^{1/a}$. □

As a corollary we find how $\eta_{v,R}$ varies with $R > 0$:

COROLLARY 2. *If $0 < r < R$ then $\eta_{v,R} \leq \eta_{v,r} \leq (R/r)^{1/a} \eta_{v,R}$.*

Proof. Let $x, y \in \partial\tilde{M} - \xi$ and $t = -\log(\eta_{v,r}(x, y))$. Then $d_t(\pi_t(x), \pi_t(y)) = r$ hence $\eta_{v,R}(x, y) \geq \eta_{v,r}(x, y)(r/R)^{1/a}$ by Lemma 1. Moreover clearly $\eta_{v,R} \leq \eta_{v,r}$. □

COROLLARY 3. $\eta^a : (x, y) \rightarrow (\eta(x, y))^a$ is a distance on $\partial\tilde{M} - \xi$.

Proof. We have to check the triangle inequality. For this let $x, y, z \in \partial\tilde{M} - \xi$ and $t = -\log(\eta(x, y))$, i.e. $d_t(\pi_t(x), \pi_t(y)) = R$. Then

$$\eta^a(x, y) \leq e^{-at}(d_t(\pi_t(x), \pi_t(z)) + d_t(\pi_t(z), \pi_t(y)))/R$$

hence the claim follows from Lemma 1. □

Using the identification of $W^{su}(v)$ with $\partial\tilde{M} - \varphi_v(-\infty)$ via the map π , $\eta_{v,R}^a$ can be viewed as a distance on $W^{su}(v)$. Let h be the topological entropy of the geodesic flow on SM . Our aim is to prove the following

THEOREM. *The measure $\tilde{\mu}^{su}$ on $W^{su}(v)$ equals up to a constant the h/a -dim. spherical measure associated to $\eta_{v,R}^a$.*

It will be convenient to show first the analogous theorem for a slightly different function $\rho = \rho_{v,R} : (\partial\tilde{M} - \xi) \times (\partial\tilde{M} - \xi) \rightarrow \mathbb{R}_+$ ($v \in SM, R > 0$) which is defined as $\eta_{v,R}$ but using the distance d on \tilde{M} which is induced by the Riemannian metric: For $x, y \in \partial\tilde{M} - \xi$ let $\tilde{f}(x, y) = \sup \{t \in \mathbb{R} \mid d(\pi_t(x), \pi_t(y)) \leq R\}$ and $\rho(x, y) = e^{-\tilde{f}(x,y)}$. Clearly $\rho_{v,R} = \rho_{w,R}$ if $w \in W^{su}(v)$.

ρ is related to η as follows:

LEMMA 4. *There is a number $\nu > 0$ such that $\nu\eta \leq \rho \leq \eta$ on $\partial\tilde{M} - \xi$.*

Proof. If $x, y \in \partial\tilde{M} - \xi$ and $d(\pi_t(x), \pi_t(y)) = R$ for some $t \in \mathbb{R}$, then $d_t(\pi_t(x), \pi_t(y)) \geq R$ which implies $\rho \leq \eta$. To show the first inequality, assume again $d(\pi_t(x), \pi_t(y)) = R$. Since the curvature K on \tilde{M} is bounded from below by $-b^2$, it follows from [4] that $d_t(\pi_t(x), \pi_t(y)) \leq 2/b \sinh(\frac{1}{2}bR)$, i.e. if we define $r = 2b^{-1} \sinh(\frac{1}{2}bR)$, then $\eta_{v,r} \leq \rho_{v,R}$. The claim now follows from Corollary 2. □

COROLLARY 5. *There is a number $c > 0$ such that $\rho(x, z) \leq \varepsilon$, $\rho(z, y) \leq \varepsilon$ implies $\rho(x, y) \leq c\varepsilon$.*

Proof. If $\rho(x, z) \leq \varepsilon$ and $\rho(z, y) \leq \varepsilon$, then by Lemma 4 $\eta(x, z)$ and $\eta(z, y)$ are not larger than ε/ν . Since η^a satisfies the triangle inequality, this implies $\eta^a(x, y) \leq 2(\varepsilon/\nu)^a$. Thus by Lemma 4 $\rho(x, y) \leq 2^{1/a}\varepsilon/\nu$. \square

LEMMA 6. *If $0 < r < R$ then $\rho_{v,R} \leq \rho_{v,r} \leq ((\sinh \frac{1}{2}aR)/(\sinh \frac{1}{2}ar))^{1/a}\rho_{v,R}$.*

Proof. Assume that for all $x, y \in \partial\tilde{M} - \varphi_v(-\infty)$ and all $t \in \mathbb{R}$, $s \geq 0$

$$(*) \quad d(\pi_{t+s}(x), \pi_{t+s}(y)) \geq \frac{2}{a} \sinh^{-1} \left(e^{as} \sinh \frac{a}{2} d(\pi_t(x), \pi_t(y)) \right)$$

(here again $\pi_t = \pi_{t,v}$). With

$$\tau = \frac{1}{a} \log \left(\left(\sinh \frac{a}{2} R \right) / \left(\sinh \frac{a}{2} r \right) \right)$$

we then obtain $d(\pi_{t+\tau}(x), \pi_{t+\tau}(y)) \geq R$ whenever $d(\pi_t(x), \pi_t(y)) \geq r$, i.e. $\rho_{v,r} \leq e^\tau \rho_{v,R}$. Since $\rho_{v,R} \leq \rho_{v,r}$ is obvious, it rests to prove formula (*). Consider a comparison situation in the hyperbolic plane H_a of constant curvature $-a^2$, given by a point $\bar{\xi} \in \partial H_a$, a Busemann function $\bar{\theta}$ at $\bar{\xi}$ and geodesic lines $\bar{\gamma}, \bar{\varphi}$ in H_a such that $\bar{\gamma}(-\infty) = \bar{\xi} = \bar{\varphi}(-\infty)$, $\bar{\theta}\bar{\gamma}(0) = 0 = \bar{\theta}\bar{\varphi}(0)$ and $d(\bar{\gamma}(0), \bar{\varphi}(0)) = d(\pi_t(x), \pi_t(y))$. Then

$$d(\bar{\gamma}(s), \bar{\varphi}(s)) = \frac{2}{a} \sinh^{-1} \left(e^{as} \sinh \frac{a}{2} d(\bar{\gamma}(0), \bar{\varphi}(0)) \right)$$

(see [4]) and the comparison arguments in [4] show $d((\pi_{t+s}(x), \pi_{t+s}(y)) \geq d(\bar{\gamma}(s), \bar{\varphi}(s))$. \square

LEMMA 7. *Let $v \in SM$, $\Omega \subset \partial\tilde{M} - \varphi_v(-\infty)$ compact and $\varepsilon > 0$. Then there is a neighbourhood U of v in \tilde{SM} such that*

$$(1 - \varepsilon)\rho_{w,R}(x, y) \leq \rho_{v,R}(x, y) \leq (1 + \varepsilon)\rho_{w,R}(x, y) \quad \text{for all } w \in U \quad \text{and } x, y \in \Omega.$$

Proof. Choose an open, relative compact neighbourhood D of Ω in $\partial\tilde{M} - \varphi_v(-\infty)$ and an open neighbourhood V of v in \tilde{SM} . Since $\Omega \subset \partial\tilde{M} - \varphi_v(-\infty)$ is compact, $\mu = \sup \{\rho_v(x, y) \mid x, y \in \Omega\}$ is finite. Let $\tau = \log(1/\mu)$ and define $\Psi: D \times V \rightarrow \tilde{M}$ by $\Psi(x, w) = \pi_{\tau,w}(x)$. Since Ψ is clearly continuous there is for a fixed number $\delta > 0$ and every $y \in \Omega$ an open neighbourhood $D(y)$ of y in D and an open neighbourhood $U(y)$ of v in V such that $d(\Psi(z, w), \Psi(y, v)) < \delta/2$ for every $z \in D(y)$ and $w \in U(y)$. By compactness Ω can be covered by finitely many of the sets $D(y)$, say $\Omega \subset \bigcup_{i=1}^k D(y_i)$ for some $y_i \in \Omega$. $U = \bigcap_{i=1}^k U(y_i)$ is an open neighbourhood of v in V . If $y \in \Omega$, then $y \in D(y_i)$ for some $i \in \{1, \dots, k\}$, hence

$$d(\pi_{\tau,v}(y), \pi_{\tau,w}(y)) \leq d(\pi_{\tau,v}(y), \pi_{\tau,v}(y_i)) + d(\pi_{\tau,v}(y_i), \pi_{\tau,w}(y)) < \delta \quad \text{for all } w \in U \subset U(y_i).$$

Now for all $y \in \Omega$ and $w \in U$ the function $t \rightarrow d(\pi_{t,v}(y), \pi_{t,w}(y))$ is decreasing. Thus given $y, z \in \Omega$ and $t \geq \tau$ we have

$$d(\pi_{t,v}(y), \pi_{t,v}(z)) - 2\delta \leq d(\pi_{t,w}(y), \pi_{t,w}(z)) \leq d(\pi_{t,v}(y), \pi_{t,v}(z)) + 2\delta$$

and consequently

$\rho_{w,R+2\delta}(y, z) \leq \rho_{v,R}(y, z) \leq \rho_{w,R-2\delta}(y, z)$ for all $y, z \in \Omega$ such that $\rho_v(y, z) \leq e^{-\tau}$, i.e. for all $y, z \in \Omega$ by the choice of τ . Since $\delta > 0$ was arbitrary, the claim now follows from Lemma 6. □

The function $\rho = \rho_{v,R}$ determines a family of balls $B_\rho(x, \varepsilon) = \{y \in \partial\tilde{M} - \xi \mid \rho(x, y) < \varepsilon\}$ ($x \in \partial\tilde{M} - \xi, \varepsilon > 0$). Our aim is to show that these balls together with their radii give rise by Carathéodory's construction (see [3]) to a Borel measure on $\partial\tilde{M} - \xi$ which is finite on compact and positive on nontrivial open subsets of $\partial\tilde{M} - \xi$. This fact is derived from the analogous property of an auxiliary function $\beta = \beta^{v,R}$ which is defined on the subsets of $\partial\tilde{M} - \xi$ in the following way: For a compact set $\Omega \subset \partial\tilde{M} - \xi$ and $\varepsilon > 0$ let $q_\varepsilon(\Omega)$ be the maximal cardinality of a subset E of Ω with the property that $B_\rho(x, \varepsilon) \cap B_\rho(y, \varepsilon) = \emptyset$ if $x, y \in E$ and $x \neq y$. As above denote by h the topological entropy of the geodesic flow on SM and define $\beta_\varepsilon(\Omega) = q_\varepsilon(\Omega) \cdot \varepsilon^h$ and $\beta(\Omega) = \limsup_{\varepsilon \rightarrow 0} \beta_\varepsilon(\Omega)$. If $\Omega_1, \Omega_2 \subset \partial\tilde{M} - \xi$ are compact and $\Omega_1 \subset \Omega_2$, then $\beta\Omega_1 \leq \beta\Omega_2$. Thus for $A \subset \partial\tilde{M} - \xi$ arbitrary we can define $\beta A = \sup \{\beta\Omega \mid \Omega \subset A \text{ compact}\}$.

Notice that β may not be subadditive, i.e. β may not be a measure on $\partial\tilde{M} - \xi$. However β has the following properties:

- (1) If $A \subset B$, then $\beta A \leq \beta B$.
- (2) If $\Omega_i (i \in \mathbb{Z})$ are compact and $\Omega \subset \bigcup_i \Omega_i$, then $\beta\Omega \leq \sum_i \beta\Omega_i$.

We will need the following lemma which is due to Margulis (it is essentially proved in [6]):

LEMMA 8. *For every $r > 0$ there are numbers $0 < \alpha_1(r) < \alpha_2(r) < \infty$ such that $\alpha_1(r) \leq \tilde{\mu}^u \{w \in W^u(v) \mid d(Pw, Pv) < r\} \leq \alpha_2(r)$ for all $v \in SM$.*

Lemma 8 shows in particular that $\tilde{\mu}^u$ is finite on compact and positive on nontrivial open subsets of $W^u(v)$.

For $p \in \tilde{M}$ let $B_d(p, r)$ be the open r -ball around p in (\tilde{M}, d) .

LEMMA 9. *If $p \in \theta^{-1}(t)$ then $\pi_\infty B_d(p, R/2) \subset \pi_\infty(B_d(p, R) \cap \theta^{-1}(t))$.*

Proof. Let $y \in \partial\tilde{M} - \xi$ such that $d(p, \pi_t(y)) \geq R$. Determine a number $\tau \in \mathbb{R}$ with the property that $d(p, \pi_\tau(y))$ realizes the distance of p to the geodesic $s \rightarrow \pi_s(y)$. Then $\pi_\tau(y) \in \theta^{-1}(\tau)$, hence $d(p, \pi_\tau(y)) \geq |t - \tau| = d(\pi_t(y), \pi_\tau(y))$ and $2d(p, \pi_\tau(y)) \geq d(p, \pi_\tau(y)) + d(\pi_\tau(y), \pi_t(y)) \geq R$. But this shows $y \notin \pi_\infty B_d(p, R/2)$ which is the claim. □

Recall that the geodesic flow g^t on SM transforms $\tilde{\mu}^u$ by $\tilde{\mu}^u \circ g^t = e^{ht} \tilde{\mu}^u$ (h as in the theorem). This and Lemma 9 is used in the proof of

LEMMA 10. *β is finite on compact subsets of $\partial\tilde{M} - \xi$.*

Proof. Identify \tilde{M} with $W^u(v)$, the set of all unit tangent vectors of geodesics γ in \tilde{M} with $\gamma(-\infty) = \varphi_v(-\infty) = \xi$. With respect to this identification the geodesic flow g^t acts on \tilde{M} by $w \in \theta^{-1}(s) \rightarrow g^t w = \pi_{s+t} w \in \theta^{-1}(s+t)$. The restriction of $\tilde{\mu}^u$ to $W^u(v)$ can be viewed as a measure on \tilde{M} .

Let $\Omega \subset \partial\tilde{M} - \xi$ be compact and $B_1 = \{y \in \partial\tilde{M} - \xi \mid \rho(y, z) \leq 1 \text{ for some } z \in \Omega\}$. Then

$$B_2 = \left\{ \pi_t(w) \mid w \in B_1, -\frac{R}{2} \leq t \leq \frac{R}{2} \right\}$$

is a compact subset of \tilde{M} , hence $\lambda = \tilde{\mu}^u B_2 < \infty$.

Let $\varepsilon \in (0, 1)$ and $\{x_1, \dots, x_q\} \subset \Omega$ be a set of maximal cardinality such that the balls $B_\rho(x_i, \varepsilon)$ are pairwise disjoint. This means

$$B_d(\pi_{\log 1/\varepsilon}(x_i), R) \cap B_d(\pi_{\log 1/\varepsilon}(x_j), R) \cap \theta^{-1}\left(\log \frac{1}{\varepsilon}\right) = \emptyset \text{ for } i \neq j.$$

By Lemma 9, the balls $B_d(\pi_{\log 1/\varepsilon}(x_i), R/2)$ are pairwise disjoint and moreover they are contained in $g^{\log 1/\varepsilon} B_2$ by the definition of B_2 . With $\alpha = \alpha_1(R/2)$ as in Lemma 8 this implies $q\alpha \leq \tilde{\mu}^u g^{\log 1/\varepsilon} B_2 = (1/\varepsilon)^R \cdot \lambda$ and $q \cdot \varepsilon^h \leq \lambda/\alpha$. Since $\varepsilon \in (0, 1)$ was arbitrary, this is the claim. \square

LEMMA 11. β is positive on nontrivial open subsets of $\partial\tilde{M} - \xi$.

Proof. It suffices to show that β is positive on compact sets B with nonempty interior. Define $B_3 = \{\pi_t y \mid y \in B, -R \leq t \leq 0\}$ and $\lambda = \tilde{\mu}^u B_3 > 0$. For $\varepsilon > 0$ let $\{x_1, \dots, x_q\} \subset B$ be a subset of maximal cardinality such that the balls $B_\rho(x_i, \varepsilon)$ are pairwise disjoint. By the definition of ρ this means that the balls $B_d(\pi_{\log 1/\varepsilon}(x_i), 2R)$ cover $\pi_{\log 1/\varepsilon} B$, hence the balls $B_d(\pi_{\log 1/\varepsilon}(x_i), 3R)$ cover $g^{\log 1/\varepsilon} B_3$. If $\alpha = \alpha_2(3R)$ is chosen as in Lemma 9, then $q\alpha \geq (1/\varepsilon)^h \lambda$ and $q \cdot \varepsilon^h \geq \lambda/\alpha$ which yields the lemma. \square

Remark. In fact we have shown that $\liminf_{\varepsilon \rightarrow 0} \beta_\varepsilon \Omega > 0$ for all nontrivial open subsets Ω of $\partial\tilde{M} - \xi$.

For a fixed number $R > 0$ we investigate now how $\beta^v = \beta^{v,R}$ varies with $v \in \tilde{S}\tilde{M}$.

LEMMA 12. Let $\Omega \subset \partial\tilde{M}$ be a compact subset with nonempty complement. Then the map $v \rightarrow \beta^v \Omega$ is continuous on $\tilde{S}\tilde{M} - \{w \mid \varphi_w(-\infty) \in \Omega\}$.

Proof. We show first that $v \rightarrow \beta^v \Omega$ is upper semi-continuous on its domain of definition.

Let $v \in \tilde{S}\tilde{M} - \{w \mid \varphi_w(-\infty) \in \Omega\}$; since $\beta^v \Omega < \infty$ by Lemma 10 it suffices to find for every $\delta > 0$ an open neighbourhood U of v in $\tilde{S}\tilde{M} - \{w \mid \varphi_w(-\infty) \in \Omega\}$ such that $\beta^w \Omega \leq (1 + \delta)\beta^v \Omega$ for all $w \in U$.

Since $\Omega \subset \partial\tilde{M} - \varphi_v(-\infty)$ is compact, $A = \{y \in \partial\tilde{M} \mid \rho_{v,R}(x, y) \leq 1 \text{ for some } x \in \Omega\}$ is a compact subset of $\partial\tilde{M} - \varphi_v(-\infty)$. By Lemma 7 there is for $\lambda = (1/(1 + \delta))^{1/h}$ a neighbourhood U of v in $\tilde{S}\tilde{M}$ such that for all $x, y \in A$ and all $w \in U$ $\lambda \rho_{w,R}(x, y) \leq \rho_v(x, y)$. Let $\varepsilon \in (0, 1)$ and $\{y_1, \dots, y_m\} \subset \Omega$ be a subset of maximal cardinality with the property that the balls $B_{\rho_{w,R}}(y_i, \varepsilon)$ are pairwise disjoint. Then the sets $B_{\rho_{w,R}}(y_i, \lambda\varepsilon) \cap A$ are pairwise disjoint. But by the choice of A for every $y \in \Omega$ the $\rho_{v,R}$ -ball of radius $\lambda\varepsilon < 1$ centred at y is contained in A . This implies $\beta_\varepsilon^w(\Omega) \leq (1 + \delta)\beta_{\lambda\varepsilon}^v(\Omega)$ and since $\varepsilon \in (0, 1)$ was arbitrary, $\beta^w(\Omega) \leq (1 + \delta)\beta^v(\Omega)$. The lower semi-continuity of the map is shown similarly. \square

Remark. The proof of Lemma 12 yields the following fact: If $\Omega \subset \partial\tilde{M} - \varphi_v(-\infty)$ is compact and $\beta^v(\Omega) = 0$, then $\beta^w(\Omega) = 0$ for all $w \in \partial\tilde{M} - \Omega$.

For $\rho = \rho_{v,R}$ let $\bar{B}_\rho(x, \varepsilon)$ ($x \in \partial\tilde{M}$) be the closure of $B_\rho(x, \varepsilon)$ in $\partial\tilde{M}$. Let $\beta = \beta^{v,R}$ be as above and $\xi = \varphi_v(-\infty)$.

COROLLARY 13. *There is a number $\kappa > 0$ such that for all $x \in \partial\tilde{M} - \xi$ and all $\varepsilon > 0$, $\kappa\varepsilon^h \leq \beta\bar{B}_\rho(x, \varepsilon) \leq \kappa^{-1}\varepsilon^h$.*

Proof. Use the notations of the proof of Lemma 12. Let $D \subset \tilde{M}$ be a compact fundamental domain for $\Gamma = \pi_1 M$ and define $u: \tilde{SM}|_D \rightarrow \mathbb{R}$, $w \rightarrow u(w) = \beta^w \bar{B}_{\rho_w}(\varphi_w(\infty), 1)$.

Let $\{w_j\} \subset \tilde{SM}|_D$ be a sequence such that $u(w_j) \rightarrow \sup\{u(w) | w \in \tilde{SM}|_D\}$. By the compactness of $\tilde{SM}|_D$ we may assume that $\{w_j\}$ converges to $w \in \tilde{SM}|_D$ as $j \rightarrow \infty$.

Since by Lemma 7 ρ_w depends continuously on $w \in \tilde{SM}$, there is a number $i_0 > 0$ such that the closed ball of radius 1 around $\varphi_{w_i}(\infty)$ with respect to ρ_{w_i} is contained in $B = \bar{B}_{\rho_w}(\varphi_w(\infty), 2)$. Thus $u(w_i) \leq \beta^w B$ for all $i \geq i_0$ and Lemma 12 shows $\limsup_{j \rightarrow \infty} u(w_j) \leq \beta^w B < \infty$. A similar argument yields $\inf\{u(w) | w \in \tilde{SM}|_D\} > 0$, i.e. there is a number $\kappa > 0$ such that $\kappa \leq u(w) \leq 1/\kappa$ for all $w \in \tilde{SM}|_D$.

For $\rho = \rho_{v,R}$ and $x \in \partial\tilde{M} - \varphi_v(-\infty)$, $\bar{B}_\rho(x, \varepsilon)$ is the projection in $\partial\tilde{M} - \varphi_v(-\infty)$ of the set $\bar{B}_d(\pi_{\log 1/\varepsilon, v} x, R) \cap \theta_v^{-1}(\log 1/\varepsilon)$ along the geodesics which are tangent to $W^u(v)$. Choose $\Phi \in \Gamma$ such that $\Phi(\pi_{\log 1/\varepsilon, v} x) \in D$. If $w \in \tilde{SM}$ is the tangent at $\log 1/\varepsilon$ of the geodesic $t \rightarrow \Phi(\pi_{t, v} x)$, then $\Phi\bar{B}_\rho(x, \varepsilon) = \bar{B}_{\rho_w}(\Phi x, 1)$ and $\Phi\bar{B}_\rho(y, \delta) = \bar{B}_{\rho_w}(\Phi y, \varepsilon^{-1}\delta)$ for all $y \in \bar{B}_\rho(x, \varepsilon)$ and all $\delta > 0$ (recall that Γ acts on $\partial\tilde{M}$ in a natural way). By the definition of β this means $\beta\bar{B}_\rho(x, \varepsilon) = \varepsilon^h \beta^w \bar{B}_{\rho_w}(\Phi x, 1)$, hence $\kappa\varepsilon^h \leq \beta\bar{B}_\rho(x, \varepsilon) \leq \kappa^{-1}\varepsilon^h$. □

Recall the definition of the *h-dim. spherical measure* $\sigma = \sigma^{v,R} = \sigma_\rho$ on $\partial\tilde{M} - \xi = \partial\tilde{M} - \varphi_v(-\infty)$ associated to $\rho = \rho_{v,R}$ (see [3]). For $\Omega \subset \partial\tilde{M} - \xi$, $\sigma(\Omega) = \sup_{\varepsilon > 0} \sigma_\varepsilon(\Omega)$ where $\sigma_\varepsilon(\Omega) = \inf\{\sum_{j=1}^\infty \varepsilon_j^h | \varepsilon_j \leq \varepsilon \text{ and } \Omega \subset \bigcup_{j=1}^\infty \bar{B}_\rho(x_j, \varepsilon_j) \text{ for some } x_j \in \Omega\}$. Corollary 5 implies that σ is a Borel regular measure, i.e. $\sigma(\Omega) = \sup\{\sigma(B) | B \subset \Omega \text{ compact}\}$ for every Borel-subset Ω of $\partial\tilde{M} - \xi$ (compare the argument in [3] for spherical measures associated to distances).

COROLLARY 14. *Let $c > 0$ be as in Corollary 5 and $\kappa > 0$ be as in Corollary 13. Then $c^h \beta(\Omega) \geq \sigma(\Omega) \geq \kappa\beta(\Omega)$ for every Borel set $\Omega \subset \partial\tilde{M} - \xi$.*

Proof. By the definition of β and the fact that σ is Borel-regular it suffices to show the claim for compact subsets Ω of $\partial\tilde{M} - \xi$.

Let $\Omega \subset \partial\tilde{M} - \xi$ be compact, let $\varepsilon > 0$ and $\{x_1, \dots, x_q\} \subset \Omega$ be a set of maximal cardinality with the property that the balls $B_\rho(x_i, \varepsilon)$ are pairwise disjoint. Then for every $y \in \Omega$ there is $i \in \{1, \dots, q\}$ and $z \in B_\rho(x_i, \varepsilon)$ such that $\rho(y, z) < \varepsilon$. Hence by Corollary 5 the balls $B_\rho(x_i, c\varepsilon)$ cover Ω , which shows $\sigma_{c\varepsilon}(\Omega) \leq q\varepsilon^h c^h = c^h \beta_\varepsilon(\Omega)$. Thus $\sigma(\Omega) \leq c^h \beta(\Omega)$. On the other hand, for each $\delta > 0$ there is a covering of Ω by balls $\bar{B}_\rho(x_i, \varepsilon_i)$ ($i \geq 1$) such that $\sum_{i=1}^\infty \varepsilon_i^h \leq \sigma(\Omega) + \delta$. Corollary 13 and property (2) of β implies

$$\beta(\Omega) \leq \frac{1}{\kappa} \sum \varepsilon_i^h \leq \frac{1}{\kappa} \sigma(\Omega) + \frac{\delta}{\kappa}$$

Since $\delta > 0$ was arbitrary, this is the claim. □

Now we are left with showing that the measures $\sigma^{v,R}$ indeed give rise to the Bowen–Margulis measure on \tilde{SM} .

For a fixed $R > 0$ recall that $\sigma^v = \sigma^{v,R}$ depends on the choice of $v \in \tilde{SM}$ and $\sigma^v = \sigma^w$ if $w \in W^{su}(v)$. Now the strong unstable manifold $W^{su}(v)$ has a canonical identification with $\partial\tilde{M} - \varphi_v(-\infty)$ via the map $\pi : w \rightarrow \varphi_w(\infty)$. Thus σ^v can be viewed as a Borel measure on $W^{su}(v)$. In this way we obtain a Borel measure μ^{su} on the leaves of the foliation W^{su} . If $v \in \tilde{SM}$ and $t \in \mathbb{R}$, then $\theta_v = \theta_{g^t v} + t$, hence $\rho_{g^t v} = e^t \rho_v$ and $\mu^{su} \circ g^t = e^{ht} \mu^{su}$.

We have to construct a measure on \tilde{SM} which is invariant under the geodesic flow and the isometry group of \tilde{M} and restricts to the measures μ^{su} on the leaves of W^{su} . We first define a Borel measure μ^u on the leaves of the foliation W^u as follows: For $A \subset W^u$ let $\mu^u(A)$ be the infimum of all numbers $\sum_{j=1}^\infty \int_{T_j} \mu^{su}(g^t A_j) dt$ corresponding to all families of Borel sets $T_j \subset \mathbb{R}$, $A_j \subset W^{su}$ with $A \subset \bigcup_{j=1}^\infty (\bigcup_{t \in T_j} g^t A_j)$. μ^u can be viewed as a weighted product measure on $W^u(v) \approx W^{su}(v) \times \mathbb{R}$ ($v \in A$; see [3] p. 114). If $A = \bigcup_{s \in T} g^s \tilde{A}$ for some Borel-set $\tilde{A} \subset W^{su}$ and a Borel-set $T \subset \mathbb{R}$, then $\mu^u(A) = \int_T \mu^{su}(g^s \tilde{A}) ds = \mu^{su}(\tilde{A}) \int_T e^{hs} ds$ (this follows as the analogous statement for product measures, see [3]). Furthermore $\mu^u(g^t A) = e^{ht} \mu^u(A)$ for all $t \in \mathbb{R}$.

For $v, w \in \tilde{SM}$ such that $\varphi_v(-\infty) \neq \varphi_w(\infty)$ there is a geodesic γ joining $\varphi_v(-\infty) = \gamma(-\infty)$ to $\varphi_w(\infty) = \gamma(\infty)$, and γ is unique up to reparametrization. Thus the intersection $W^u(v) \cap W^{su}(w)$ consists of a unique point. Following Margulis ([6]) we call sets $A_1 \subset W^u$, $A_2 \subset W^u(w)$ *equivalent* if $A_2 = \{W^u(w) \cap W^{su}(v) \mid v \in A_1\}$. If $A_1 \subset W^u$ and $w \in \tilde{SM}$ is such that $\varphi_w(-\infty) \notin \pi A_1$, then A_1 is equivalent to a subset of $W^u(w)$.

For equivalent sets $A_1, A_2 \subset W^u$ there is a homeomorphism $\Psi : A_1 \rightarrow A_2$ such that $\Psi(v) \in W^{su}(v)$ for all $v \in A_1$. A_1 and A_2 are called ε -*equivalent* if A_1 and A_2 are equivalent and if furthermore the homeomorphism $\Psi : A_1 \rightarrow A_2$ satisfies $d(Pw, P\Psi w) < \varepsilon$ for all $w \in A_1$. If A_1 and A_2 are ε -equivalent for some $\varepsilon > 0$, then $\tilde{\mu}^u(A_1) = \tilde{\mu}^u(A_2)$ ([6]). This is also true for μ^u .

LEMMA 15. *If $A_1, A_2 \subset W^u$ are relatively compact and equivalent, then $\mu^u A_1 = \mu^u A_2$.*

Proof. We want to show $\mu^u A_1 \geq \mu^u A_2$ if A_1, A_2 are as above. Since μ^u is Borel regular we may assume that A_1 is compact. Denote by W_i^u the leaf of the foliation W^u which contains A_i .

Let $\bar{v} \in A_1$, $w \in A_2$ and choose a compact subset Ω of $W^{su}(\bar{v})$ such that $\pi\Omega$ is a compact neighbourhood of πA_1 in $\partial\tilde{M} - \varphi_w(-\infty)$. Then there is a number $\tau > 0$ such that $V = \bigcup_{-\tau \leq t \leq \tau} g^t \Omega$ is a compact neighbourhood of A_1 in W_1^u .

By the choice of Ω , V is equivalent to a subset of W_2^u . This means that there is a homeomorphism Ψ of V onto a compact neighbourhood ΨV of A_2 in W_2^u such that $\Psi A_1 = A_2$ and $\Psi(v) \in W^{su}(v) \cap W_2^u$ for all $v \in V$.

By the definition of μ^u , for every $\delta > 0$ there are Borel sets $S_j \subset (-\tau, \tau)$, $\Omega_j \subset \Omega$ ($j \geq 1$) such that $A_1 \subset \bigcup_{j=1}^\infty (\bigcup_{s \in S_j} g^s \Omega_j) \subset V$ and $\mu^u(A_1) \geq \sum_{j=1}^\infty \int_{S_j} \mu^{su}(g^s \Omega_j) ds - \delta$. Since $\mu^u(A_2) \leq \sum_{j=1}^\infty \mu^u(\Psi(\bigcup_{s \in S_j} g^s \Omega_j))$ and $\mu^u(\bigcup_{s \in S_j} g^s \Omega_j) = \int_{S_j} \mu^{su}(g^s \Omega_j) ds$, it thus suffices to show $\mu^u(B_1) \geq \mu^u(\Psi B_1)$ for every subset B_1 of V of the form $B_1 = \bigcup_{s \in S} g^s B$ with Borel sets $S \subset [-\tau, \tau]$, $B \subset \Omega$.

Let $\delta > 0$, B_1 as above and $\lambda = \mu^{su}(g^\tau \Omega) < \infty$. Since the Lebesgue-measure on coincides with the 1-dim. spherical measure with respect to the Euclidean distance there are countably many closed intervals $S_j \subset [-\tau, \tau] (j \geq 1)$ such that $\int_S dt - \sum_{j=1}^\infty \int_{S_j} dt - \delta/\lambda$ and $S \subset \bigcup_{j=1}^\infty S_j$. Write $T_j = (S \cap S_j) \setminus \bigcup_{i=1}^{j-1} S_i$; then $S = \bigcup_{j=1}^\infty T_j$ and

$$\int_S dt = \sum_{j=1}^\infty \int_{T_j} dt \geq \sum_{j=1}^\infty \int_{T_j} dt + \sum_{j=1}^\infty \int_{S_j - T_j} dt - \delta/\lambda.$$

Thus the choice of λ yields

$$\sum_{j=1}^\infty \int_{S_j} \mu^{su}(g^t B) dt \leq \int_S \mu^{su}(g^t B) dt + \delta.$$

Since the sets $\bigcup_{s \in S_j} g^s B (j \geq 1)$ cover B_1 , it follows as above that we need only consider sets $B_1 = \bigcup_{t \in T} g^t B$ where $B \subset \Omega$ is Borel and $T \subset [-\tau, \tau]$ is a closed interval.

Assume without loss of generality that $B_1 = \bigcup_{-\nu \leq s \leq \nu} g^s B$ for some $\nu > 0$. If eventually enlarging Ω we may also suppose that the closure \bar{B} of B is contained in the interior of Ω . Define $B_2 = \Psi B_1$ and let $\varepsilon > 0$. By continuity there is for every $v \in \bar{B}$ an open neighbourhood $U(v)$ of v in Ω such that $d(P\Psi(w), P\Psi(v)) < \varepsilon$ for all $w \in U(v)$. The compact set \bar{B} admits a finite cover by open sets $U(v_i) (v_i \in \bar{B} \text{ and } i = 1, \dots, k)$. In particular B has a Borel-partition $B = \sum_{i=1}^k C^i$ into pairwise disjoint sets $C^i \subset (U(v_i) \cap B)$.

Define $D^i = \bigcup_{-\nu \leq s \leq \nu} g^s C^i$; then $B_1 = \bigcup_{i=1}^k D^i$ and $D^i \cap D^j = \emptyset$ if $i \neq j$, i.e. $\mu^u(B_1) = \sum_{i=1}^k \mu^u(D^i)$.

For fixed $i \in \{1, \dots, k\}$ we want to compare the measures $\mu^u(D^i)$ and $\mu^u(\Psi D^i)$.

This is done by estimating the measure of a set $\tilde{E}^i \supset \Psi(D^i)$ which is defined by $\tilde{E}^i = \bigcup_{-\nu - \varepsilon < s < \nu + \varepsilon} g^s E^i$ where $E^i = \{w \in W^{su}(\Psi v_i) \mid \pi w \in \pi C^i\}$.

We have to show $\tilde{E}^i \supset \Psi(D^i)$: Indeed, for every $v \in C^i$ there is a number $s(v) \in [-\nu, \nu]$ such that $g^{s(v)} \Psi(v) \in E^i$. Then $s(v_i) = 0$ and consequently $s(v) \leq d(P\Psi(v), P\Psi(v_i)) < \varepsilon$ for all $v \in C^i$ by the choice of C^i . Since $\pi^{-1}(\pi v) \cap \Psi(D^i) \subset \bigcup_{-\nu \leq s \leq \nu} g^s \Psi(v)$ this implies $\Psi(D^i) \subset \tilde{E}^i$.

In order to estimate $\mu^u(\tilde{E}^i)$ we have to estimate $\mu^{su}(E^i)$. For this purpose let $v \in C^i, w \in E^i$ and $\rho = \rho_{v,R}, \tilde{\rho} = \rho_{w,R}$. Since \bar{B} is compact and $\Psi(g^s v) = g^s \Psi(v)$ for all $s \in [-\nu, \nu]$, there is a number $t_0 \in \mathbb{R}$ such that $g^t B_1$ and $g^t B_2$ are ε -equivalent for all $t \geq t_0$, i.e. $d(Pg^t v, Pg^t \Psi(v)) < \varepsilon$ for all $v \in B_1$ (compare [6]).

Let $\delta < e^{-t_0}$. For every $x \in \pi C^i$ there are unique points $w_1(x) \in g^{\log 1/\delta} C^i, w_2(x) \in g^{\log 1/\delta} E^i$ such that $\pi w_i(x) = x (i = 1, 2)$. The choice of δ yields

$$d(Pw_1(x), Pw_2(x)) \leq d(Pw_1(x), P\Psi w_1(x)) + d(P\Psi w_1(x), Pw_2(x)) < \varepsilon + |s(g^{-\log 1/\delta} w_1(x))| < 2\varepsilon.$$

Thus $y \in \bar{B}_\rho(x, \delta) \cap \pi C^i$, i.e. $d(Pw_1(x), Pw_1(y)) \leq R$, implies $d(Pw_2(x), Pw_2(y)) \leq R + 2\varepsilon$. If we define

$$\tau(\varepsilon) = \left(\left(\sinh \frac{a}{2} R(1 + 2\varepsilon) \right) / \left(\sinh \frac{a}{2} R \right) \right)^{1/a}$$

then Lemma 6 shows as before that $\bar{B}_\rho(x, \delta) \cap \pi C^i \subset \bar{B}_\rho(x, \tau(\varepsilon)\delta) \cap \pi C^i$ for $x \in \pi C^i, \delta < \varepsilon^{-t_0}$.

Given $\delta \in (0, e^{-\delta_0})$ arbitrary, there is a covering of πC^i by balls $\bar{B}_\rho(x_j, \delta_j)$ ($x_j \in \pi C^i, j \geq 1, \delta_j \leq \delta$) such that $\sum_{j=1}^\infty \delta_j^h \leq \mu^{su}(C^i) + \delta$. By the above consideration the balls $\bar{B}_\rho(x_j, \tau(\varepsilon)\delta_j)$ cover $\pi C^i = \pi E^i$ which implies $\mu^{su}(E^i) \leq (\tau(\varepsilon))^h \mu^{su}(C^i)$.

Using this inequality we obtain

$$\begin{aligned} \mu^u(\tilde{E}^i) &= \int_{-\nu-\varepsilon}^{\nu+\varepsilon} e^{ht} \mu^{su}(E^i) dt \\ &= \frac{1}{h} (e^{h(\nu+\varepsilon)} - e^{-h(\nu+\varepsilon)}) \mu^{su}(E^i) \\ &\leq \frac{1}{h} \tau(\varepsilon)^h (e^{h\nu} e^{h\varepsilon} - e^{-h\nu} e^{-h\varepsilon}) \mu^{su}(C^i) \end{aligned}$$

hence

$$\mu^u(B_2) \leq \sum_{i=1}^k \mu^u(\tilde{E}^i) \leq \frac{1}{h} \tau(\varepsilon)^h (e^{h\nu} e^{h\varepsilon} - e^{-h\nu} e^{-h\varepsilon}) \mu^{su}(B).$$

On the other hand $\mu^u(B_1) = h^{-1}(e^{h\nu} - e^{-h\nu}) \mu^{su}(B)$; since $\varepsilon > 0$ was arbitrary and $\tau(\varepsilon) \rightarrow 1$ as $\varepsilon \rightarrow 0$, this shows $\mu^u(B_2) \leq \mu^u(B_1)$ and finishes the proof of the lemma. □

For every $v \in \tilde{SM}$, the leaf $W^{ss}(v)$ of the strong stable foliation has a canonical identification with $W^{su}(-v)$. Thus μ^{su} induces a measure μ^{ss} on the leaves of W^{ss} . Clearly $\mu^{ss} \circ g^t = e^{-ht} \mu^{ss}$.

As in [6], Lemma 15 yields the existence of a g^t -invariant measure μ on \tilde{SM} which restricts to μ^i on the leaves of W^i ($i = ss, u, su$). If $A \subset \tilde{SM}$ is compact and if $W^u(v) \cap A$ is equivalent to $W^u(w) \cap A$ for all $v, w \in A$, then we have

$$\mu(A) = \int_{W^u(v) \cap A} \mu^{ss}(W^{ss}(w) \cap A) d\mu^u$$

where $v \in A$ is arbitrary. Now μ^u and μ^{ss} are clearly invariant under the action of Γ on \tilde{SM} , hence the same is true for μ . Thus μ induces a finite Borel measure on SM which is positive on all open subsets of SM . The standard computation (see [2]) shows that the measure-theoretic entropy of this measure equals the topological entropy h of the geodesic flow on SM , so μ coincides indeed (up to a constant) with the Bowen–Margulis measure $\tilde{\mu}$. In particular the construction of μ and $\tilde{\mu}$ shows $\tilde{\mu}^{su} = \mu^{su}$ on the leaves of W^{su} .

Now let $\bar{\sigma} = \bar{\sigma}^{v,R}$ be the h -dim. spherical measure associated to $\eta = \eta_{v,R}$. Lemma 4 yields $\nu^h \bar{\sigma} \leq \sigma^{v,R} \leq \bar{\sigma}$; in particular $\bar{\sigma}$ is finite on compact subsets of $\partial \tilde{M} - \varphi_v(-\infty)$ and determines the same measure class as σ^v . The proof of Lemma 15 can easily be modified to be valid for the measure $\bar{\mu}^u$ on the leaves of W^u which is induced by the measures $\bar{\sigma}^{v,R}$ on $W^{su}(v) \approx \partial \tilde{M} - \varphi_v(-\infty)$. As above we obtain a measure $\bar{\mu}$ on \tilde{SM} in the measure class of μ which is invariant under g^t and Γ and restricts to $\bar{\sigma}^{v,R}$ on $W^{su}(v)$. By the ergodicity of the geodesic flow on SM with respect to μ , $\bar{\mu}$ equals μ up to a constant. This finishes the proof of the theorem.

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