Ergod. Th. & Dynam. Sys. (2017), **37**, 1862–1886
doi:10.1017/etds.2015.126

Embedding subshifts of finite type into the Fibonacci–Dyck shift

TOSHIHIRO HAMACHI† and WOLFGANG KRIEGER‡

† *Faculty of Mathematics, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka 819-0395, Japan*
(e-mail: hamachi@math.kyushu-u.ac.jp)
‡ *Institute for Applied Mathematics, University of Heidelberg, Im Neuenheimer Feld 294, 69120 Heidelberg, Germany*
(e-mail: krieger@math.uni-heidelberg.de)

(*Received* 29 *June* 2015 *and accepted in revised form* 16 *October* 2015)

Abstract. A necessary and sufficient condition is given for the existence of an embedding of an irreducible subshift of finite type into the Fibonacci–Dyck shift.

1. *Introduction*

Let Σ be a finite alphabet, and let S be the shift on the shift space $\Sigma^{\mathbb{Z}}$,

$$S((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}, \quad (x_i)_{i\in\mathbb{Z}} \in \Sigma^{\mathbb{Z}}.$$

An S-invariant closed subset X of $\Sigma^{\mathbb{Z}}$ is called a subshift. For an introduction to the theory of subshifts see [**Ki**] or [**LM**]. A word is called admissible for the subshift $X \subset \Sigma^{\mathbb{Z}}$ if it appears in a point of X. A subshift is uniquely determined by its language of admissible words.

Among the first examples of subshifts are the subshifts of finite type. A subshift of finite type is constructed from a finite set $\mathcal{F}$ of words in the alphabet Σ as the subshift that contains the points in $\Sigma^{\mathbb{Z}}$ in which no word in $\mathcal{F}$ appears. Other prototypical examples of subshifts are the Dyck shifts. To recall the construction of the Dyck shifts [**Kr1**], let $N > 1$ and let

$$\alpha_-(n), \alpha_+(n), \quad 0 \le n < N,$$

be the generators of the Dyck inverse monoid [**NP**] $\mathcal{D}_N$ with the relations

$$\alpha_-(n)\alpha_+(m) = \begin{cases} \mathbf{1} & \text{if } n = m, \\ 0 & \text{if } n \ne m. \end{cases}$$

The Dyck shifts are defined as the subshifts

$$D_N \subset (\{\alpha_-(n) : 0 \le n < N\} \cup \{\alpha_+(n) : 0 \le n < N\})^{\mathbb{Z}}$$

with the admissible words $(\sigma_i)_{1\le i\le I}$, $I \in \mathbb{N}$, of D_N, $N > 1$, given by the condition

$$\prod_{1\le i\le I} \sigma_i \neq 0.$$

In [**HI**], a necessary and sufficient condition was given for the existence of an embedding of an irreducible subshift of finite type into a Dyck shift. In [**HIK**], this result was extended to a wider class of target shifts that contains the $\mathcal{D}_N$-presentations. With the semigroup $\mathcal{D}_N^-$ ($\mathcal{D}_N^+$), that is freely generated by $\{\alpha_-(n) : 0 \le n < N\}$ ($\{\alpha_+(n) : 0 \le n < N\}$), $\mathcal{D}_N$-presentations can be described as arising from a finite irreducible directed labelled graph with vertex set $\mathcal{V}$, edge set Σ and a label map

$$\lambda : \Sigma \to \mathcal{D}_N^- \cup \{\mathbf{1}\} \cup \mathcal{D}_N^+$$

that extends to directed paths $b = (b_i)_{1\le i\le I}$, $I > 1$, in the directed graph $(\mathcal{V}, \Sigma)$, by $\lambda(b) = \prod_{1\le i\le I} \lambda(b_i)$. It is required that there exists, for $U, W \in \mathcal{V}$ and for $\beta \in \mathcal{D}_N$, a path b from U to W such that $\lambda(b) = \beta$. The $\mathcal{D}_N$-presentation $X(\mathcal{V}, \Sigma, \lambda)$ is the subshift with alphabet Σ and with the set of admissible words given by the set of directed finite paths b in the graph $(\mathcal{V}, \Sigma, \lambda)$ such that $\lambda(b) \neq 0$.

For Dyck shifts, the notion of a multiplier was introduced in [**HI**]. A multiplier of the Dyck shift D_N, $N \in \mathbb{N}$, or, more generally, of a $\mathcal{D}_N$-presentation [**HIK**], is an equivalence class of primitive words in the symbols $\alpha(n)$, $0 \le n \le N$. Here a word is called primitive if it is not the power of another word, and two primitive words are equivalent if one is a cyclic permutation of the other. The multipliers are the primitive necklaces of combinatorics [**BP**, §4]. We use one of its representatives as a notation for a multiplier.

We denote the period of a periodic point p of a $\mathcal{D}_N$-presentation $X(\mathcal{V}, \Sigma, \lambda)$ by $\Pi(p)$. A periodic point $p = (p_i)_{i\in\mathbb{Z}}$ of a $\mathcal{D}_N$-presentation $X(\mathcal{V}, \Sigma, \lambda)$, and its orbit, are said to be neutral if there exists an $i \in \mathbb{Z}$ such that $\lambda((p_j)_{i\le j<i+\Pi(p)}) = \mathbf{1}$, and they are said to have negative (positive) multiplier if there exists an $i \in \mathbb{Z}$ such that $\lambda((p_j)_{i\le j<i+\Pi(p)}) \in \mathcal{D}_N^-(\mathcal{D}_N^+)$. More precisely, given a multiplier μ, a periodic point $p = (p_i)_{i\in\mathbb{Z}}$ of a $\mathcal{D}_N$-presentation $X(\mathcal{V}, \Sigma, \lambda)$, and its orbit, are said to have (negative) multiplier μ_- if there exists an $i \in \mathbb{Z}$ and a representative $(\alpha(n_j))_{1\le j\le J}$ of μ such that $\lambda((p_j)_{i\le j<i+\Pi(p)})$ is equal to $\prod_{1\le j\le J} \alpha_-(n_j)$, and are said to have (positive) multiplier μ_+ if there exists an $i \in \mathbb{Z}$ and a representative $(\alpha(n_j))_{1\le j\le J}$ of μ such that $\lambda((p_j)_{i\le j<i+\Pi(p)})$ is equal to $\prod_{J\ge j\ge 1} \alpha_+(n_j)$. Denote the set of periodic orbits of length n of the $\mathcal{D}_N$-presentation $X(\mathcal{V}, \Sigma, \lambda)$ with negative (positive) multiplier by $\mathcal{O}_n^-(X(\mathcal{V}, \Sigma, \lambda)$ ($\mathcal{O}_n^+(X(\mathcal{V}, \Sigma, \lambda))$, and denote the set of its periodic orbits of length n with multiplier $\mu_-(\mu_+)$ by $\mathcal{O}_n(\mu_-)(X(\mathcal{V}, \Sigma, \lambda)$ ($\mathcal{O}_n(\mu_+)(X(\mathcal{V}, \Sigma, \lambda))$. The notion of an exceptional multiplier was introduced in [**HI**]. A multiplier μ of a $\mathcal{D}_N$-presentation $X(\mathcal{V}, \Sigma, \lambda)$ is said to be exceptional at period $n \in \mathbb{N}$ if

$$\begin{aligned}\operatorname{card}(&\mathcal{O}_n(\mu_-)(X(\mathcal{V}, \Sigma, \lambda)) \cup \mathcal{O}_n(\mu_+)(X(\mathcal{V}, \Sigma, \lambda)))\\ &> \max\{\operatorname{card}(\mathcal{O}_n^-(X(\mathcal{V}, \Sigma, \lambda))), \operatorname{card}(\mathcal{O}_n^+(X(\mathcal{V}, \Sigma, \lambda)))\}.\end{aligned}$$

The Fibonacci–Dyck shift is the Markov–Dyck shift [**M**] of the Fibonacci graph. We introduce the Fibonacci graph as the directed graph with vertex set $\{0, 1\}$ and edge set $\{\beta(0), \beta, \beta(1)\}$: $\beta(0)$ is a loop at vertex 0, the edge $\beta(1)$ goes from vertex 1 to vertex 0

https://doi.org/10.1017/etds.2015.126 Published online by Cambridge University Press

and the edge β from vertex 0 to vertex 1. Let $(\{0, 1\}, \{\beta^-(0), \beta^-, \beta^-(1)\})$ be a copy of the Fibonacci graph, and let $(\{0, 1\}, \{\beta^+(0), \beta^+, \beta^+(1)\})$ be its reversal. The Fibonacci–Dyck shift is given by a $\mathcal{D}_2$-presentation, with

$$\mathcal{V} = \{0, 1\}, \quad \Sigma = \{\beta^-(0), \beta^-, \beta^-(1), \beta^+(0), \beta^+, \beta^+(1)\},$$

and

$$\lambda : \Sigma \to \{\alpha_-(0), \alpha_-(1), \alpha_+(0), \alpha_+(1)\},$$

given by

$$\lambda(\beta^-(0)) = \alpha_-(0), \quad \lambda(\beta^+(0)) = \alpha_+(0),$$
$$\lambda(\beta^-(1)) = \alpha_-(1), \quad \lambda(\beta^+(1)) = \alpha_+(1),$$
$$\lambda(\beta_-) = \lambda(\beta_+) = \mathbf{1}.$$

For this $\mathcal{D}_2$-presentation we choose a label map λ that assigns the label $\mathbf{1}$ to the edges β^- and β^+, since these edges arise from the splitting of the edge β in the Fibonacci graph. The edge β is contracted to a vertex in the procedure that turns the Fibonacci graph into the 1-vertex graph with two loops, whose graph inverse semigroup is $\mathcal{D}_2$. (For this part of the theory see [**Kr2**, **HK**]. In [**HK**, §2], it is shown that the Fibonacci–Dyck shift has Property (A) and, in [**HK**, §3], it is shown that its associated semigroup is $\mathcal{D}_2$. The procedure is also described in [**HK**, §3].) In this paper we consider, exclusively, the Fibonacci–Dyck shift F.

The label map λ can be written in the form of a matrix with entries in the semigroup ring of $\mathcal{D}_2$, that is

$$\begin{pmatrix} a^-(0) + a^+(0) & \mathbf{1} + a^+(1) \\ \mathbf{1} + a^-(1) & 0 \end{pmatrix}.$$

Taking the adjoint and applying the involution of the semigroup ring of $\mathcal{D}_2$ to its entries, does not change the matrix. This symmetry property of the matrix makes visible the time reversal ρ of F that is obtained by setting

$$\chi(\beta^-) = \beta^+, \qquad \chi(\beta^+) = \beta^-,$$
$$\chi(\beta^-(0)) = \beta^+(0), \; \chi(\beta^+(0)) = \beta^-(0),$$
$$\chi(\beta^-(1)) = \beta^+(1), \; \chi(\beta^+(1)) = \beta^-(1),$$

and

$$\rho((x_i)_{i\in\mathbb{Z}}) = (\chi(x_{-i}))_{i\in\mathbb{Z}}, \quad x \in F.$$

We denote the set of multipliers of F by $\mathcal{M}$. The time reversal ρ maps the set $\mathcal{O}_n(\mu_-)$ bijectively onto the set $\mathcal{O}_n(\mu_+)$ and we can note the following lemma.

LEMMA (a).

$$\operatorname{card}(\mathcal{O}_n(\mu_-)) = \operatorname{card}(\mathcal{O}_n(\mu_+)), \quad n \in \mathbb{N}, \mu \in \mathcal{M}.$$

We also note orbit counts of the Fibonacci–Dyck shift for small periods as a lemma.

https://doi.org/10.1017/etds.2015.126 Published online by Cambridge University Press

LEMMA (b).

$$\begin{aligned}\operatorname{card}(\mathcal{O}_3(\alpha^-(0))) &= 2,\\ \operatorname{card}(\mathcal{O}_5(\alpha^-(0))) &= 9,\\ \operatorname{card}(\mathcal{O}_5(\alpha^-(0)\alpha^-(1))) &= 5.\end{aligned}$$

$$\begin{aligned}\operatorname{card}(\mathcal{O}_4(\alpha^-(0))) &= 2,\\ \operatorname{card}(\mathcal{O}_4(\alpha^-(1))) &= 3,\\ \operatorname{card}(\mathcal{O}_6(\alpha^-(0))) &= 10,\\ \operatorname{card}(\mathcal{O}_6(\alpha^-(1))) &= 13.\end{aligned}$$

We note a consequence of Lemma (a), also as a lemma.

LEMMA (c). *A multiplier $\mu \in \mathcal{M}$ is exceptional at period $n \in \mathbb{N}$ if and only if*

$$\operatorname{card}(\mathcal{O}_n(\mu_-)) > \operatorname{card}\bigg(\bigcup_{\widetilde{\mu}\in\mathcal{M}\setminus\{\mu\}} \mathcal{O}_n(\widetilde{\mu}_-)\bigg).$$

The Fibonacci–Dyck shift has the exceptional multiplier $\alpha(0)$ that is exceptional at period one (and, in view of Lemma (b), also at period three), and the exceptional multiplier $\alpha(1)$ that is exceptional at period two. After introducing notation and terminology in §2, we show in §3 that the multiplier $\alpha(0)\alpha(1)$ is not exceptional. In §4, we prove that the remaining multipliers are not exceptional. Based on these results, and on the results of [**HIK**], we give, in §5, a necessary and sufficient condition for the existence of an embedding of an irreducible subshift of finite type into the Fibonacci–Dyck shift. The multipliers (especially the exceptional multiplier) enter here in an essential way. Moreover, we show in §6 that the multiplier $\alpha(0)$ is exceptional only at periods one and three, and we show in §7 that the multiplier $\alpha(1)$ is exceptional only at period two.

We denote the set of periodic points $p \in F$ with smallest period $n \in \mathbb{N}$ by P_n°, and we denote the set of points $p \in P_n^\circ$ with negative multiplier $\mu \in \mathcal{M}$ by $P_n^\circ(\mu_-)$. In the proofs of §§3, 4, 6 and 7 we construct, for the multiplier μ in question and for a suitably chosen period $k \in \mathbb{N}$, shift commuting injections

$$\eta_n : P_n^\circ(\mu_-) \to \bigcup_{\tilde{\mu}\in\mathcal{M}\setminus\{\mu\}} P_n^\circ(\tilde{\mu}_-), \quad n > k.$$

We do this by first constructing a partition of $P_n^\circ(\mu)$, $n > k$, (some sets of which may be empty), together with a shift commuting injection of each set of the partitions into $\bigcup_{\tilde{\mu}\in\mathcal{M}\setminus\{\mu\}} P_n^\circ(\tilde{\mu}_-)$, where we show injectivity on each set by describing how a point can be reconstructed from its image under η_n. Then we show that the images under η_n of the sets of the partition are disjoint.

2. *Preliminaries*

We denote the set of admissible words of the Fibonacci–Dyck shift by $\mathcal{L}$. We denote the empty word by ϵ, and the length of a word by ℓ.

We denote by $\mathcal{C}(0)$ the circular code of words $c = (c_i)_{1\le i\le I} \in \mathcal{L}$, $I > 1$, that begin with the symbol $\beta^-(0)$ and are such that $\lambda(c) = \mathbf{1}$, and there is no index J, $1 < J < I$, for

https://doi.org/10.1017/etds.2015.126 Published online by Cambridge University Press

which

$$\prod_{1\le j\le J} \lambda(c_j) = \mathbf{1}.$$

Also, we denote by $\mathcal{C}(1)$ the circular code [**BPR**, §7] of words $c = (c_i)_{1\le i\le I} \in \mathcal{L})$, $I > 3$ that begin with the symbol β^- and end with the word $\beta^+(1)\beta^+$ and are such that $\lambda(c) = \mathbf{1}$ and there is no index J, $1 < J < I$, for which $c_J = \beta^+$ and

$$\prod_{1\le j\le J} \lambda(c_j) = \mathbf{1}.$$

We also have the circular codes

$$\mathcal{C} = \mathcal{C}(0) \cup \{\beta^-\beta^+\} \cup \mathcal{C}(1)$$

and

$$\mathcal{C}^\circ(1) = \beta^-(1)\mathcal{C}^*\beta^+(1).$$

Note that

$$\mathcal{C}(0) = \beta^-(0)\mathcal{C}^*\beta^+(0), \quad \mathcal{C}(1) = \beta^-(\mathcal{C}^\circ(1)^*\setminus\{\epsilon\})\beta^+.$$

A bijection

$$\Psi_\circ : \mathcal{C}^\circ(1) \to \mathcal{C}(0)$$

is given by

$$\Psi_\circ(\beta^-(1) f \beta^+(1)) = \beta^-(0) f \beta^+(0), \quad f \in \mathcal{C}^*.$$

The bijection $\Psi_\circ : \mathcal{C}^\circ(1) \to \mathcal{C}(0)$ extends to a bijection

$$\Psi : \mathcal{C}^\circ(1)^* \to \mathcal{C}(0)^*$$

by

$$\Psi((c_k^\circ)_{1\le k\le K}) = ((\Psi_\circ(c_k^\circ))_{1\le k\le K}, \quad c_k^\circ \in \mathcal{C}^\circ(1),\ 1 \le k \le K,\ K \in \mathbb{N}.$$

We set

$$\mathcal{B}(1) = \beta^-\mathcal{C}^\circ(1)^*\beta^-(1),$$

and we define a bijection

$$\Xi : \mathcal{B}(1) \to \mathcal{C}(0)$$

by

$$\Xi(\beta^- f^\circ \beta^-(1)) = \beta^-(0)\Psi(f^\circ)\beta^+(0), \quad f^\circ \in \mathcal{C}^\circ(1)^*.$$

We also set

$$\mathcal{B}(0, 0) = \beta^-(0)\mathcal{C}^*\beta^-(0),$$

and we define a bijection

$$\Phi_0 : \mathcal{C}(0) \to \mathcal{B}(0, 0)$$

by

$$\Phi_0(\beta^-(0) f \beta^+(0)) = \beta^-(0) f \beta^-(0), \quad f \in \mathcal{C}^*.$$

We also set

$$\mathcal{B}(1, 1) = \mathcal{B}(1)\mathcal{C}^*\beta^-\beta^-(1),$$

https://doi.org/10.1017/etds.2015.126 Published online by Cambridge University Press

and we define a bijection

$$\Phi_1 : \mathcal{C}(1) \to \mathcal{B}(1, 1),$$

by

$$\Phi_1(\beta^- f^\circ \beta^-(1) f \beta^+(1) \beta^+) = \beta^- f^\circ \beta^-(1) f \beta^- \beta^-(1), \quad f^\circ \in \mathcal{C}^\circ(1)^*, \ f \in \mathcal{C}^*.$$

We set

$$\mathcal{Q}_0 = (\mathcal{C}(0) \cup \{\beta^-\beta^+\})^* \backslash \{\beta^-\beta^+\}^*,$$

and we define an injection

$$\Delta_0 : \mathcal{Q}_0 \to \mathcal{L}.$$

For this we let $f \in \mathcal{Q}_0$,

$$f = (c_k)_{1 \le k \le K}, \quad c_k \in \mathcal{C}(0) \cup \{\beta^-\beta^+\}, 1 \le k \le K, \ K \in \mathbb{N},$$

set

$$k_\circ(f) = \max\{k \in [1, K] : c_k \in \mathcal{C}(0)\},$$

and set

$$\Delta_0(f) = ((c_k)_{1 \le k < k_\circ(f)}, \Phi_0(c_{k_\circ(f)}), (c_k)_{k_\circ(f) < k \le K}).$$

We also set

$$\mathcal{Q}_1 = \mathcal{C}^* \backslash (\mathcal{C}(0) \cup \{\beta^-\beta^+\})^*,$$

and define an injection

$$\Delta_1 : \mathcal{Q}_1 \to \mathcal{L}.$$

For this we let $f \in \mathcal{Q}_1$,

$$f = (c_k)_{1 \le k \le K}, \quad c_k \in \mathcal{C}, 1 \le k \le K, \ K > 1,$$

set

$$k_\circ(f) = \max\{k \in [1, K] : c_k \in \mathcal{C}(1)\},$$

and set

$$\Delta_1(f) = ((c_k)_{1 \le k < k_\circ(f)}, \quad \Phi_1(c_{k_\circ(f)}), (c_k)_{k_\circ(f) < k \le K}).$$

We put a linear order on the alphabet of F. The resulting lexicographic order on $\mathcal{L}$ will be used to single out an element of $\mathbb{Z}/n\mathbb{Z}$ when constructing the shift commuting maps

$$\eta_n : P_n^\circ(\mu_-) \to \bigcup_{\tilde{\mu} \in \mathcal{M} \backslash \{\mu\}} P_n^\circ(\tilde{\mu}_-), \quad n > k.$$

If a word appears in a point $p \in P_n^\circ$ with its last symbol at index $i \in \mathbb{Z}$, then we say that the word appears at index i. For $p \in F$, we denote by $\mathcal{I}^{(0)}(p)$ ($\mathcal{I}^{(1)}(p)$) the set of indices $i \in \mathbb{Z}$ such that $p_i = \beta_-(0)(\beta_-(1))$ and

$$\lambda(p_{[i,i+k]}) \ne \mathbf{1}, \quad k \in \mathbb{N}.$$

We say that a word appears openly in $p \in F$ if it appears at an index $i \in \mathcal{I}^{(0)}(p) \cup \mathcal{I}^{(1)}(p)$.

For an element γ of the free monoid that is generated by $\alpha(0)$ and $\alpha(1)$ (or by $\alpha_-(0)$ and $\alpha_-(1)$), for example, for

$$\gamma = \prod_{0 \le n < N} \alpha(0)^{K(0,n)} \alpha(1)^{K(1,n)}, \quad K(0, n), K(1, n) \in \mathbb{Z}_+, 0 \le n < N,$$

https://doi.org/10.1017/etds.2015.126 Published online by Cambridge University Press

we use the notation

$$\nu_0(\gamma) = \sum_{0 \le n < N} K(0, n), \quad \nu_1(\gamma) = \sum_{0 \le n < N} K(1, n),$$

and, choosing for γ any representative of the multiplier $\mu \in \mathcal{M}$, we set

$$\nu_0(\mu) = \nu_0(\gamma), \quad \nu_1(\mu) = \nu_1(\gamma).$$

A point $p \in P_n^{\circ}(\mu_-)$, $\mu \in \mathcal{M}$, determines a $\kappa_p \in \mathbb{N}$ by

$$(\nu_0(\lambda(p_{[0,n)})), \nu_1(\lambda(p_{[0,n)}))) = \kappa_p(\nu_0(\mu), \nu_1(\mu)).$$

For

$$b = \beta^- f^{\circ} \beta^-(1) \in \mathcal{B}(1),$$

we set

$$\Lambda(b) = \ell(f^{\circ}),$$

and, for

$$b = \beta^-(0) f \beta^-(0) \in \mathcal{B}(0, 0),$$

we set

$$\Lambda(b) = \ell(f).$$

We also define a subset $\mathcal{D}(1, 1)$ of $\mathcal{B}(1)\mathcal{C}^*\mathcal{B}(1)$ by

$$\begin{aligned}\mathcal{D}(1, 1) = \{&\beta^- f^{\circ,-} \beta^-(1) f \beta^- f^{\circ,+} \beta^-(1) \in \beta^- \mathcal{C}^{\circ}(1)^* \beta^-(1) \mathcal{C}^* \beta^-(0) \mathcal{C}^{\circ}(1)^* \beta^-(1) : \\ &\ell(f) \ge \ell(f^{\circ,-}), \ell(f^{\circ,+})\},\end{aligned}$$

and a subset $\mathcal{D}(0, 1)$ of $\beta^-(0)\mathcal{C}^*\mathcal{B}(1)$ by

$$\mathcal{D}(0, 1) = \{\beta^-(0) f \beta^- f^{\circ} \beta^-(1) \in \beta^-(0)\mathcal{C}^* \beta^- \mathcal{C}^{\circ}(1)^* \beta^-(1) : \ell(f) \ge \ell(f^{\circ})\},$$

as well as a subset $\mathcal{D}(1, 0)$ of $\mathcal{B}(1)\mathcal{C}^*\beta^-(0)$ by

$$\mathcal{D}(1, 0) = \{\beta^- f^{\circ} \beta^-(1) f \beta^-(0) \in \beta^- \mathcal{C}^{\circ}(1)^* \beta^-(1)\mathcal{C}^* \beta^-(0) : \ell(f) \ge \ell(f^{\circ})\},$$

and, for

$$d = \beta^- f^{\circ,-} \beta^-(1) f \beta^- f^{\circ,+} \beta^-(1) \in \mathcal{D}(1, 1),$$

$$d = \beta^-(0) f \beta^- f^{\circ} \beta^-(1) \in \mathcal{D}(0, 1),$$

and

$$d = \beta^- f^{\circ} \beta^-(1) f \beta^-(0) \in \mathcal{D}(1, 0),$$

we set

$$\Lambda(d) = \ell(f).$$

For a point $p \in P_n^{\circ}(F)$, we denote by $\Lambda(p)$ the maximal value of $\Lambda(d)$ of words

$$d \in \mathcal{B}(0, 0) \cup \mathcal{B}(1) \cup \mathcal{D}(1, 1) \cup \mathcal{D}(0, 1) \cup \mathcal{D}(1, 0)$$

that appear openly in p, and we denote by $\mathcal{J}^{(0,0)}(p)(\mathcal{J}^{(1)}(p), \mathcal{J}^{(1,1)}(p), \mathcal{J}^{(0,1)}(p), \mathcal{J}^{(1,0)}(p)$ the set of indices at which there appears openly in p a word $d \in \mathcal{B}(0, 0)$ $(d \in \mathcal{B}(1), d \in \mathcal{D}(1, 1), d \in \mathcal{D}(0, 1), d \in \mathcal{D}(1, 0))$ such that $\Lambda(p) = \Lambda(d)$. Also, we denote by $\mathcal{J}_{\circ}^{(0,0)}(p)$ $(\mathcal{J}_{\circ}^{(1)}(p), \mathcal{J}_{\circ}^{(1,1)}(p), \mathcal{J}_{\circ}^{(0,1)}(p), \mathcal{J}_{\circ}^{(1,0)}(p))$ the set of indices $j_{\circ} \in \mathcal{J}^{(0,0)}(p)(\mathcal{J}^{(1)}(p), \mathcal{J}^{(1,1)}(p), \mathcal{J}^{(0,1)}(p), \mathcal{J}^{(1,0)}(p))$ such that the word $p_{(j_{\circ}-n, j_{\circ}]}$ is lexicographically the smallest one among the words $p_{(j-n,j]}$, $j \in \mathcal{J}^{(0,0)}(p)(\mathcal{J}^{(1)}(p), \mathcal{J}^{(1,1)}(p), \mathcal{J}^{(0,1)}(p), \mathcal{J}^{(1,0)}(p))$.

https://doi.org/10.1017/etds.2015.126 Published online by Cambridge University Press

3. *The multiplier $\alpha(0)\alpha(1)$*

LEMMA 1. *The multiplier $\alpha(0)\alpha(1)$ is not exceptional for the Fibonacci–Dyck shift.*

Proof. By Lemma (b), the multiplier $\alpha(0)\alpha(1)$ is not exceptional for the Fibonacci–Dyck shift for periods three and five.

We construct shift commuting injections

$$\eta_n : P_n^\circ(\alpha_-(0)\alpha_-(1)) \to \bigcup_{\tilde{\mu}\in\mathcal{M}\setminus\{\alpha(0)\alpha(1)\}} P_n^\circ(\tilde{\mu}_-), \quad n > 5.$$

Let $m > 2$. Let $P^{(0)}_{2m+1}$ be the set of $p \in P^\circ_{2m+1}(\alpha_-(0)\alpha_-(1))$ such that $\kappa_p = 1$, which means that

$$p_{(i-2m-1,i]} \in \mathcal{C}^*\beta^-(0)\mathcal{C}^*\beta^-\mathcal{C}^\circ(1)^*\beta^-(1), \quad i \in \mathcal{I}^{(1)}(p),$$

and, for $p \in P^{(0)}_{2m+1}$, let the words

$$f^-(p),\ f^+(p) \in \mathcal{C}^*, \quad f^\circ(p) \in \mathcal{C}^\circ(1)^*,$$

be given by writing

$$p_{(i-2m-1,i]} = f^-(p)\beta^-(0)f^+(p)\beta^-f^\circ(p)\beta^-(1), \quad i \in \mathcal{I}^{(1)}(p).$$

We set

$$P^{[1]}_{2m+1} = \{p \in P^{(0)}_{2m+1} : f^+(p) \in \mathcal{Q}_1\}.$$

The shift commuting map η_{2m+1} is to map a point $p \in P^{[1]}_{2m+1}$ to the point $q \in P^\circ_{2m+1}$ that is given by

$$q_{(i-2m-1,i]} = f^-(p)\beta^-(0)\Delta_1(f^+(p))\beta^-f^\circ(p)\beta^-(1), \quad i \in \mathcal{I}^{(1)}(p).$$

For $q \in \eta_{2m+1}(P^{[1]}_{2m+1})$,

$$q_{(i-2m-1,i]} \in \mathcal{C}^*\mathcal{B}(1,1)\mathcal{C}^*\mathcal{B}(1)\mathcal{C}^*\beta^-(0), \quad i \in \mathcal{I}^{(0)}(q),$$

and, with the words

$$g(q),\ g^-(q) \in \mathcal{C}^*,\ g^+(q) \in (\mathcal{C}(0) \cup \{\beta^-\beta^+\})^*, \quad b(q) \in \mathcal{B}(1,1),\ h(q) \in \mathcal{B}(1)$$

that are given by

$$q_{(i-2m-1,i]} = g^-(q)b(q)g^+(q)h(q)g(q)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(q),$$

the point $p \in P^{[0]}_{2m+1}$ can be reconstructed from its image q under η_{2m+1} as the point in $P^{(0)}_{2m+1}$ that is given by

$$p_{(i-2m-1,i]} = g^-(q)\Phi_1^{-1}(b(q))g^+(q)h(q)g(q)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(q).$$

We note that

$$\nu_0(\lambda(\eta_{2m+1}(p)_{[0,2m+1)})) = 1, \quad \nu_1(\lambda(\eta_{2m+1}(p)_{[0,2m+1)})) = 3, \quad p \in P^{[1]}_{2m+1}. \quad \text{(P.01[1])}$$

We set

$$P^{[0]}_{2m+1} = \{p \in P^{(0)}_{2m+1} : f^+(p) \in \mathcal{Q}_0\}.$$

https://doi.org/10.1017/etds.2015.126 Published online by Cambridge University Press

The shift commuting map η_{2m+1} is to map a point $p \in P^{[0]}_{2m+1}$ to the point $q \in P^{\circ}_{2m+1}(F)$ that is given by

$$q_{(i-2m-1,i]} = f^-(p)\beta^-(0)\Delta_0(f^+(p))\beta^- f^\circ(p)\beta^-(1), \quad i \in \mathcal{I}^{(1)}(p).$$

For $q \in \eta_{2m+1}(P^{[0]}_{2m+1})$,

$$q_{(i-2m-1,i]} \in \mathcal{C}^*\beta^-(0)\mathcal{B}(0,0)\mathcal{C}^*\beta^-\mathcal{C}^\circ(1)^*\beta^-(1), \quad i \in \mathcal{I}^{(1)}(q).$$

and, with the words

$$g(q) \in \mathcal{C}^*,\ g^-(q) \in (\mathcal{C}(0) \cup \{\beta^-\beta^+\})^*,\ g^+(q) \in \{\beta^-\beta^+\})^*,$$
$$b(q) \in \mathcal{B}(0,0),\ g^\circ(q) \in \mathcal{C}^\circ(1)^*$$

that are given by

$$q_{(i-2m-1,i]} = g(q)\beta^-(0)g^-(q)b(q)g^+(q)\beta^- g^\circ(q)\beta^-(1), \quad i \in \mathcal{I}^{(1)}(q),$$

the point $p \in P^{[0]}_{2m+1}$ can be reconstructed from its image q under η_{2m+1} as the point in $P^{(0)}_{2m+1}$ that is given by

$$p_{(i-2m-1,i]} = g(q)\beta^-(0)g^-(q)\Phi_0^{-1}(b(q))g^+(q)\beta^- g^\circ(q)\beta^-(1), \quad i \in \mathcal{I}^{(1)}(q).$$

We note that

$$\nu_0(\lambda(\eta_{2m+1}(p)_{[0,2m+1)})) = 3, \quad \nu_1(\lambda(\eta_{2m+1}(p)_{[0,2m+1)})) = 1, \quad p \in P^{[1]}_{2m+1}. \quad \text{(P.01[0])}$$

We set

$$P^{[\beta]}_{2m+1} = P^{(0)}_{2m+1} \setminus (P^{[1]}_{2m+1} \cup P^{[0]}_{2m+1}).$$

The shift commuting map η_{2m+1} is to map a point $p \in P^{[\beta]}_{2m+1}$ to the point $q \in P^{\circ}_{2m+1}(F)$ that is given by

$$q_{(i-2m-1,i]} = f^-(p)\beta^-(0)f^+(p)\Xi(\beta^- f^\circ(p)\beta^-(1)), \quad i \in \mathcal{I}^{(1)}(p).$$

With words

$$g^{(\beta)}(q) \in \{\beta^-\beta^+\}^*, \quad c(q) \in \mathcal{C}(0), \quad g(q) \in \mathcal{C}^*$$

that are given by

$$q_{(i-2m-1,i]} = g^{(\beta)}(q)c(q)g(q)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(q),$$

a point $p \in P^{[\beta]}_{2m+1}$ can be reconstructed from its image q under η_{2m+1} as the point $p \in P^{(\circ)}_{2m+1}$ that is given by

$$p_{(i-2m-1,i]} = g^{(\beta)}(q)\Xi^{-1}(c(q))g(q)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(q).$$

We note that

$$\nu_0(\lambda(\eta_{2m+1}(p)_{[0,2m+1)})) = 1, \quad \nu_1(\lambda(\eta_{2m+1}(p)_{[0,2m+1)})) = 0, \quad p \in P^{[\beta]}_{2m+1}. \quad \text{(P.01[}\beta\text{])}$$

We set

$$P^{(0)}_{2m} = \emptyset, \quad m > 3,$$

and, for $n > 5$, we set

$$P^{(1)}_n = \{p \in P^{\circ}_n(\alpha_-(0)\alpha_-(1)) \backslash P^{(0)}_n : \mathcal{J}^{(1)}(p) \neq \emptyset\}.$$

The shift commuting map η_n is to map a point $p \in P^{(1)}_n$ to the point $q \in P_n(F)$ that is obtained by replacing, in the point p, each of the words $b \in \mathcal{B}(1)$, that appear at the indices in $\mathcal{J}^{(1)}_{\circ}(p)$, by the word $\Xi(b)$.

A point $p \in P^{(1)}_n$ can be reconstructed from its image q under η_n by replacing, in q, the word $c(q) \in \mathcal{C}(0)$, that is identified as the unique word in $\mathcal{C}^*$ of maximal length that appears in q, by the word $\Xi^{-1}(c(q))$.

We note that

$$(\nu_0(\lambda(\eta_n(p)_{[0,n)}), \nu_1(\lambda(\eta_n(p)_{[0,n)})) = (\kappa_p, \kappa_p - 1), \quad p \in P^{(1)}_n. \tag{P.01.1}$$

We set

$$P^{(0,1)}_n = \{p \in P^{\circ}_n(\alpha_-(0)\alpha_-(1)) \backslash (P^{(0)}_n \cup P^{(1)}_n) : \mathcal{J}^{(0,1)}(p) \neq \emptyset\}.$$

The shift commuting map η_n is to map a point $p \in P^{(0,1)}_n$ to the point $q \in P_n$ that is obtained by replacing, in the point p, the words $b \in \mathcal{B}(1)$, that appear in p at the indices in $\mathcal{J}^{(0,1)}_{\circ}(p)$, by the word $\Phi_0(\Xi(b))$.

A point $p \in P^{(0,1)}_n$ can be reconstructed from its image q under η_n by replacing, in q, the word

$$\beta^-(0)h(q)b(q) \in \beta^-(0)\mathcal{C}^*\mathcal{B}(0, 0),$$

whose prefix $\beta^-(0)h(q)\beta^-(0)$ is identified as the unique word in $\mathcal{B}(0, 0)$ of maximal length that appears openly in q, by the word

$$\beta^-(0)h(q)\Xi^{-1}(\Phi^{-1}_0(b(q))).$$

We note that

$$(\nu_0(\lambda(\eta_n(p)_{[0,n)}), \nu_1(\lambda(\eta_n(p)_{[0,n)})) = (\kappa_p + 2, \kappa_p - 1), \quad p \in P^{(0,1)}_n. \tag{P.01.01}$$

We set

$$P^{(1,0)}_n = P^{\circ}_n(\alpha^-(0)\alpha^-(1)) \backslash (P^{(0)}_n \cup P^{(1)}_n \cup P^{(0,1)}_n).$$

With the words $b(p) \in \mathcal{B}(1)$, $f(p) \in \mathcal{C}^*$, that are given by writing the word in $\mathcal{D}(1, 0)$ that appears at the indices in $\mathcal{J}^{(1,0)}_{\circ}(p)$ as $b(p)f(p)\beta^-(0)$, the shift commuting map η_n is to map a point $p \in P^{(1,0)}_n$ to the point $q \in P_n(F)$ that is obtained by replacing, in the point p, the words in $\mathcal{D}(1, 0)$, that appear at the indices in $\mathcal{J}^{(1,0)}_{\circ}(p)$, by the word $\Phi_0(\Xi(b(p)))f(p)\beta^-(0)$.

A point $p \in P^{(1,0)}_n$ can be reconstructed from its image q under η_n by replacing, in q, the word

$$b(q)\beta^-(0)h(q)\beta^-(0) \in \mathcal{B}(0, 0)\mathcal{C}^*\beta^-(0),$$

whose suffix $\beta^-(0)h(q)\beta^-(0))$ is identified as the unique word of maximal length in $\mathcal{B}(0, 0)$ that appears openly in q, by the word

$$\Xi^{-1}(\Phi^{-1}_0(b(q)))h(q)\beta^-(0).$$

https://doi.org/10.1017/etds.2015.126 Published online by Cambridge University Press

We note that

$$(\nu_0(\lambda(\eta(p)_{[0,n)}), \nu_1(\lambda(\eta(p)_{[0,n)})) = (\kappa_p + 2, \kappa_p - 1), \quad p \in \eta_n(P_n^{(1,0)}). \quad \text{(P.01.10)}$$

We have produced a partition

$$P_n^{\circ}(\alpha(0)\alpha(0)) = P_n^{(0)} \cup P_n^{(1)} \cup P_n^{(0,1)} \cup P_n^{(1,0)}. \quad \text{(P.01)}$$

In points $q \in \eta_n(P_n^{(0,1)})$, the unique word in $\beta^-(0)\mathcal{C}^*\beta^-(0)$ of maximal length that appears openly in q is followed by a word in $\mathcal{C}^*\ \beta^-(0)$, whereas, in points $q \in \eta_n(P_n^{(1,0)})$, the unique word in $\beta^-(0)\mathcal{C}^*\beta^-(0)$ of maximal length that appears openly in q is followed by a word in $\mathcal{C}^*\ \beta^-$. From this observation and from (P.01[0]), (P.01[1]), (P.01[β]) and (P.01.1), (P.01.01), (P.01.10) it follows that the images under η_n of the sets of the partition (P.01) are disjoint. From (P.01[0]), (P.01[1]), (P.01[β]) and (P.01.1), (P.01.01), (P.01.10) it follows also that

$$\eta_n(P_n^{\circ}(\alpha_-(0)\alpha_-(1)) \cap P_n^{\circ}(\alpha_-(0)) = \emptyset.$$

We have shown that

$$\operatorname{card}(\mathcal{O}_n(\alpha_-(0)\alpha_-(1))) \le \operatorname{card}\Bigg(\bigcup_{\widetilde{\mu} \in \mathcal{M} \setminus \{\alpha(0)\alpha(1)\}} \mathcal{O}_n(\widetilde{\mu}^-)\Bigg).$$

Apply Lemma (c). □

4. *The remaining multipliers*

LEMMA 2. *Besides the multipliers $\alpha(0)$ and $\alpha(1)$, the Fibonacci–Dyck shift has no exceptional multipliers.*

Proof. Consider a multiplier

$$\mu \notin \{\alpha(0), \alpha(1), \alpha(0)\alpha(1)\},$$

of F. We construct shift commuting injections

$$\eta_n : P_n^{\circ}(\mu_-) \to \bigcup_{\tilde{\mu} \in \mathcal{M} \setminus \{\mu\}} P_n^{\circ}(\tilde{\mu}_-), \quad n > 4.$$

We set

$$P_n^{(0,0)} = \{p \in P_n^{\circ}(\mu) : \mathcal{J}^{(0,0)}(p) \neq \emptyset\}.$$

With the word $b(p) \in \mathcal{B}(0, 0)$, that appears in p at the indices in $\mathcal{J}_{\circ}^{(0,0)}(p)$, the shift commuting map η_n is to map a point $p \in P_n^{(0,0)}$ to the point $q \in P_n$ that is obtained by replacing, in p, the words $b(p)$ in $\mathcal{B}(0, 0)$, that appear in p at the indices in $\mathcal{J}_{\circ}^{(0,0)}(p)$, by the word $\Phi_0^{-1}(b(p))$.

The point p can be reconstructed from its image q under η_n by replacing, in q, the word $c(q)$, that is identified as the unique word in $\mathcal{C}(0)$ of maximal length that appears in q, by the word $\Phi_0(c(q))$.

We note that

$$(\nu_0(\lambda(\eta_n(p)_{[0,n)}), \nu_1(\lambda(\eta_n(p)_{[0,n)}))) = (\kappa_p\nu_0(\mu) - 2, \kappa_p\nu_1(\mu)), \quad p \in P_n^{(0,0)}. \quad (0.0)$$

https://doi.org/10.1017/etds.2015.126 Published online by Cambridge University Press

We set

$$P_n^{(1,1)} = \{p \in P_n^\circ(\mu)\backslash P_n^{(0,0)} : \mathcal{J}^{(1,1)}(p) \neq \emptyset\}.$$

With the words

$$b(p), \in \mathcal{B}(1), \quad f(p) \in \mathcal{C}^*, \quad f^\circ(p) \in \mathcal{C}^\circ(1)^*,$$

that are given by writing the word in $\mathcal{D}(1, 1)$ that appears in p at the indices in $\mathcal{J}_\circ^{(1,1)}(p)$ as

$$b(p) f(p)\beta^- f^\circ(p)\beta^-(1),$$

the shift commuting map η_n is to map a point $p \in P_n^{(1,1)}$ to the point $q \in P_n(F)$, that is obtained by replacing, in p, the words in $\mathcal{D}(1, 1)$, that appear in p at the indices in $\mathcal{J}_\circ^{(1,1)}(p)$, by the word

$$\Phi_0(\Xi(b(p)) f(p)\beta^+(0)\Psi(f^\circ(p))\beta^-(0).$$

A point $p \in P_n^{(1,1)}$ can be reconstructed from its image q under η_n by replacing, in q, the word

$$\beta^-(0)h^-(q)\beta^-(0)h(q)\beta^+(0)h^+(q)\beta^-(0),$$

that is identified as the word with the uniquely determined word $\beta^-(0)h(q)\beta^+(0) \in \mathcal{C}(0)$ of maximal length that appears in q as infix, a uniquely determined openly in q appearing word $h^+(q)\beta^-(0) \in \mathcal{C}(0)^*\beta^-(0)$ as suffix, and a uniquely determined word $\beta^-(0)h^-(q) \in \beta^-(0)\mathcal{C}(0)^*$ as prefix, by the word

$$\beta^-\Psi^{-1}(h^-(q))\beta^-(1)h(q)\beta^-\Psi^{-1}(h^+(q))\beta^-(0).$$

We note that

$$(\nu_0(\lambda(\eta_n(p)_{[0,n)})), \nu_1(\lambda(\eta_n(p)_{[0,n)}))) = (\kappa_p\nu_0(\mu) + 2, \kappa_p\nu_1(\mu) - 2), \ p \in P_n^{(1,1)}. \quad (1.1)$$

We denote by $P_n^{(0,1,0)}$ the set of points in

$$P_n^\circ(\mu)\backslash(P_n^{(0,0)} \cup P_n^{(1,1)})$$

such that $\mathcal{J}^{(1)}(p) \neq \emptyset$, and such that the word in $\mathcal{B}(1)$ that appears at the indices in $\mathcal{J}_\circ^{(1)}(p)$ is preceded in p by a word in $\beta^-(0)\mathcal{C}^*$, and followed in p by a word in $\mathcal{C}^*\beta^-(0)$.

With the word $f(p) \in \mathcal{C}^*$, that is given by writing the openly appearing word in $\mathcal{C}^*\beta^-(0)$, that follows the word $b(p) \in \mathcal{B}(1)$ that appears in p at the indices in $\mathcal{J}_\circ^{(1)}(p)$, as $f(p)\beta^-(0)$, the shift commuting map η_n is to map a point $p \in P_n^{(0,1,0)}$ to the point $q \in P_n(F)$ that is obtained by replacing, in the point p, the words in $\mathcal{B}(1)$, that appear in p at the indices in $\mathcal{J}_\circ^{(1)}(p)$ together with the openly appearing words in $\mathcal{C}^*\beta^-(0)$ that follow them, by the word $\Xi(b(p)) f(p)\beta^+(0)$.

Denoting a point $p \in P_n^{(0,1,1)}$ by $q' \in F$ the point that is obtained from its image q under η_n by replacing, in q, the unique word $c(q) \in \mathcal{C}(0)$ of maximal length, that appears in q, by the word $\Phi(c(q))$, one sees that the point p can be reconstructed from q by replacing, in the point q', the unique word $c(q') \in \mathcal{C}(0)$ of maximal length, that appears in q', by the word $\Xi^{-1}(c(q'))$.

We note that

$$(\nu_0(\lambda(\eta_n(p)_{[0,n)})),\ \nu_1(\lambda(\eta_n(p)_{[0,n)}))) = (\kappa_p\nu_0(\mu) - 2,\ \kappa_p\nu_1(\mu) - 1),\ p \in P_n^{(0,1,0)}. \tag{0.1.0}$$

We denote by $P_n^{(\bullet,1,1)}$ the set of points p in

$$P_n^{\circ}(\mu_-)\backslash(P_n^{(0,0)} \cup P_n^{(0,1,0)})$$

such that $\mathcal{J}^{(1)}(p) \neq \emptyset$, and such that the words in $\mathcal{B}(1)$ that appear at the indices in $\mathcal{J}_\circ^{(1)}(p)$ are followed in p by a word in $\mathcal{C}^*\beta^-\mathcal{C}^\circ(1)^*\beta^-(1)$.

The shift commuting map η_n is to map a point $p \in P_n^{(\bullet,1,1)}$ to the point $q \in P_n(F)$ that is obtained by replacing, in p, the words $b(p) \in \mathcal{B}(1)$, that appear in p at the indices in $\mathcal{J}_\circ^{(1)}(p)$, by the word $\Phi_0(\Xi(b(p))$.

A point $p \in P_n^{(\bullet,1,1)}$ can be reconstructed from its image q under η_n by replacing, in q, the word $h(q)$, that is identified as the unique word in $\mathcal{B}(0,0)$ of maximal length that appears in q, by the word $\Xi^{-1}(\Phi_0^{-1}(h(q)))$.

We note that

$$(\nu_0(\lambda(\eta_n(p)_{[0,n)}),\ \nu_1(\lambda(\eta_n(p)_{[0,n)}) = (\kappa_p\nu_0(\mu) + 2,\ \kappa_p\nu_1(\mu) - 1),\ p \in P_n^{(\bullet,1,1)}. \tag{$\bullet$.1.1}$$

We denote by $P_n^{(1,1,\bullet)}$ the set of points in

$$P_n^{\circ}(\mu_-)\backslash(P_n^{(0,0)} \cup P_n^{(0,1,0)} \cup P_n^{(\bullet,1,1)})$$

such that $\mathcal{J}^{(1)}(p) \neq \emptyset$, and such that the word in $\mathcal{B}(1)$ that appears at indices in $\mathcal{J}_\circ^{(1)}(p)$ is preceded in p by a word in $\beta^-\mathcal{C}^\circ(1)^*\beta^-(1)\mathcal{C}^*$.

The shift commuting map η_n is to map a point $p \in P_n^{(1,1,\bullet)}$ to the point $q \in P_n(F)$ that is obtained by replacing, in p, the words $b(p) \in \mathcal{B}(1)$, that appear in p at the indices in $\mathcal{J}_\circ^{(1)}(p)$, by the word $\Xi(b(p))$.

A point $p \in P_n^{(1,1,\bullet)}$ can be reconstructed from its image q under η_n by replacing, in q, the word $c(q)$, that is identified as the unique word in $\mathcal{C}(0)$ of maximal length that appears in q, by the word $\Xi^{-1}(c(q))$.

We note that

$$(\nu_0(\lambda(\eta_n(p)_{[0,n)}),\ \nu_1(\lambda(\eta_n(p)_{[0,n)}) = (\kappa_p\nu_0(\mu),\ \kappa_p\nu_1(\mu) - 1), \quad p \in P_n^{(1,1,\bullet)}. \tag{1.1.$\bullet$}$$

We set

$$P_n^{(0,1)} = \{p \in P_n^{\circ}(\mu_-)\backslash(P_n^{(0,0)} \cup P_n^{(0,1,0)} \cup P_n^{(\bullet,1,1)}) \cup P^{(1,1,\bullet)}) : \mathcal{J}^{(1,0)}(p) \neq \emptyset\}.$$

The shift commuting map η_n is to map a point $p \in P_n^{(0,1)}$ to the point $q \in P_n(F)$ that is obtained by replacing, in p, the words $b(p) \in \mathcal{B}(1)$, that appear at the indices in $\mathcal{J}_\circ^{(0,1)}(p)$, by the word $\Phi_0(\Xi(b(p)))$.

The point $p \in P_n^{(0,1)}$ can be reconstructed from its image q under η_n by replacing, in q, the word $\beta^-(0)g(q)\beta^-(0)$, that is identified as the unique word in $\mathcal{B}(0,0)$ of maximal length that appears in q, by the word $\beta^-(0)g(q)\beta^-$, and the open appearances $h(q)\beta^-(0)$ of a word in $\mathcal{C}(0)^*\beta^-(0)$, that follow the word $\beta^-(0)g(q)\beta^-(0)$ in q, by the word $\Psi^{-1}(h(q))\beta^-(1)$.

We note that

$$(\nu_0(\lambda(\eta(p)_{[0,n)})), \nu_1(\lambda(\eta(p)_{[0,n)}))) = (\kappa_p \nu_0(\mu) + 2, \kappa_p \nu_1(\mu) - 1), \quad p \in P_n^{(0,1)}. \quad (0.1)$$

We set

$$P_n^{(1,0)} = P_n^{\circ}(\mu_-) \setminus (P_n^{(0,0)} \cup P_n^{(0,1,0)} \cup P_n^{(\bullet,1,1)} \cup P^{(1,1,\bullet)} \cup P_n^{(0,1)}).$$

With the words

$$b(p) \in \mathcal{B}, \quad f(p) \in \mathcal{C}^*,$$

that are given by writing the word in $\mathcal{D}(1, 0)$ that appears at the indices in $\mathcal{J}_{\circ}^{(1,0)}(p)$ as

$$b(p) f(p) \beta^-(0),$$

the shift commuting map η_n is to map a point $p \in P_n^{(1,0)}$ to the point $q \in P_n(F)$, that is obtained by replacing, in p, the words $b(p) f(p) \beta^-(0)$, that appear at the indices in $\mathcal{J}_{\circ}^{(1,0)}(p)$, by the word

$$\Phi_0(\Xi(b(p)) f(p) \beta^+(0).$$

The point $p \in P_n^{(1,0)}$ can be reconstructed from its image q under η_n by replacing, in q, the word $\beta^-(0) g(q) \beta^+(0)$, that is identified as the unique word in $\mathcal{C}(0)$ of maximal length that appears in q, by the word $\beta^-(1) g(q) \beta^-(0)$, and the word $\beta^-(0) h(q) \in \beta^-(0)\mathcal{C}^*$, that precedes the word $\beta^-(0) g(q) \beta^+(0)$ in q, by the word $\beta^- \Psi^{-1}(h(q))$.

We note that

$$(\nu_0(\lambda(\eta(p)_{[0,n)})), \nu_1(\lambda(\eta(p)_{[0,n)}))) = (\kappa_p \nu_0(\mu), \kappa_p \nu_1(\mu) - 1), \quad p \in P_n^{(1,0)}. \quad (1.0)$$

We have produced a partition

$$P_n^{\circ}(\mu_-) = P_n^{(0,0)} \cup P_n^{(0,1,0)} \cup P_n^{(\bullet,1,1)} \cup P_n^{(1,1,\bullet)} \cup P_n^{(0,1)} \cup P_n^{(1,0)}. \quad (\mathrm{P})$$

In a point $q \in \eta_n(P_n^{(\bullet,1,1)})$, the word in $\mathcal{B}(0, 0)$ of maximal length that appears in q is followed by an open appearance of a word in $\mathcal{C}^*\mathcal{B}(1)$, whereas, in the points $q \in \eta_n(P_n^{(0,1)})$, this word is followed by an open appearance of a word in $\mathcal{C}^*\beta^-(0)$. Also, in a point $q \in \eta_n(P_n^{(1,1,\bullet)})$, the word in $\mathcal{C}(0)$ of maximal length that appears in q is preceded by a word in $\mathcal{B}(1)\mathcal{C}^*$, whereas, in a point $q \in \eta_n(P_n^{(1,0)})$, this word is preceded by a word in $\beta^-(0)\mathcal{C}^*$.

It follows from these observations and from (00), (0,1,0), (•,1,1), (1,1,•), (0,1) and (1,0) that the images under η_n of the elements of the partition (P) are disjoint. From (0,0), (0,1,0), (•,1,1), (1,1,•), (0,1) and (1,0) it also follows that

$$\eta_n(P_n^{\circ}(\mu_-)) \cap P_n^{\circ}(\mu_-) = \emptyset.$$

We have shown that

$$\operatorname{card} \mathcal{O}_n(\mu_-) \le \operatorname{card}\Big(\bigcup_{\widetilde{\mu} \in \mathcal{M} \setminus \{\mu^-\}} \mathcal{O}_n(\widetilde{\mu}_-) \Big).$$

The lemma follows now from Lemma (c) and Lemma 1. □

https://doi.org/10.1017/etds.2015.126 Published online by Cambridge University Press

5. *An embedding theorem*

Let $P_k(\alpha(0))$ denote the set of points of F of period k with multiplier $\alpha(0)$, let $P_k(\alpha(1))$ denote the set of points of F, of period k, with multiplier $\alpha(1)$ and let $P_k(\mathbf{1})$ denote the set of points of F, of period k, that are neutral. Denote by $\zeta_{\mathbf{1}}$ the zeta function of the neutral periodic points of F, by $\zeta_{\alpha(0)}$ the zeta function of the periodic points of F with multiplier $\alpha(0)$ and by $\zeta_{\alpha(1)}$ the zeta function of the periodic points of F with multiplier $\alpha(1)$.

LEMMA 3.
$$\liminf_{k\to\infty} \frac{1}{k} \log \operatorname{card}(P_k(\mathbf{1}) \cup P_k(\alpha(0)))$$
$$= \liminf_{k\to\infty} \frac{1}{k} \log \operatorname{card}(P_k(\mathbf{1}) \cup P_k(\alpha(1))) = \frac{3}{2}\log 3 - \log 2.$$

Proof. Set
$$\xi(z) = \frac{2}{\sqrt{3}} \sin\left(\frac{1}{3} \arcsin \frac{3\sqrt{3}}{2} z\right), \quad 0 \le z \le \frac{2}{3\sqrt{3}}.$$
By [**KM**, (4.8)] we have the generating functions
$$g_{\mathcal{C}^\circ(1)^\star}(z) = \frac{\xi(z)}{z} \tag{5.1}$$
and
$$g_{\mathcal{C}^\star}(z) = \frac{\xi(z)^2}{z^2}, \tag{5.2}$$
and it follows that
$$\zeta_{\mathbf{1}}(z) = g_{\mathcal{C}^\circ(1)^\star}(z) g_{\mathcal{C}^\star}(z) = \frac{\xi(z)^3}{z^3}. \tag{5.3}$$
Using the circular code $\mathcal{C}^\star\beta^-(0)$ one finds from (5.2) that
$$\zeta_{\alpha(0)}(z) = \left(\frac{z}{z - \xi(z)^2}\right)^2, \tag{5.4}$$
and using the circular code $\beta^-(1)\mathcal{C}^\star\beta^-\mathcal{C}^\circ(1)^\star$ one finds, from (5.1) and (5.2), that
$$\zeta_{\alpha(1)}(z) = \left(\frac{z}{z - \xi(z)^3}\right)^2. \tag{5.5}$$
(See, for example, [**P**, §5] or [**KM**, §2].) The lemma follows from (5.4), (5.5) and (5.3). □

Let P_k^+ denote the set of points of F of period k with positive multiplier, where $k \in \mathbb{N}$. Let $\mathcal{O}_k^+$ denote the set of orbits of F of length k with positive multiplier, let $\mathcal{O}_k(\alpha(0))$ denote the set of orbits of F of length k with multiplier $\alpha(0)$, let $\mathcal{O}_k(\alpha(1))$ denote the set of orbits of F of length k with multiplier $\alpha(1)$ and let $\mathcal{O}_k(\mathbf{1})$ denote the set of orbits of F of length k that are neutral, where $k \in \mathbb{N}$.

THEOREM. *Let Y be an irreducible subshift of finite type. Let $\mathcal{O}_k(Y)$ be its set of periodic orbits of length $k \in \mathbb{N}$, and let h_Y be its entropy. An embedding of Y into the Fibonacci–Dyck shift exists if and only if at least one of the following conditions is satisfied.*

https://doi.org/10.1017/etds.2015.126 Published online by Cambridge University Press

(a)

$$\mathrm{card}(\mathcal{O}_k(Y)) \le \mathrm{card}(\mathcal{O}_k(\mathbf{1}) \cup \mathcal{O}_k(\alpha(0))), \quad k \in \mathbb{N},$$

and

$$h(Y) < \tfrac{3}{2}\log 3 - \log 2.$$

(b)

$$\mathrm{card}(\mathcal{O}_k(Y)) \le \mathrm{card}(\mathcal{O}_k(\mathbf{1}) \cup \mathcal{O}_k(\alpha(1))), \quad k \in \mathbb{N},$$

and

$$h(Y) < \tfrac{3}{2}\log 3 - \log 2.$$

(c)

$$\mathrm{card}(\mathcal{O}_k(Y)) \le \mathrm{card}(\mathcal{O}_k(\mathbf{1}) \cup \mathcal{O}_k^{+}), \quad k \in \mathbb{N},$$

and

$$h(Y) < 3\log 2 - \log 3.$$

Proof. The theorem results from an application of [**HIK**, Theorem 5.8] that uses Lemma (a) and Lemmas 1–3 and that takes into account that only the exceptional multipliers (in this case the multipliers $\alpha(0)$ and $\alpha(1)$) contribute to the possibility of an embedding beyond the case of negative (or positive) multipliers. In [**HIK**, Theorem 5.8], the entropy condition in (a) reads

$$h(Y) < \liminf_{k\to\infty} \frac{1}{k} \log \mathrm{card}(P_k(\mathbf{1}) \cup P_k(\alpha(0))),$$

the entropy condition in (b) reads

$$h(Y) < \liminf_{k\to\infty} \frac{1}{k} \log \mathrm{card}(P_k(\mathbf{1}) \cup P_k(\alpha(1)))$$

and the entropy condition in (c) reads

$$h(Y) < \liminf_{k\to\infty} \frac{1}{k} \log \mathrm{card}(P_k(\mathbf{1}) \cup P_k^{+}).$$

For (a) and (b) apply Lemma 3. For (c) note that, by Lemma (a), the right-hand side of the inequality is equal to the topological entropy of F, which is known to be $3\log 2 - \log 3$ [**KM**, §4]. □

By Lemma (a),

$$\begin{aligned} \mathrm{card}(\mathcal{O}_k^{+}) &= \tfrac{1}{2}\,\mathrm{card}(\mathcal{O}_k \setminus \mathcal{O}_k(\mathbf{1})), \\ \mathrm{card}(\mathcal{O}_k(\alpha(0))) &= 2\,\mathrm{card}(\mathcal{O}_k(\alpha^{-}(0))), \\ \mathrm{card}(\mathcal{O}_k(\alpha(1))) &= 2\,\mathrm{card}(\mathcal{O}_k(\alpha^{-}(1))), \quad k \in \mathbb{N}. \end{aligned} \tag{5.6}$$

Denote the set of points of F of period k by P_k, $k \in \mathbb{N}$, and denote by ζ the zeta function of F. The sequence $(\mathrm{card}(\mathcal{O}_k))_{k\in\mathbb{N}}$ can be obtained by Möbius inversion from the sequence $(\mathrm{card}(P_k)_{k\in\mathbb{N}}$ that enters into the zeta function $\zeta(z) = e^{\sum_{n\in\mathbb{N}}((\mathrm{card}(P_n)z^n)/n)}$ of F. The same applies to the sequences

$$\mathrm{card}(\mathcal{O}_k(\mathbf{1}))_{k\in\mathbb{N}}, \quad \mathrm{card}(P_k(\alpha(0)))_{k\in\mathbb{N}}, \quad \mathrm{card}(\mathcal{O}_k(\alpha(1)))_{k\in\mathbb{N}}.$$

https://doi.org/10.1017/etds.2015.126 Published online by Cambridge University Press

Therefore, by (5.6), the information that is relevant for each of the conditions (a), (b) and (c) of Theorem (5.1) is (in principle) contained in the zeta functions ζ, $\zeta_{\mathbf{1}}$, and $\zeta_{\alpha(0)}$, $\zeta_{\alpha(1)}$. ζ is also known. From [**KM**, (4.12)],

$$\zeta(z) = \frac{\xi(z)}{z(2\xi(z)^2 + \xi(z) - 1)^2}.$$

We note that the zeta function $\zeta_+(z)$ of the periodic point with positive multiplier is also known: by Lemma (a),

$$\zeta_+(z) = \sqrt{\zeta_{\mathbf{1}}(z)^{-1}\zeta(z)} = \frac{\xi(z)^2}{z^2(2\xi(z)^2 + \xi(z) - 1)}.$$

6. *The multiplier $\alpha(0)$*

PROPOSITION M0. *The multiplier $\alpha(0)$ is exceptional only at periods one and three.*

Proof. By Lemma (b) the multiplier $\alpha(0)$ is is not exceptional at periods two, four, five and six. Let

$$n \geq 7. \tag{a}$$

We construct a shift commuting injection

$$\eta_n : P_n^\circ(\alpha_-(0)) \to \bigcup_{\widetilde{\mu} \in \mathcal{M}\setminus\{\alpha(0)\}} P_n^\circ(\widetilde{\mu}_-).$$

Set

$$I(i) = \max\{i^{(0)} \in \mathcal{I}^{(0)} : i^{(0)} < i\}, \quad i \in \mathcal{I}^{(0)}(p), \quad p \in P_n^\circ(\alpha^-(0)).$$

Let $P_n^{(1)}$ be the set of points $p \in P_n^\circ(\alpha^-(0))$ such that the set $\mathcal{I}^{\langle 1\rangle}$ of indices $i^0 \in \mathcal{I}^{(0)}$ and such that $p_{(I(i),i)} \in \mathcal{Q}_1$ is not empty.

For $p \in P_n^{(1)}$, we denote by $\mathcal{I}_\circ^{\langle 1\rangle}(p)$ the set of indices $i_\circ^{(1)} \in \mathcal{I}^{\langle 1\rangle}(p)$ such that the word $p_{(I(i_\circ^{(1)}),i_\circ^{(1)}]}$ is lexicographically the smallest among the words

$$p_{(I(i^{(1)}),i^{(1)}]}, \quad i^{(1)} \in \mathcal{I}^{\langle 1\rangle}(p).$$

With the word $f(p) \in \mathcal{Q}_1$ that is given by writing

$$p_{(I(i_\circ^{(1)}),i_\circ^{(1)}]} = f(p)\beta_-(0), \quad i_\circ^{(1)} \in \mathcal{I}_\circ^{\langle 1\rangle}(p),$$

the shift commuting map η_n is to map a point $p \in P_n^{(1)}$ to the point $p \in P_n^\circ(F)$ that is obtained by replacing, in the point p, the words $f(p)\beta_-(0)$, that appear at the indices in $\mathcal{I}_\circ^{\langle 1\rangle}(p)$, by $\Delta_1(f)\beta_-(0)$.

For $q \in \eta_n(P_n^{(1)})$, there is a unique word $b(q) \in \mathcal{B}(1, 1)$ that appears openly in q, and a point $p \in P_n^{(1)}$ that can be reconstructed from its image q under η_n by replacing, in q, the word $b(q)$, when it appears openly in q, by the word $\Phi_1^{-1}(b(q))$.

We note that

$$(\nu_0(\lambda(\eta(p)_{[0,n)})), \nu_1(\lambda(\eta(p)_{[0,n)}))) = (\kappa_p, 2), \quad p \in P_n^{(1)}. \tag{1}$$

https://doi.org/10.1017/etds.2015.126 Published online by Cambridge University Press

Let $P_n^{(\beta)}$ be the set of points

$$p \in P_n^{\circ}(\alpha^-(0)) \backslash P_n^{(1)})$$

such that the set $\mathcal{I}^{\langle\beta\rangle}(p)$ of indices $i^{(\beta)} \in \mathcal{I}^{(0)}(p)$, at which there appears a word in $\beta^-\beta^+(\mathcal{C}(0)^* \backslash \{\epsilon\})\beta^-(0)$, is not empty.

For $p \in P_n^{(\beta)}$, we denote by $\mathcal{I}_{\circ}^{\langle\beta\rangle}(p)$ the set of indices $i_{\circ}^{(\beta)} \in \mathcal{I}^{\langle\beta\rangle}(p)$ such that the word $p_{(i_{\circ}^{(\beta)}-n, i_{\circ}^{(\beta)}]}$ is lexicographically the smallest among the words

$$p_{(i^{(\beta)}-n, i^{(\beta)}]}, \quad i^{(\beta)} \in \mathcal{I}^{\langle\beta\rangle}(p).$$

With the word $f(p) \in \mathcal{C}(0)^*$ that is given by writing

$$p_{(I(i_{\circ}^{(\beta)}), i_{\circ}^{(\beta)}]} = f(p), \quad i_{\circ}^{(\beta)} \in \mathcal{I}_{\circ}^{\langle\beta\rangle}(p),$$

the shift commuting map η_n is to map a point $p \in P_n^{(\beta)}$ to the point $p \in P_n^{\circ}(F)$ that is obtained by replacing, in the point p, the word $\beta^-\beta^+ f(p)$, that appears in p at the indices in $\mathcal{I}_{\circ}^{\langle\beta\rangle}(p) - 1$, by the word $\beta^-\Psi(f(p))\beta^-(1)$.

For a point $q \in \eta_n(P_n^{(\beta)})$ there is a unique word $b(q) \in \mathcal{B}(1)$ that appears openly in q, and a point $p \in P_n^{(\beta)}$ that can be reconstructed from its image q under η_n by replacing, in q, the word $b(q)$, when it appears openly in q, by the word $\beta^-\beta^+\Psi^{-1}(b(q))$.

We note that

$$(\nu_0(\lambda(\eta(p)_{[0,n)})), \nu_1(\lambda(\eta(p)_{[0,n)}))) = (\kappa_p, 2), \quad p \in P_n^{(\beta)}. \qquad (\beta)$$

Let $P_n^{(\beta,0)}$ be the set of points

$$p \in P_n^{\circ}(\alpha^-(0)) \backslash (P_n^{(1)} \cup P_n^{(\beta)})$$

such that the set $\mathcal{I}^{\langle\beta,0\rangle}(p)$ of indices $i^{(\beta,0)} \in \mathcal{I}^{(0)}(p)$, at which there appears openly the word $\beta^-\beta^+\beta^-(0)$, is not empty.

For $p \in P_n^{(\beta,0)}$, we denote by $\mathcal{I}_{\circ}^{\langle\beta,0\rangle}(p)$ the set of indices $i_{\circ}^{(\beta,0)} \in \mathcal{I}^{\langle\beta,0\rangle}(p)$ such that the word $p_{(i_{\circ}^{(\beta,0)}-n, (i_{\circ}^{(\beta,0)}]}$ is lexicographically the smallest among the words

$$p_{(i^{(\beta,0)}-n, i^{(\beta,0)}]}, \quad i^{(\beta,0)} \in \mathcal{I}^{\langle\beta,0\rangle}(p).$$

The shift commuting map η_n is to map a point $p \in P_n^{(\beta,0)}$ to the point $p \in P_n^{\circ}(F)$ that is obtained by replacing, in the point p, the word $\beta^-\beta^+$, that appears in p at the indices in $\mathcal{I}_{\circ}^{\langle\beta,0\rangle}(p) - 1$, by the word $\beta^-\beta^-(1)$.

In a point of $\eta_n(P_n^{(\beta,0)})$ the word $\beta^-\beta^-(1)$ appears openly, and a point $p \in P_n^{(\beta,0)}$ can be reconstructed from its image q under η_n by replacing, in q, the word $\beta^-\beta^-(1)$, when it appears openly in q, by the word $\beta^-\beta^+$.

We note that

$$(\nu_0(\lambda(\eta(p)_{[0,n)})), \nu_1(\lambda(\eta(p)_{[0,n)}))) = (\kappa_p, 2), \quad p \in P_n^{(\beta,0)}. \qquad (\beta.0)$$

Let $P_n^{(0,2)}$ be the set of points

$$p \in P_n^{\circ}(\alpha^-(0)) \backslash (P^{(1)} \cup P_n^{(\beta)} \cup P_n^{(\beta,0)})$$

https://doi.org/10.1017/etds.2015.126 Published online by Cambridge University Press

such that

$$\nu_0(p) \geq 2.$$

For $p \in P_n^{(0,2)}$, we denote by $\mathcal{I}_\circ^{\langle 0,2\rangle}(p)$ the set of indices $i \in \mathcal{I}^{(0)}(p)$ such that

$$i - I(i) > 1.$$

For $p \in P_n^{(0,2)}$, we denote by $\mathcal{I}_\circ^{\langle 0,2\rangle}(p)$ the set of indices $i_\circ^{(0,2)} \in \mathcal{I}^{\langle 0,2\rangle}(p)$ such that the word $p_{(i_\circ^{(0,2)}-n,i_\circ^{(0,2)}]}$ is lexicographically the smallest among the words

$$p_{(i^{(0,2)}-n,i^{(0,2)}]}, \quad i^{(0,2)} \in \mathcal{I}^{\langle 0,2\rangle}(p).$$

With the word $f(p) \in \mathcal{C}(0)^*$, that is given by writing

$$p_{[I(i_\circ^{\langle 0,2\rangle}),i_\circ^{\langle 0,2\rangle}]} = \beta^-(0)f(p)\beta^-(0), \quad i^{(0,2)} \in \mathcal{I}_\circ^{\langle 0,2\rangle}(p),$$

the shift commuting map η_n is to map a point $p \in P_n^{(0,2)}$ to the point $p \in P_n^\circ(F)$ that is obtained by replacing, in the point p, the words $\beta^-(0)f(p)\beta^-(0)$, that appear in p at the indices in $\mathcal{I}_\circ^{\langle 0,2\rangle}(p)$, by the word $\beta^-\Psi^{-1}(f(p))\beta^-(1)$.

For a point $q \in \eta_n(P_n^{(0,2)})$ there is a unique word $b(q) \in \mathcal{B}(1)$ that appears openly in q, and a point $p \in P_n^{(0,2)}$ that can be reconstructed from its image q under η_n by replacing, in q, the word $b(q)$, when it appears openly in q, by the word $\beta^-\Psi^{-1}(b(q))\beta^-$.

We note that

$$(\nu_0(\lambda(\eta(p)_{[0,n)})), \nu_1(\lambda(\eta(p)_{[0,n)}))) = (\kappa_p - 2, 1), \quad p \in P_n^{(0,2)}. \tag{0.2}$$

Let $P_n^{(0,1,l)}$ be the set of points

$$p \in P_n^\circ(\alpha^-(0)) \backslash (P^{(1)} \cup P_n^{(\beta)} \cup P_n^{(\beta,0)} \cup P_n^{(0,2)})$$

such that

$$p_{(i-n,i]} \in \beta^-(0)\beta^+(0)\mathcal{C}(0)^*\beta^-(0), \quad i \in \mathcal{I}^{(0)}(p).$$

With the word $f(p) \in \beta^-(0)\beta^+(0)\mathcal{C}(0)^*\beta^-(0)$, that is given by writing

$$p_{(i-n,i]} = \beta^-(0)\beta^+(0)f(p)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(p),$$

the shift commuting map η_n is to map a point $p \in P_n^{(0,1,l)}$ to the point $q \in P_n^\circ(F)$ that is given by

$$q_{(i-n,i]} = \beta^-\beta^-(1)f(p)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(p).$$

For a point $q \in \eta_n(P_n^{(0,1,l)})$,

$$q_{(i-n,i]} \in \beta^-\beta^-(1)\mathcal{C}(0)^*\beta^-(0), \quad i \in \mathcal{I}^{(0)}(q),$$

and with the word $a(q) \in \mathcal{C}(0)^*$, that is given by writing

$$q_{(i-n,i]} = \beta^-\beta^-(1)a(q)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(q),$$

a point $p \in P_n^{(0,1,l)}$ can be reconstructed from its image q under η_n as the point in $P_n^\circ(F)$ that is given by

$$p_{(i-n,i]} = \beta^-(0)\beta^+(0)a(q))\beta^-(0), \quad i \in \mathcal{I}^{(0)}(q).$$

We note that

$$(\nu_0(\lambda(\eta(p)_{[0,n)})),\ \nu_1(\lambda(\eta(p)_{[0,n)}))) = (1,\ 1), \quad p \in P_n^{(0,1,l)}. \tag{0.1.l}$$

Let $P_n^{(0,1,m)}$ be the set of points

$$p \in P_n^{\circ}(\alpha^-(0)) \backslash (P_n^{(1)} \cup P_n^{(\beta)} \cup P_n^{(\beta,0)} \cup P_n^{(0,2)})$$

such that

$$p_{(i-n,i]} \in \beta^-(0)(\mathcal{C}(0)^* \backslash \{\epsilon\})\beta^+(0)(\mathcal{C}(0)^* \backslash \{\epsilon\})\beta^-(0), \quad i \in \mathcal{I}^{(0)}(p).$$

With the words $f(p) \in \mathcal{C}(0)^*$, $g(p) \in \mathcal{C}(0)^*$, that are given by writing

$$p_{(i-n,i]} = \beta^-(0) f(p)\beta^+(0) g(p)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(p),$$

the shift commuting map η_n is to map a point $p \in P_n^{(0,1,m)}$ to the point $q \in P_n^{\circ}(F)$ that is given by

$$q_{(i-n,i]} = \beta^-(1) f(p)\beta^-(0) g(p)\beta^-, \quad i \in \mathcal{I}^{(0)}(p).$$

For a point $q \in \eta_n(P_n^{(0,1.m)})$,

$$q_{(i-n,i]} \in \mathcal{C}(0)^* \beta^-(0)\mathcal{C}(0)^* \beta^- \beta^-(1), \quad i \in \mathcal{I}^{(1)}(q),$$

and with the words $a(q) \in \mathcal{C}(0)^*$, $b(q) \in \mathcal{C}^*$, that are given by writing

$$q_{[i,i+n} = a(q)\beta^-(0) b(q)\beta^- \beta^-(1). \quad i \in \mathcal{I}^{(1)}(q),$$

a point $p \in P_n^{(0,1,0)}$ can be reconstructed from its image q under η_n as the point that is given by

$$p_{(i-n,i]} = a(q)\beta^+(0) b(q)\beta^-(0)\beta^-(0), \quad i \in \mathcal{I}^{(1)}(q).$$

We note that

$$(\nu_0(\lambda(\eta(p)_{[0,n)})),\ \nu_1(\lambda(\eta(p)_{[0,n)}))) = (1,\ 1), \quad p \in P_n^{(0,1,0)}. \tag{0.1.m}$$

Let $P_n^{(0,1,r)}$ be the set of points

$$p \in P_n^{\circ}(\alpha^-(0)) \backslash (P_n^{(1)} \cup P_n^{(\beta)} \cup P_n^{(\beta,0)} \cup P_n^{(0,2)})$$

such that

$$p_{(i-n,i]} \in \beta^-(0)\beta^-(0)(\mathcal{C}(0)^* \backslash \{\epsilon\})\beta^+(0)\mathcal{C}(0)^* \beta^+(0)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(p).$$

With the words $f(p) \in \mathcal{C}(0)^*$, $g(p) \in \mathcal{C}(0)^*$, that are given by writing

$$p_{(i-n,i]} = \beta^-(0) f(p)\beta^+(0) g(p)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(p),$$

the shift commuting map η_n is to map a point $p \in P_n^{(0,1,r)}$ to the point $q \in P_n^{\circ}(F)$ that is given by

$$q_{(i-n,i]} = \beta^- \beta^-(1) f(p)\beta^-(0) g(p)\beta^-(0)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(p).$$

For a point $q \in \eta_n(P_n^{(0,1,r)})$,

$$q_{(i-n,i]} \in \mathcal{C}(0)^* \beta^-(0)\mathcal{C}(0)^* \beta^-(0)\beta^-(0)\beta^- \beta^-(1), \quad i \in \mathcal{I}^{(1)}(q),$$

and with the words $a(q) \in \mathcal{C}(0)^*$, $b(q) \in \mathcal{C}(0)^*$, that are given by writing

$$q_{[i,i+n} = a(q)\beta^-(0)b(q)\beta^-(0)\beta^-(0)\beta^-\beta^-(1). \quad i \in \mathcal{I}^{(1)}(q),$$

a point $p \in P_n^{(0,1,r)}$ can be reconstructed from its image q under η_n as the point that is given by

$$p_{(i-n,i]} = a(q)\beta^-(0)b(q)\beta^-(0)\beta^-\beta^-(1), \quad i \in \mathcal{I}^{(1)}(q).$$

We note that

$$(\nu_0(\lambda(\eta(p)_{[0,n)})), \nu_1(\lambda(\eta(p)_{[0,n)}))) = (3, 1), \quad p \in P_n^{(0,1,r)}. \tag{0.1.r}$$

Let $P_n^{(0,1,r,\beta)}$ be the set of points

$$p \in P_n^\circ(\alpha^-(0)) \backslash (P^{(1)} \cup P_n^{(\beta)} \cup P_n^{(\beta,0)} \cup P_n^{(0,2)})$$

such that

$$p_{(i-n,i]} \in \beta^-(0)\beta^-(0)\beta^+(0)(\mathcal{C}(0)^* \backslash \mathcal{C}(0)^*)\beta^+(0)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(p).$$

With the word $f(p) \in \mathcal{C}(0)^*$, that is given by writing

$$p_{(i-n,i]} = \beta^-(0)\beta^-(0)\beta^+(0)f(p)\beta^+(0)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(p),$$

the shift commuting map η_n is to map a point $p \in P_n^{(0,1,r,\beta))}$ to the point $q \in P_n^\circ(F)$ that is given by

$$q_{(i-n,i]} = \beta^-(0)\beta^-\beta^-(1)f(p)\beta^-(0)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(p).$$

For a point $q \in \eta_n(P_n^{(0,1,r,\beta)})$,

$$q_{(i-n,i]} \in \mathcal{C}(0)^*\beta^-(0)\beta^-(0)\beta^-(0)\beta^-\beta^-(1), \quad i \in \mathcal{I}^{(1)}(q),$$

and with the word $a(q) \in \mathcal{C}(0)^*$, that is given by writing

$$q_{(i-n,i]} = a(q)\beta^-(0)\beta^-\beta^-(1), \quad i \in \mathcal{I}^{(0)}(q),$$

a point $p \in P_n^{(0,1,r,\beta)}$ can be reconstructed from its image q under η_n as the point in $P_n^\circ(F)$ that is given by

$$p_{(i-n,i]} = a(q)\beta^+(0)\beta^-(0)\beta^-(0)\beta^+(0), \quad i \in \mathcal{I}^{(0)}(q).$$

We note that

$$(\nu_0(\lambda(\eta(p)_{[0,n)})), \nu_1(\lambda(\eta(p)_{[0,n)}))) = (3, 1), \quad p \in P_n^{(0,1,r,\beta)}. \tag{0.1.r.β}$$

Let $P_n^{(0,1,r,1)}$ be the set of points

$$p \in P_n^\circ(\alpha^-(0)) \backslash (P^{(1)} \cup P_n^{(\beta)} \cup P_n^{(\beta,0)} \cup P_n^{(0,2)})$$

such that

$$p_{(i-n,i]} \in \beta^-(0)\beta^-(0)\beta^+(0)\mathcal{C}(0)^*\beta^+(0)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(p).$$

With the word $f(p) \in \beta^-(0)\beta^+(0)\mathcal{C}(0)^*\beta^-(0)$, that is given by writing

$$p_{(i-n,i]} = \beta^-(0)\beta^-(0)\beta^+(0)f(p)\beta^+(0)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(p),$$

https://doi.org/10.1017/etds.2015.126 Published online by Cambridge University Press

the shift commuting map η_n is to map a point $p \in P_n^{(0,1,r,1))}$ to the point $q \in P_n^{\circ}(F)$ that is given by

$$q_{(i-n,i]} = \beta^-(1)\beta^-\Psi^{-1}(f(p))\beta^-(1)\beta^-(0)\beta^- \quad i \in \mathcal{I}^{(0)}(p).$$

For a point $q \in \eta_n(P_n^{(0,1,r,1)})$,

$$q_{(i-n,i]} \in \beta^-\beta^-(1)\beta^-\mathcal{C}(0)^*\beta^-(1)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(q),$$

and with the word $a(q) \in \mathcal{C}(0)^*$, that is given by writing

$$q_{(i-n,i]} = \beta^-\beta^-(1)a(q)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(q),$$

a point $p \in P_n^{(0,1,r,\beta))}$ can be reconstructed from its image q under η_n as the point in $P_n^{\circ}(F)$ that is given by

$$p_{(i-n,i]} = \beta^-\beta^-(1)\beta^-\Psi(a(q))\beta^-(1)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(q).$$

We note that

$$(\nu_0(\lambda(\eta(p)_{[0,n)})), \nu_1(\lambda(\eta(p)_{[0,n)}))) = (1, 2), \quad p \in P_n^{(0,1,r,0)}. \tag{0.1.r.0}$$

An inspection of the definition of $P_n^{(0,1,r,1)}$ shows that we have produced a partition

$$\begin{aligned} P_n^{\circ}(\alpha_-(0)) = P_n^{(1)} \cup P_n^{(\beta,\beta)} \cup P_n^{(\beta)} \cup P_n^{(\beta,0)} \cup P^{(0,2)} \\ \cup P_n^{(0,1,l)} \cup P_n^{(0,1,m)} \cup P_n^{(0,1,r)} \cup P_n^{(0,1,r,\beta)} \cup P_n^{(0,1,r,0)}. \end{aligned} \tag{P.0}$$

The points of $\eta_n(P_n^{(1)})$ are the only ones in $\eta_n(P_n^{\circ}(\alpha_-(0))$ in which there appears openly a word in $\mathcal{B}(1.1)$. In the points of $\eta_n(P_n^{(\beta)} \cup P_n^{(0,2)})$, the word $\beta^-\beta^-(1)$ does not appear openly, whereas, in the words of $\eta_n(P_n^{(0,1,l)} \cup P_n^{(0,1,m)} \cup P_n^{(0,1,r)} \cup P_n^{(0,1,r,\beta)} \cup P_n^{(0,1,r,0)})$, this word does appear openly. In the points of $\eta_n(P_n^{(\beta,0)})$, the word $\beta^-\beta^-(1)\beta^-(0)$ appears openly, whereas, in the points of $\eta_n(P_n^{(0,1,l)} \cup P_n^{(0,1,m)} \cup P_n^{(0,1,r)})$, this word does not appear openly. However, in the points of $\eta_n(P_n^{(0,1,l)} \cup P_n^{(0,1,m)} \cup P_n^{(0,1,r)})$, the word $\beta^-\beta^-(1)$ does appear openly. The points of $\eta_n(P_n^{(0,1,r,\beta)})$ are the only ones in $\eta_n(P_n^{\circ}(\alpha_-(0))$ in which there appears openly a word in $(\mathcal{C}^*\backslash\mathcal{C}(0)^*)\beta^-(0)$, and the points of $\eta_n(P_n^{(0,1,r,1)})$ are the only ones in $\eta_n(P_n^{\circ}(\alpha_-(0))$ in which there appears openly a word in $\beta^-\beta^-(1)\mathcal{C}(0)^*\mathcal{B}(1)\mathcal{C}(0)^*$.

From these observations and from (1), (β), (β.0), (0.2), and (0.1.l), (0.1.m), (0.1.r), (0.1.r.β), (0.1.r.0) it follows that the images under η_n of the sets in the partition (P.0) are disjoint. Also, $\eta_n(P_n^{\circ}(\alpha_-(0))) \cap P_n^{\circ}(\alpha_-(0)) = \emptyset$. We have shown that

$$\operatorname{card} \mathcal{O}_n(\alpha^-(0)) \leq \operatorname{card}\Bigg(\bigcup_{\widetilde{\mu}\in\mathcal{M}\backslash\{\alpha^-(0)\}} \mathcal{O}_n(\widetilde{\mu}^-)\Bigg).$$

Apply Lemma (c). □

https://doi.org/10.1017/etds.2015.126 Published online by Cambridge University Press

7. *The multiplier* $\alpha(1)$

PROPOSITION M1. *The multiplier* $\alpha(1)$ *is exceptional only at period two.*

Proof. We construct shift commuting injections

$$\eta_n : P_n^\circ(\alpha_-(1)) \to \bigcup_{\tilde{\mu}\in\mathcal{M}\setminus\{\alpha_-(1)\}} P_n^\circ(\tilde{\mu}^-), \quad n > 2.$$

Let $n > 2$. Denote by $P_n^{(1)}$ the set of $p \in P_n^\circ(\alpha^-(1))$ such that the word $\beta^-\beta^-(1)$ appears openly in p. The shift commuting map η_n is to map a point $p \in P_n^{(1)}$ to the point $q \in P_n(F)$ that is obtained by replacing, in p, the word $\beta^-\beta^-(1)$, when it appears openly in p, by the word $\beta^-(0)\beta^-(0)$. A point $p \in P_n^{(1)}$ can be reconstructed from its image q under η_n by replacing, in q, every word $\beta^-(0)^{2K}$, that appears in q openly and that is neither preceded nor followed in q by an open appearance of the symbol $\beta^-(0)$, by the word $(\beta^-\beta^-(1))^K$, $K \in \mathbb{N}$.

Denote by $P_n^{(2)}$ the set of

$$p \in P_n^\circ(\alpha^-(1))\setminus P_n^{(1)}$$

such that words in $\beta^-\mathcal{C}^\circ(1)^*\beta^-(1)$ appear openly in p at least twice during a period. Denote by $\mathcal{J}(p)$, for $p \in P_n^{(2)}$, the set of indices $j \in \mathcal{I}^{(1)}(p)$ such that the word $p_{(n-j,n]}$ is lexicographically the smallest one among the words $p_{(n-i,n]}$, $i \in \mathcal{I}^{(1)}(p)$. With the word $f^\circ(p) \in \mathcal{C}^\circ(1)^*$, that is given by writing the word in $\beta^-(\mathcal{C}^\circ(1)^*\setminus\{\epsilon\})\beta^-(1)$ that appears in p at an index in $\mathcal{J}(p)$ as

$$\beta^- f^\circ(p)\beta^-(1), \tag{a}$$

the shift commuting map η_n is to map a point $p \in P_n^{(2)}$ to the point $q \in P_n(F)$ that is obtained by replacing each of the words in the point p (a), that appear in p at an index in $\mathcal{J}(p)$, by the word

$$\beta^-(0)\Psi(f^\circ(p)(p))\beta^-(0).$$

With the word $f(q) \in \mathcal{C}(0)^*$ such that the word $\beta^-(0)f(q)\beta^-(0)$ appears openly in q, the point p can be reconstructed from its image q under η_n by replacing, in q, the word

$$\beta^-(0)f(q)\beta^-(0),$$

when it appears openly in q, by the word

$$\beta^-\Psi^{-1}(f(q))\beta^-(1).$$

Denote by $P_n^{(3)}$ the set of

$$p \in P_n^\circ(\alpha^-(1))\setminus(P_n^{(1)} \cup P_n^{(2)})$$

such that

$$p_{(i-n,i]} \in (\mathcal{C}^*\setminus\{\epsilon\})\beta^-\mathcal{C}^\circ(1)^*\beta^-(1).$$

With the words

$$f(p) \in \mathcal{C}^*, \quad f^\circ(p) \in \mathcal{C}^\circ(1)^*,$$

that are given by writing

$$p_{(i-n,i]} = f(p)\beta^- f^\circ(p)\beta^-(1),$$

https://doi.org/10.1017/etds.2015.126 Published online by Cambridge University Press

and denoting by $\Psi'(c)$ the word that is obtained by removing from the word $\Psi(f^\circ(p))$ its last symbol, the shift commuting map ψ_n is to map a point $p \in P_n^{(3)}$ to the point $q \in P_n^\circ(F)$ that is given by

$$q_{(i-n,i]} = f(p)\beta^-(0)\Psi'(f^\circ(p))\beta^-\beta^-(1), \quad i \in \mathcal{I}^{(1)}(q).$$

With the words

$$f(q) \in \mathcal{C}^* \backslash \{\epsilon\}, \quad f'(q) \in \mathcal{C}(0)^*\beta^-(0)\mathcal{C}(0)^*,$$

that are given by writing

$$q_{(i-n,i]} = f(q)\beta^-(0)f'(q)\beta^-\beta^-(1), \quad i \in \mathcal{I}^{(1)}(q),$$

the point p can be reconstructed from its image q under η_n as the point that is given by

$$p_{(i-n,i]} = f(q)\beta^-\Psi^{-1}(f'(q)\beta^-(0))\beta^-(1), \quad i \in \mathcal{I}^{(1)}(q).$$

Set

$$P_n^{(4)} = P_n^\circ(\alpha^-(1)) \backslash (P_n^{(1)} \cup P_n^{(2)} \cup P_n^{(3)}).$$

One has

$$p_{(i-n,i]} \in \beta^-(1)\beta^-\mathcal{C}^\circ(1)^*, \quad i \in \mathcal{I}^{(1)}(p), \quad p \in P_n^{(4)}.$$

With the words

$$c(p) \in \mathcal{C}^\circ(1), \quad f^\circ(p) \in \mathcal{C}^\circ(1)^*,$$

that are given by writing

$$p_{(i-n,i]} = \beta^-(1)\beta^-c(p)f^\circ(p), \quad i \in \mathcal{I}^{(1)}(p),$$

the shift commuting map η_n is to map a point $p \in P_n^{(4)}$ to the point $q \in P_n(F)$ that is given by

$$q_{(i-n,i]} = \beta^-(1)\Phi_0(\Psi(c(p)))\Psi(f^\circ(p))\beta^-, \quad i \in \mathcal{I}^{(1)}(p).$$

With the words

$$b(q) \in \mathcal{B}(0,0), \quad g(q) \in \mathcal{C}(0)^*,$$

that are given by writing

$$q_{(i-n,i]} = \beta^-(1)b(q)g(q)\beta^-, \quad i \in \mathcal{I}^{(1)}(q),$$

the point p can be reconstructed from its image q under η_n as the point that is given by

$$p_{(i-n,i]} = \beta^-(1)\beta^-\Psi^{-1}(\Phi_0^{-1}(b(q)))\Psi^{-1}(g(q)), \quad i \in \mathcal{I}^{(1)}(q).$$

We have produced a partition

$$P_n^\circ(\alpha_-(1)) = \bigcup_{1 \le l \le 4} P_n^{(l)}. \tag{P.1}$$

In the points in $\eta_n(P_n^{(1)})$, the word $\beta^-(0)\beta^-(0)$ appears openly, the word $\beta^-\beta^-(1)$ does not appear openly and, in the points in $\eta_n(P_n^{(2)})$, neither the word $\beta^-(0)\beta^-(0)$ nor the

https://doi.org/10.1017/etds.2015.126 Published online by Cambridge University Press

word $\beta^-\beta^-(1)$ appear openly. In the points in $\eta_n(P_n^{(3)})$ and $\eta_n(P_n^{(4)})$, the word $\beta^-\beta^-(1)$ appears openly. Also,

$$(\mathcal{I}^{(1)}(q)+1)\cap\mathcal{I}^{(0)}(q)=\emptyset, \quad q\in\eta_n(P_n^{(3)}),$$
$$\mathcal{I}^{(1)}(q)+1\subset\mathcal{I}^{(0)}, \quad q\in\eta_n(P_n^{(4)}).$$

From these observations it follows that the images under η_n of the sets of the partition (P.1) are disjoint.

We have shown that

$$\operatorname{card}(\mathcal{O}_n(\alpha^-(0)))\le\operatorname{card}\biggl(\bigcup_{\widetilde{\mu}\in\mathcal{M}\setminus\{\alpha(1)\}}\mathcal{O}_n(\widetilde{\mu}_-)\biggr).$$

Apply Lemma (c). □

REFERENCES

[**BP**] J. Berstel and D. Perrin. The origins of combinatorics on words. *European J. Combin.* **28** (2007), 996–1022.

[**BPR**] J. Berstel, D. Perrin and Ch. Reutenauer. *Codes and Automata*. Cambridge University Press, Cambridge, 2010.

[**HI**] T. Hamachi and K. Inoue. Embeddings of shifts of finite type into the Dyck shift. *Monatsh. Math.* **145** (2005), 107–129.

[**HIK**] T. Hamachi, K. Inoue and W. Krieger. Subsystems of finite type and semigroup invariants of subshifts. *J. Reine Angew. Math.* **632** (2009), 37–61.

[**HK**] T. Hamachi and W. Krieger. A construction of subshifts and a class of semigroups. *Preprint*, 2013, arXiv:1303.4158 [math.DS].

[**Ki**] B. P. Kitchens. *Symbolic Dynamics*. Springer, New York, 1998.

[**KM**] W. Krieger and K. Matsumoto. Zeta functions and topological entropy of the Markov–Dyck shifts. *Münster J. Math.* **4** (2011), 171–184.

[**Kr1**] W. Krieger. On the uniqueness of the equilibrium state. *Math. Syst. Theory* **8** (1974), 97–104.

[**Kr2**] W. Krieger. On a syntactically defined invariant of symbolic dynamics. *Ergod. Th. & Dynam. Sys.* **20** (2000), 501–516.

[**LM**] D. Lind and B. Marcus. *An Introduction to Symbolic Dynamics and Coding*. Cambridge University Press, Cambridge, 1995.

[**M**] K. Matsumoto. C^*-algebras arising from Dyck systems of topological Markov chains. *Math. Scand.* **109** (2011), 31–54.

[**NP**] M. Nivat and J.-F. Perrot. Une généralisation du monoîde bicyclique. *C. R. Math. Acad. Sci. Paris* **271** (1970), 824–827.

[**P**] D. Perrin. Algebraic combinatorics on words. *Algebraic Combinatorics and Computer Science*. Eds. H. Crapo and D. Senato. Springer, Berlin, Heidelberg, New York, 2001, pp. 391–430.