MINIMAL DEPENDENT SETS

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1. Introduction

The subject matter of this note is the notion of a dependence structure on an abstract set. There are a number of different approaches to this topic and it is known that many of these lead to precisely the same structure. Axioms are given here to specify the minimal dependent sets for such a structure. They are closely related to conditions introduced by Hassler Whitney in [1] and to a certain “elimination axiom” given by A. P. Robertson and J. D. Weston in [2]. Theorem 1 shows that a dependence structure may equally well be defined by means of axioms for the independent sets. Axiom (I) is adapted from a condition due to R. Rado [3]. Theorem 2 links our minimal dependent sets with Whitney’s “circuits”. Theorem 3 is an “elimination” theorem which generalizes the statement of our axiom (C2). Theorem 4 is due to Rado ([3], Theorem 3, p. 307), and A. W. Ingleton [4]. It is shown here to follow from Theorem 3.

2. Axioms and theorems

Let $X$ be a set. Let $\mathcal{C}$ be a set of non-empty finite subsets of $X$. Furthermore, let $\mathcal{C}$ satisfy the following two conditions.

(C1) No proper subset of a member of $\mathcal{C}$ is a member of $\mathcal{C}$.

(C2) If $E$ and $F$ are distinct members of $\mathcal{C}$ and $x \in E \cap F$, then $E \cup F$ has a subset belonging to $\mathcal{C}$ but not containing $x$.

Axiom (C2) is the “elimination axiom” of Robertson and Weston [2] who apply it to a set $\mathcal{R}$ of finite subsets of a set $X$ and use no other condition. For the case where the empty set is not a member of $\mathcal{R}$, it can be seen that the members of $\mathcal{C}$ are precisely the minimal members of $\mathcal{R}$, for it is clear that the “elimination axiom” must hold with $\mathcal{R}$ replaced by the set of its minimal members. In [2] the authors define “pure sets” as those non-empty subsets of $X$ which fail to contain members of $\mathcal{R}$. Here we define independent sets to be those subsets of $X$ which fail to contain

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1 A study of the various axioms for a dependence structure formed the topic of a M. Sc. thesis by the author at Monash University.
Theorem 1. A set \( \mathcal{U} \) of subsets of a set \( X \) is the set of independent sets defined by a set \( \mathcal{C} \) of non-empty finite subsets of \( X \) satisfying \((C_1)\) and \((C_2)\) if, and only if \( \mathcal{U} \) is non-empty, \( \mathcal{U} \) has the inductive property and \( \mathcal{U} \) satisfies the following condition.

\[ (I) \text{ If } A \text{ and } B \text{ are subsets of } X \text{ such that } A \notin \mathcal{U}, B \notin \mathcal{U}, \text{ and } A \cap B \in \mathcal{U}, \text{ then for all elements } x \in A \cup B \text{ it follows that } (A \cup B) \setminus \{x\} \notin \mathcal{U}. \]

Proof. Firstly, let \( \mathcal{U} \) be the set of independent sets. Then \( \mathcal{U} \) is non-empty since \( \emptyset \in \mathcal{U} \), where \( \emptyset \) is the empty set. Now if \( A \in \mathcal{U} \), then clearly every subset of \( A \) belongs to \( \mathcal{U} \). If \( A \notin \mathcal{U} \), then a member of \( \mathcal{C} \) contained in \( A \) is a finite subset of \( A \) which fails to belong to \( \mathcal{U} \). Thus \( \mathcal{U} \) has the inductive property. To verify condition \((I)\) let \( A \notin \mathcal{U}, B \notin \mathcal{U}, A \cap B \in \mathcal{U}, \) \( x \in A \cup B \). Then there exist sets \( C \in \mathcal{C}, D \in \mathcal{C} \) such that \( C \subseteq A, D \subseteq B. \) Also \( C \neq D, \) since \( A \cap B \in \mathcal{U}. \) If \( x \notin C \cap D, \) then either \( C \subseteq (A \cup B) \setminus \{x\} \) or \( D \subseteq (A \cup B) \setminus \{x\} \) and so \( (A \cup B) \setminus \{x\} \notin \mathcal{U}. \) If \( x \in C \cap D, \) then from \((C_2)\) there exists \( E \in \mathcal{C} \) with \( E \subseteq (C \cup D) \setminus \{x\} \subseteq (A \cup B) \setminus \{x\} \) so again \( (A \cup B) \setminus \{x\} \notin \mathcal{U}. \) Thus condition \((I)\) holds. One may observe that the set \( \mathcal{C} \) is precisely the set of subsets of \( X \) minimal with respect to not belonging to \( \mathcal{U}. \)

Secondly, let \( \mathcal{U} \) be a set of subsets of \( X \) having the properties stated in the theorem, and let \( \mathcal{C} \) be the set of subsets of \( X \) minimal with respect to not belonging to \( \mathcal{U}. \) Since \( \mathcal{U} \) is non-empty and possesses the inductive property, it follows that \( \emptyset \in \mathcal{U}. \) Hence \( \emptyset \notin \mathcal{C}. \) Also from the inductive property, any subset not in \( \mathcal{U} \) contains a finite subset not in \( \mathcal{U}. \) Hence \( \mathcal{C} \) consists of non-empty finite sets. That \((C_1)\) holds is clear from the definition of \( \mathcal{C}. \) In order to verify \((C_2)\), let \( E \in \mathcal{C}, F \in \mathcal{C}, E \neq F, \) and \( x \in E \cap F. \) Then \( E \notin \mathcal{U}, F \notin \mathcal{U} \) and also \( E \cap F \in \mathcal{U} \) because \( E \cap F \) is a proper subset of \( E. \) It follows from condition \((I)\) that \( (E \cup F) \setminus \{x\} \notin \mathcal{U}. \) Then since it is a finite set, we have that \( (E \cup F) \setminus \{x\} \) must contain a member of \( \mathcal{C}. \) To complete the proof of the theorem one observes that the set \( \mathcal{U} \) consists precisely of those subsets of \( X \) which fail to contain members of \( \mathcal{C}. \)

In the paper [1], Whitney uses the following axiom \((C_2')\) together with \((C_1)\) and refers to the members of \( \mathcal{C} \) by the name "circuits". Also he restricts attention to the case where the set \( X \) is finite.

\[ (C_2') \text{ If } E \text{ and } F \text{ are distinct members of } \mathcal{C}, \text{ if } x \in E \cap F \text{ and if } y \in E \setminus F, \text{ then } E \cup F \text{ has a subset belonging to } \mathcal{C} \text{ which contains } y \text{ but fails to contain } x. \]

Since \((C_2')\) seems to impose a stronger condition on the set \( \mathcal{C} \) than \((C_2)\), the following theorem may be of some interest. In any case it provides the link between the two systems.
Theorem 2. If $\mathcal{C}$ is a set of non-empty finite subsets of a set $X$, then conditions $(C_1)$ and $(C_2)$ together are equivalent to conditions $(C')_1$ and $(C')_2$ together.

Proof. It is clear that $(C_2)$ follows from $(C')_2$. Suppose now that $(C_1)$ and $(C_2)$ hold but that $(C')_2$ fails to hold and let $m$ be the least integer such that for some pair of sets $E$ and $F$ belonging to $\mathcal{C}$ and satisfying $|E \cup F| = m$ there exist elements $x \in E \cap F$ and $y \in E \setminus F$ such that $E \cup F$ contains no member $G$ of $\mathcal{C}$ satisfying $y \in G$ and $x \notin G$. By supposition such an integer exists, and we may assume that $E$, $F$, $x$ and $y$ have the stated properties. Then by $(C_2)$ there exists a subset $G$ of $E \cup F$ belonging to $\mathcal{C}$ and failing to contain $x$. But then $y \notin G$. From $(C_1)$ we may choose $z \in G \setminus E$, and then using the fact that $|G \cup F| < m$ and the minimality of $m$ we may apply $(C')_2$ to the sets $G$ and $F$ and the elements $x \in G \setminus F$ and $y \in E \setminus G$. Thus there exists a subset $H$ of $G \cup F$ belonging to $\mathcal{C}$ and containing $x$ but not containing $z$. But then $|E \cup H| < m$ since $z \notin E \cup H$ and we may apply $(C')_2$ to the sets $E$ and $H$ and the elements $x \in E \cap H$ and $y \in E \setminus H$ to show the existence of a subset $J$ of $E \cup H$ belonging to $\mathcal{C}$, containing $y$ and failing to contain $x$. This, however, is a contradiction.

Theorem 3. If $A_1, A_2, \ldots, A_n$ are members of $\mathcal{C}$, if $n \geq 2$ and if

$$A_i \notin \bigcup \{A_j : j < i\}, \quad i = 2, \ldots, n$$

holds, then for each subset $B$ of $X$ with $|B| = r < n$, there exist members $C_1, C_2, \ldots, C_{n-r}$ of $\mathcal{C}$ such that $C_i \subseteq \bigcup \{A_k : k = 1, \ldots, n\}\setminus B$ and $C_i \notin \bigcup \{C_j : j \neq i\}$ hold for $i = 1, 2, \ldots, n-r$.

Proof. Let $(n, r)$ denote the case of the theorem for which $n$ members of the set $\mathcal{C}$ are considered and the set $B$ consists of $r$ elements.

Case $(n, 0)$. By the hypothesis and since $\square \notin \mathcal{C}$ we may choose $x_1 \in A_1$ and elements $x_i \in A_i \setminus \bigcup \{A_j : j < i\}$ for $i = 2, \ldots, n$. The following sets $B_{i,j}$ are defined for the indices $j = i+1, \ldots, n$ and $i = 1, 2, \ldots, n$. For $j = 2, \ldots, n$, if $x_i \notin A_j$, put $B_{i,j} = A_j$, and if $x_i \in A_j$ using $(C'_2)$ choose $B_{i,j} \in \mathcal{C}$ such that $B_{i,j} \subseteq A_i \cup A_j$, $x_1 \notin B_{i,j}$, and $x_j \in B_{i,j}$. Then by induction, if $x_i \notin B_{i-1,j}$ define $B_{i,j} = B_{i-1,j}$, and if $x_i \in B_{i-1,j}$ choose $B_{i,j} \in \mathcal{C}$ such that $B_{i,j} \subseteq B_{i-1,j} \cup B_{i-1,j}$, $x_i \notin B_{i,j}$ and $x_j \in B_{i,j}$ where the choice is made possible by $(C'_2)$ and by the fact that $x_i \in B_{i-1,j} \cap B_{i-1,j}$ and $x_j \in B_{i-1,j} \setminus B_{i-1,j}$. Then let $C_1 = A_1$, $C_2 = B_{1,2}, \ldots, C_n = B_{n-1,n}$. Then for $i = 1, \ldots, n$ we have $x_i \in C_i$ and $x_i \notin \bigcup \{C_j : j = i+1, \ldots, n\}$. Also one observes that for $i = 2, \ldots, n$ we have $x_i \notin \bigcup \{C_j : j < i\}$. Thus for $i = 1, \ldots, n$ the relation $C_i \notin \bigcup \{C_j : j \neq i\}$ holds. Finally it is clear that
$C_i \subseteq \bigcup \{A_k : k = 1, \ldots, n\}$ must also hold and hence the sets $C_1, \ldots, C_n$ satisfy the requirements of the theorem in this case.

Case $(n, r)$. This case of the theorem is shown to depend on the case $(n-1, r-1)$ and so ultimately on the case $(n-r, 0)$ which is a case already proved. Let $B \subseteq X$ and $|B| = r$. By the case $(n, 0)$ it may be assumed that the sets $A_1, \ldots, A_n$ already satisfy the relation $A_i \notin \bigcup \{A_j : j \neq i\}$. Let $x \in B$. If $x \notin \bigcup \{A_i : i = 1, \ldots, n\}$, the case reduces immediately to the case $(n-1, r-1)$. Otherwise, by the symmetry it may be assumed that $x \in A_n$. Now choose elements $y_i \in A_i \bigcup \{A_j : j \neq i\}$ for $i = 1, 2, \ldots, n-1$. Using $(C'_2)$ where necessary, there exist members $B_1, B_2, \ldots, B_{n-1}$ of $\mathcal{C}$ such that for $j = 1, 2, \ldots, n-1$ we have $B_j \subseteq A_n \cup A_j$, $x \notin B_j$ and $y_j \in B_j$. One then notices that $y_j \notin \bigcup \{B_i : i \neq j\}$ and so $B_j \notin \bigcup \{A_i : j \neq i\}$ for $j = 1, 2, \ldots, n-1$. Applying the theorem for the case $(n-1, r-1)$ to these members $B_1, \ldots, B_{n-1}$ of $\mathcal{C}$ and to the set $B \{x\}$ we obtain members $C_1, C_2, \ldots, C_n - r$ of $\mathcal{C}$ having the required properties.

**Theorem 4.** If $A_1, A_2, \ldots, A_n$ are subsets of $X$ which do not belong to the set $\mathcal{U}$ of independent sets, and if for $i = 2, 3, \ldots, n$ the sets $A_i \bigcap \bigcup \{A_j : j < i\}$ belong to $\mathcal{U}$, then for any subset $B$ of $X$ with $|B| < n$ it follows that $\bigcup \{A_i : i = 1, \ldots, n\} \setminus B$ is not a member of $\mathcal{U}$.

**Proof.** Let $B_1, B_2, \ldots, B_n$ be minimal dependent sets (i.e. members of $\mathcal{C}$) contained in $A_1, A_2, \ldots, A_n$ respectively. Then for $i = 1, 2, \ldots, n$ we have

$$B_i \bigcap \bigcup \{B_j : j < i\} \subseteq A_i \bigcap \bigcup \{A_j : j < i\}.$$

Then since $B_i \notin \mathcal{U}$ it follows that $B_i \notin \bigcup \{B_j : j < i\}$ for $i = 2, 3, \ldots, n$. By theorem 3, if $|B| = r$, there are $(n-r) > 0$ minimal dependent sets contained in $\bigcup \{A_i : i = 1, 2, \ldots, n\} \setminus B$ and hence this latter set is not a member of $\mathcal{U}$.

**References**


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