Let $A$ be a commutative Banach algebra, $D$ a closed derivation defined on a subalgebra $\Delta$ of $A$, and with range in $A$. The elements of $\Delta$ may be called derivable in the obvious sense. For each integer $k \geq 1$, denote by $\Delta_k$ the domain of $D^k$ (so that $\Delta_1 = \Delta$); it is a simple consequence of Leibniz's formula that each $\Delta_k$ is an algebra. The classical example of this situation is $A = C(0, 1)$ under the supremum norm with $D$ ordinary differentiation, and here $\Delta_k = C^k(0, 1)$ is a Banach algebra under the norm $\| \cdot \|_k$:

$$\|x\|_k = \sum_{n=0}^{k} \frac{1}{n!} \sup_{t \in [0, 1]} |x^{(n)}(t)|.$$ 

Furthermore, the maximal ideals of $\Delta_k$ are precisely those subsets of $\Delta_k$ of the form $M \cap \Delta_k$ where $M$ is a maximal ideal of $A$, and $\overline{M \cap \Delta_k} = M$, the bar denoting closure in $A$. In the present note we show how this extends to the general case.

If $A$ is a commutative Banach algebra then $\| \cdot \|_A$, $\rho_A(\cdot), \mathcal{M}(A)$ will denote the norm, spectral radius and maximal ideal space of $A$ respectively. The author is indebted to the referee for the present proof of the following result.

**Theorem 1.** Let $A$, $B$ be commutative Banach algebras, with $B$ a dense subalgebra of $A$ in the norm topology of $A$. Suppose that there is a constant $K$ such that $\rho_B(x) \leq K \rho_A(x)$ for $x \in B$. Then the map $\Gamma : \mathcal{M}(A) \rightarrow \mathcal{M}(B) : M \mapsto M \cap B$ is a homeomorphism of $\mathcal{M}(A)$ onto $\mathcal{M}(B)$ (and so $\rho_B(x) = \rho_A(x)$ for $x \in B$).

**Proof.** If $\psi$ is a multiplicative linear functional on $A$ then $\psi|_B$ is clearly such a functional on $B$. Conversely, if $\phi$ is a multiplicative linear functional on $B$, the given inequality shows that $\phi$ is continuous in the norm topology of $A$, and so has a unique continuous extension, also multiplicative linear, to all of $A$. From the correspondence between multiplicative linear functionals and maximal modular ideals it follows that $\Gamma$ is bijective. That $\Gamma$ is a homeomorphism is an immediate con-
sequence of the fact that \( B \) is dense in \( A \). The last statement is clear from the form of \( \Gamma \).

We now turn to the situation at hand.

**Lemma 1.** Let \( A \) be a Banach algebra with norm \( ||\cdot|| \), \( D \) a closed derivation defined on a subalgebra \( \mathcal{A} \) of \( A \), with range in \( A \). Then for each integer \( k \geq 1 \), \( \mathcal{A}_k \) is a Banach algebra under the norm \( ||\cdot||_k \):

\[
||x||_k = \sum_{n=0}^{k} \frac{1}{n!} ||D^n x||.
\]

**Proof.** As was remarked above each \( \mathcal{A}_k \) is certainly an algebra, and an application of Leibniz's formula shows that \( ||\cdot||_k \) is a norm on \( \mathcal{A}_k \). If \( \{x_n\} \subseteq \mathcal{A}_k \) is Cauchy under \( ||\cdot||_k \), then \( \{D^j x_n\} \) is Cauchy in \( A \) for \( 0 \leq j \leq k \). Setting \( y_j = \lim_n D^j x_n \), the closure of \( D \) shows that \( y_j = D^j y_0 \), whence \( y_0 \in \mathcal{A}_k \) and \( ||x_n - y_0||_k \to 0 \) as \( n \to \infty \).

**Lemma 2.** Let \( A \) be a commutative normed algebra, \( D \) a derivation defined on a subalgebra \( \mathcal{A} \) of \( A \), with range in \( A \). Denote by \( v_k(\cdot) \) the spectral radius in \( \mathcal{A}_k \) calculated from \( ||\cdot||_k \). Then if \( x \in \mathcal{A}_k \), \( v_k(x) = v_A(x) \).

**Proof.** It is clear that \( v_k(x) \geq v_A(x) \) for all \( x \in \mathcal{A}_k \). Now for \( j < n \) and \( x \in \mathcal{A}_k \),

\[
D^j x^n = \sum_{i=1}^{j} u_{i,j} x^{n-i}
\]

where the \( u_{i,j} \) are polynomials in \( D^r x \), \( 1 \leq r \leq j \), of degree \( \leq j \), the scalars concerned being polynomials in \( n \) of degree \( \leq j \). To see this, note that the formula is true for \( j = 1 \), since \( Dx^n = nx^{n-1} Dx \). Supposing by way of induction that it holds for \( j = m-1 \), we have

\[
D^m x^n = \sum_{i=1}^{m-1} \left\{ D(u_{i,m-1}) x^{n-i} + (n-i)u_{i,m-1} x^{n-i-1} Dx \right\},
\]

which is of the desired form.

Thus if \( x \in \mathcal{A}_k \) and \( n > k \),

\[
||x^n||_k = ||x^n|| + \sum_{j=1}^{k} \frac{1}{j!} \left|\sum_{i=1}^{j} u_{i,j} x^{n-i}\right|
\]

\[
\leq ||x^{n-k}|| \left\{ ||x^n|| + \sum_{j=1}^{k} \frac{1}{j!} \sum_{i=1}^{j} ||u_{i,j}|| \cdot ||x^{n-i}|| \right\}
\]

\[
\leq Kn^k ||x^{n-k}||
\]

1 The exact form is

\[
\frac{D^j x^n}{j!} = \sum_{i_1 + \cdots + i_n = j} \frac{D^{i_1} x}{i_1!} \cdots \frac{D^{i_n} x}{i_n!}.
\]
for some constant $K$, by the properties of the elements $u_{i,j}$. But this means $v_k(x) \leq v_A(x)$.

Our main result is an immediate consequence of Lemmas 1 and 2, and Theorem 1.

**Theorem 2.** Let $A$ be a commutative Banach algebra, $D$ a closed derivation on a subalgebra $\Delta$ of $A$, with range in $A$. Suppose that $\Delta_k$ is dense in $A$ for some integer $k \geq 1$. Then the map $\Gamma_j : M \rightarrow M \cap \Delta_j$ is homeomorphism of $\mathcal{M}(A)$ onto $\mathcal{M}(\Delta_j)$, $1 \leq j \leq k$.

**Corollary 1.** If $A$ has an identity $e$ then $e \in \Delta$.

**Proof.** Theorem 2 shows that $\mathcal{M}(A)$ is compact, and so by Silov’s theorem there is an idempotent $f \in \Delta$ with $\hat{f} \equiv 1$ on $\mathcal{M}(\Delta)$, and hence on $\mathcal{M}(A)$. But this means the idempotent $e - f$ is quasi-nilpotent, and hence zero.

**Corollary 2.** If $\Delta$ is dense in $A$ and $D$ has non-empty resolvent set then $\Gamma_j$ is a homeomorphism for each $j \geq 1$.

**Proof.** By Lemma VIII.2.9 of [1] $\Delta_j$ is dense in $A$ for each $j \geq 1$.

**Remark.** In the situation of Theorem 2 define, for $\alpha > 0$,

$$A_{\alpha, \alpha} = \left\{ x \in \bigcap_{k \geq 1} A_k : \|x\|_{\alpha, \alpha} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \|D^n x\| < \infty \right\}.$$  

An argument similar to that of Lemma 1 shows that $A_{\alpha, \alpha}$ is a Banach algebra under $\|\cdot\|_{\alpha, \alpha}$, however $\mathcal{M}(A)$ and $\mathcal{M}(A_{\alpha, \alpha})$ are not homeomorphic in general, even when $A_{\alpha, \alpha}$ is dense in $A$. Indeed, in the classical situation mentioned at the beginning of this paper, $\mathcal{M}(A) = [0, 1]$, while $\mathcal{M}(A_{\alpha, \alpha})$ is homeomorphic to the closed unit disc.

**Reference**


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