Spherical Modifications
And The Strong Category Of Manifolds

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Using the notion of spherical modification and results from Morse theory a general technique is described for constructing manifolds whose strong category is small (≤ 3) but whose homological structure is complex.

Unless stated otherwise an n-manifold is a compact, differentiable n dimensional manifold without boundary.

Let $V_1$ be an n-manifold and suppose $S^i$ is an i-sphere homeomorphically and smoothly imbedded in $V_1$ with a trivial normal bundle. Then $S^i$ has a neighborhood of the form $S^i \times D^{n-i}$ ($D^{n-i}$ is an $(n-i)$-disc). Clearly the boundary of $S^i \times D^{n-i} = S^i \times S^{n-i-1}$ = the boundary of $D^{i+1} \times S^{n-i-1}$. Smoothly identifying the boundary of $D^{i+1} \times S^{n-i-1}$ with the boundary of $(V_1$-interior $(S^i \times D^{n-i}))$ results in a new manifold $V_2$. $V_2$ is said to be obtained from $V_1$ by a spherical modification of type $(i, n-i-1)$. (Cf. [8] page 504). The manifold $V_2$ has a sphere $S^{n-i-1}$ (the associated sphere to $S^i$) imbedded in it with trivial normal bundle; namely,

$$\{0\} \times S^{n-i-1} \subset D^{i+1} \times S^{n-i-1} \subset V_2.$$

Clearly by reversing the procedure $V_1$ can be obtained from $V_2$ by a spherical modification of type $(n-i-1, i)$ determined by the associated sphere to $S^i$. Such a modification will be called an inverse to the given one.

Let $V_2$ be obtained from $V_1$ by performing a finite sequence $S$ of spherical modifications on $V_1$. Associated to $S$ is a $n+1$-manifold $W$ called the trace of $S$ with boundary of $W = V_1 \cup V_2$. The triple $(W, V_1, V_2)$ is a manifold triad in the sense of [6] page 2. A rearrangement theorem says that the modifications $S$ can be rearranged so that all modifications of type $(i, n-i-1)$ are performed before any of type $(i+1, n-i-2)$ and all modifications of type $(i, n-i-1)$ can be assumed to be carried out on the same manifold. Further the trace of the rearranged sequence is the same as the trace of $S$. (Cf. [8] page 514 and [6] page 44).

Assuming that the sequence $S$ of modifications leading from $V_1$ to $V_2$ is already ‘rearranged’ as above one has a sequence of manifolds $V_1 = M_0$, $M_1, \ldots M_i$, $M_{i+1}, \ldots, M_k = V_2$ where $M_{i+1}$ is obtained from $M_i$ by modifications of type $(i, n-i-1)$ only. Suppose $b_{i+1} = \text{the number of}$
(i, n—i—1) type modifications in S. Let \( \{ S^i_j \}_{j=1,\ldots,b_{i+1}} \) be the spheres in \( M_i \) determining the \( b_{i+1} \) (i, n—i—1) type modifications and let \( \{ S^{n-i-1}_j \}_{j=1,2,\ldots,b_{i+1}} \) be the associated spheres in \( M_{i+1} \). For any integer \( b \geq 0 \) let \( F(b) \) be the free abelian group on \( b \) generators (\( F(0) = 0 \)). Define \( C_i = F(b_i) \) where the \( \{ S^i_j \}_{j=1,\ldots,b_i} \) can be taken as representatives for the generators. Then \( C_{i-1} = F(b_{i-1}) \) is generated by \( \{ S^{i-2}_k \}_{k=1,\ldots,b_{i-1}} \). Define \( d_i : C_i \to C_{i-1} \) by
\[
d_i(S^{i-1}_j) = \sum_{k=1}^{b_i} A_{jk} S^{i-2}_k
\]
where \( A_{jk} = S^{i-1}_j, S^{n-i+1}_k = \) intersection number of \( S^{i-1}_j \) and \( S^{n-i+1}_k \), where \( S^{n-i+1}_k = \) associated sphere to \( S^{i-2}_k \) (note \( S^{i-1}_j \) and \( S^{n-i+1}_k \) are both spheres in \( M_{i-1} \)). It is not hard to see that \( (C_*, \partial) = (C_, d_*) \) is a chain complex.

**Theorem 1.** Let \( W = \) trace of \( S \) (performed on \( V_1 \)) then

\[
H_i(W, V_1) \cong H_i(C_*),
\]

all \( i \) (homology with integer coefficients).


**Theorem 2.** Let \( M \) be an \( n \) dimensional, compact, connected manifold of the form \( M = W \cup D^n \) where \( W = \) trace of a finite sequence \( S \) of spherical modifications on \( V_1 = \) boundary of \( D^n = S^{n-1} \) and \( D^n \) is attached to \( W \) by smoothly identifying (boundary \( D^n \)) to ((boundary \( W \)) = \( V_1 \)). Then \( H_0(M) = Z \) (integers) and for \( i > 0 \) \( H_i(M) \cong H_i(C_* \) where \( C_* \) is obtained from \( S \) as described above.

**Proof.** Consider the sequence \( H_i(M) \to H_i(M, D^n) \to H_i(W, V_1) \)

where \( j \) is from the homology sequence of the pair \( (M, D^n) \) and is thus an isomorphism for \( i > 0 \) and \( g \) is induced by excision and homotopy and is thus an isomorphism for all \( i \). Since \( M \) is connected \( H_0(M) = Z \) and for \( i > 0 \) the theorem follows from theorem 1.

**Corollary.** If \( d_i = 0 \) all \( i > 0 \) then \( H_i(M) = C_i = F(b_i) \) all \( i > 0 \) where \( b_i = \) number of \( (i-1, n-i-1) \)type modifications in \( S \).

**Proof.** Follows directly from theorem 2 and the definition of \( H_i(C_*) \).

Before getting to the main result one further definition is needed. Let \( M \) be an \( n \)-manifold. Define \( C(M) \) to be the minimum number of contractible in themselves, open sets needed to cover \( M \). (See strong category [2] page 360.)

**Theorem 3** is a statement of the main result although the technique of proof is of more interest than the theorem. (See remark following the proof of theorem 3.)
Note that if, for example, \( b_t = 0 \) for \( 0 < i \leq 33 \) and \( n = 100 \) then the \( 'C(M) \leq 3' \) portion of the theorem falls under the case mentioned in [1] page 201.

**Theorem 3.** Let \( \{b_i\}_{i=0,1,\ldots,n} \) be a sequence of non-negative integers satisfying the following conditions: \( b_0 = 1, b_i = b_{n-i} \) and if \( n = 2m \) then \( b_m = 2t \). Under these conditions there exists an orientable, connected \( n \)-manifold \( M \) with \( H_i(M) = F(b_i) \) and \( C(M) \leq 3 \).

**Proof.** Let \( N = \sum_{i=0}^{n} b_i \) if \( n = 2m+1 \) and let \( N = \sum_{i=1}^{m} b_{i-1} + t \) if \( n = 2m \) and \( b_n = 2t \). Let \( D_1^n \) be an \( n \)-disc and in the boundary of \( D_1^n = S^{n-1} = V_1 \) pick out \( N \) mutually disjoint \((n-1)\) discs. Call them \( D(b_i, j) \) where \( 1 \leq i \leq m \) and \( 1 \leq j \leq b_i \) if \( n = 2m+1 \) if \( n = 2m \), then for \( 1 \leq i \leq m-1, 1 \leq j \leq b_i \) and for \( i = m, 1 \leq j \leq t \) (note that no discs are picked if \( b_i = 0 \)). In each disc \( D(b_i, j) \) imbed an \((i-1)\)-sphere \( S_{i-1}^{j-1} \) with trivial normal bundle. (e.g. \( S^{t-1} = \text{Boundary} \))

\[
D^i \subset D^i \subset D^i \times \{0\} \subset D^i \times D^{n-i-1} = D^{n-1}.
\]

Performing on \( V_1 \) spherical modifications determined by these \( N \) different spheres gives a manifold \( V \) with \( N \) mutually disjoint spheres \( \{S_{j-1}^{n-i-1}\} \) (\( S_{j-1}^{t-1} \) is associated to \( S_j^{t-1} \)) imbedded in it. Performing on \( V, N \) spherical modifications inverse to those performed on \( V_1 \) gives a manifold \( V_2 \) which is again \( S^{n-1} \). Finally, perform on \( V_2 \) an \((n-1, -1)\) spherical modification determined by \( V_2 \) itself. Let \( W = \text{trace} \) of these \( 2N+1 \) modifications and let \( M = D_1^n \cup W \) (\( M \) is clearly a compact, connected \( n \)-manifold).

For \( n = 2m+1 \) if \( 1 \leq i \leq m \) then \( m+1 \leq n-i \leq n-1 \). Hence the number of \((i-1, n-i-1)\) modifications = number of \((n-i-1, i-1)\) modifications = \( b_i = b_{n-i} \) \((1 \leq i \leq n-1)\). For \( n = 2m \) a slight change occurs; namely, the number of \((m-1, m-1)\) modifications performed on \( V_1 = t \), and the number of \((m-1, m-1)\) modifications performed on \( V = t \), so the total number of \((m-1, m-1)\) modifications = \( 2t = b_m \). In both cases there is only one \((n-1, -1)\) modification performed thus \( C_i = F(b_i)1 \leq i \leq n \) for any \( n \).

Consider now \( d_i : C_i \to C_{i-1} \). To compute \( d_i \) it is necessary to find the intersection numbers \( S_{j-1}^{t-1}, S_{k-1}^{n-i} \) where \( S_{j-1}^{t-1} \) is a generator of \( C_i \) and \( S_{k-1}^{n-i} \) is associated to a generator \( S_{k-2}^{t-2} \) of \( C_{i-1} \). If \( n = 2m \) then \( i-1 \neq n-i \) and thus by the construction of \( W, S_{j-1}^{t-1} \cap S_{k-1}^{n-i} = 0 \) for \( 2 \leq i \leq n-1 \). However if \( n = 2m+1 \) then \( i-1 = n-i \) for \( i = m+1 \). In this case, then, the associated spheres to generators of \( C_m \) are the generators of \( C_{m+1} \).

Let \( S_{m-1}^{n-1} \) be a generator of \( C_m \) and let \( S_m^{n-1} \) (a generator of \( C_{m+1} \)) be associated to \( S_{k-1}^{n-i} \). Now \( S_{m}^{n-1} \) is identified with \( \{0\} \times S_{m}^{n-1} \subset D_{m}^{n} \times S_{m}^{n-1} \) introduced when \( S_{m-1}^{n-1} \times D_{m}^{n} \) is replaced by \( D_{m}^{n} \times S_{m}^{n-1} \) under the modification. Let \( P \neq 0 \) be a point in \( D_{m}^{n} \) and let \( S_{m}^{n-1} \) be the sphere \( \{P\} \times S_{m}^{n-1} \subset D_{m}^{n} \times S_{m}^{n-1} \). Then \( S_{m}^{n-1} \cap \tilde{S}_{k}^{n} = \emptyset \) and performing the modification with respect to \( \tilde{S}_{k}^{n} \) gives the
same result as using $S^n_k$ since $S^n_k$ and $S^n_k$ are isotopic. (Cf. [9] 776). Thus for $n$ odd or even its clear that if $2 \leq i \leq n-1$ then $d_i = 0$.

Further suppose all of the $(0, n-2)$ modifications performed on $V_1$ to be orientable (i.e. $V$ is orientable). This corresponds to identifying (boundary $S^n \times D^{n-1}$) to (boundary $D^1 \times S^{n-2}$) in such a way that the orientations on one component of the boundary are the same while those on the other component are opposite. Hence when the $(n-1, -1)$ modification is performed on $V_2$ the intersection number is $S^{n-1} \cdot S^0 = 1 - 1 = 0$ where $S^0_j$ in $V_2$ is associated to $S^{n-2}_j$ in $V$ and $S^{n-2}_j$ is associated to the o-sphere determined by $D(b_1, j)$. Also for $W$ its clear that $C_0 = 0$ and thus $d_i = 0$ for $i = 1, \ldots, n$ and by the corollary to theorem 2 $H_i(M) = H_i(C_*) = F(b_i)$. This proves the first part of theorem 3.

To see that $C(M) \leq 3$ consider the following: By [7] page 14 (the trace of the modification corresponding to $D(b_i, j)$ on $V_1$) $\cup D^n_1 = D^n_1$ with an $n$-disc $C(b_i, j)$ attached to the boundary of $D^n_1 = V_1$. Actually $C(b_i, j) \cap V_1 = S^{n-1}_i \times D^n_{-i} =$ tubular neighborhood of $S^{n-1}_i$ used to determine the spherical modification.) Thus $D^n_1 \cup$ (trace of the $N$ different spherical modifications on $V_1$) = $D^n_1 \cup C_1$ where $C_1$ is a set of $N$ mutually disjoint $n$-discs $C(b_i, j)$. Repeating the above argument on $V$ using the $N$ inverse modifications to those done on $V_1$ one obtains that $D^n_1 \cup C_1 \cup$ (trace of the $N$ inverse modifications) = $D^n_1 \cup C_1 \cup C_2$ where $C_2$ is a set of $N$ mutually disjoint $n$-discs $C(b_{n-i}, j)$. Further, when performing the inverse modification to the one determined by $D(b_i, j)$ a tubular neighborhood of small enough ‘radius’ can be used so that

$$C(b_{n-i}, j) \cap C(b_i, j) = D^i \times S^{n-i-1} \subset D^i \times S^{n-i-1} \subset V$$

where $D^i \times S^{n-i-1}$ is the set introduced into $V$ by the modification and where $D^i$ is an $i$-disc $C$ interior $D^i$. Changing the ‘radius’ does not effect the modification in any significant way (Cf. [9] page 776). Thus all the discs in $C_2$ can be taken disjoint from $D^n_1$. Finally, performing the $(n-1, -1)$ spherical modification determined by $V_2$ corresponds to attaching an $n$-disc $D^n_2$ to $D^n_1 \cup C_1 \cup C_2$ by identifying the boundary of $D^n_2$ to $V_2$. Thus $M = (D^n_1 \cup C_2) \cup C_1 \cup D^n_2$ where $(D^n_1 \cup C_2)$, $C_1$, and $D^n_2$ each consist of finitely many mutually disjoint $n$-discs.

If $(D_1, D_2, \ldots, D_k)$ is a set of mutually disjoint $n$-discs in a connected $n$-manifold (which $M$ is) then $D_1$ can be joined to $D_2$ by a smooth arc $\alpha$ so that $\alpha \cap \bigcup_{i=1}^k D_i = \text{two points}$, one in boundary $D_1$ and one in boundary $D_2$. A tubular neighborhood $T$ of $\alpha$ can be picked so that $T \cap D_i = \text{an}$ $n-1$ disc in boundary $D_i$ ($i = 1, 2$) and $T$ misses $D_3, \ldots, D_k$. Thus $D_1$ and $D_2$ can be joined to form a set $E_2$ which is contractable in itself. Repeat this construction on $(E_2, D_3, \ldots, D_k)$ starting with $E_2$ and $D_3$ to form $E_3$. Finally one obtains a set $E_k$ which is contractable in itself.
Hence \( M \) can be covered by 3 such contractable sets and if each of them is expanded slightly \( M \) can be covered by their interiors and it follows that \( C(M) \leq 3 \). This completes the proof of theorem 3.

**Remark.** The theorem only asserts the existence of a manifold of a certain type. A manifold satisfying theorem 3 can be constructed in a simple manner as indicated below. The more involved construction given in the proof of the theorem gives a general technique for constructing manifolds with \( C(M) \leq 3 \) as there are few restrictions placed on the spherical modifications involved. For example, by changing the \((0, 0)\) modification one can obtain the 2-torus, the klein bottle or the projective plane.

For \( n = 2m + 1 \) one can obtain a manifold satisfying theorem 3 as follows: Let \( N = \sum_{i=1}^{m} b_i \) and denote by \( M_i, b_i \) copies of \( S^i \times S^{n-i} \). Define \( M \) to be the connected sum of \( M' = \bigcup_{i=1}^{N} M_i \) (i.e. fix a component \( C \) of \( M' \) and connect all other components of \( M' \) to \( C \) by \((0, n-1)\)-modifications. It is not difficult to prove directly that \( M \) satisfies the theorem and is in fact a special case of the construction given in the proof of theorem 3. A similar argument holds for \( n \) even.

In theorem 3 if \( n = 2m \) then \( b_m \) is assumed to be even. This assumption can easily be removed in certain cases. Let \( b_m = 2t+1 \) and suppose there exists an \((m-1, n-m-1)\) spherical modification \( \phi \) on \( S^{n-1} \) which again yields \( S^{n-1} \). Then as in the proof of theorem 3 perform the \( N = \sum_{i=1}^{m} b_i + t \) spherical modifications on \( V_1 \) together with one more; namely \( \phi \), to obtain \( V \). Then performing the \( N \) inverse modifications on \( V \) one obtains \( V_2 = S^{n-1} \). (No inverse modification is needed to ‘cancel’ \( \phi \)). Thus the number of \((m-1, n-m-1)\) type modifications = \( 2t+1 = b_m \). Its easy to see that the rest of the proof goes through as before.

If \( n = 2, 4, 8, 16 \) such spherical modifications as \( \phi \) exist, the trace of \( \phi \) being the real projective plane with two 2-discs removed if \( n = 2 \), the complex projective plane with two 4-discs removed if \( n = 4 \), the quater-nionic projective plane with two 8-discs removed if \( n = 8 \), and the Cayley projective plane with two 16-discs removed if \( n = 16 \). (Cf. [4] page 708). However for \( n = 2 \) the \((0, 0)\) modification is non-orientable but for \( n = 4, 8 \) and 16 one has:

**Corollary 1.** If \( n = 4, 8, 16 \) the restriction that \( b_m \) be even in theorem 3 can be removed.

**Corollary 2.** If \( b_1 \neq 0 \) and \( n \geq 2 \) in theorem 3 then \( C(M) = 3 \).

**Proof.** This follows from [2] page 258 theorem 29.3.

Let \( f \) be a Morse function on an \( n \)-manifold \( M \) (\( f : M \to R \) (reals) with a finite number of critical points all of which are non-degenerate). To each critical point of \( f \) is attached an index \( i \) (an integer \( 0 \leq i \leq n \)) (Cf. [7]
Define $\mu(M)$ to be the minimum number of different indices appearing in $f$ as $f$ ranges over all Morse functions on $M$. It is well known that $C(M) \leq \mu(M) \leq n + 1$ (Cf. [3] or [5]). Further if the number of points with index $i$ for a Morse function $f$ on $M$ is zero then $H_i(M) = 0$ (Cf. [7] page 20). Hence if $b_i > 0$, $i = 0, \cdots, n$ in theorem 3 then the manifold $M$ constructed there has the property that $C(M) \leq 3$ and $\mu(M) = n + 1$. Thus $\mu(M) - C(M) \geq n - 2$ and, in view of corollary 2, $n - 2$ is the maximum difference if $b_1 \neq 0$ and $n \geq 2$.

**Corollary 3.** If $n \geq 2$ then there exists an $n$-manifold $M$ (actually there exists infinitely many non-diffeomorphic such $n$-manifolds) such that $\mu(M) - C(M) = n - 2$ and if $n = 1$ then clearly $\mu(S^1) - C(S^1) = 0$ where $S^1$ is the 1-sphere.

**References**