A NON-CYCLIC ONE-RELATOR GROUP
ALL OF WHOSE FINITE QUOTIENTS ARE CYCLIC

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To Bernhard Hermann Neumann on his 60th birthday
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Let \( G \) be a group on two generators \( a \) and \( b \) subject to the single
defining relation \( a = [a, ab] \):
\[
G = (a, b; a = [a, ab]).
\]
As usual \( [x, y] = x^{-1}y^{-1}xy \) and \( x^y = y^{-1}xy \) if \( x \) and \( y \) are elements of a
group. The object of this note is to show that every finite quotient of \( G \) is
cyclic. This implies that every normal subgroup of \( G \) contains the derived
group \( G' \). But by Magnus’ theory of groups with a single defining relation
\( G' \neq 1 \) ([1], §4.4). So \( G \) is not residually finite. This underlines the fact
that groups with a single defining relation need not be residually finite
(cf. [2]).

In order to prove that \( G \) has the described properties let us put
\[
a_i = b^{-i}a b^i.
\]
Then the normal closure \( N \) of \( a \) in \( G \) is generated by the elements
\( \cdots, a_{-1}, a_0, a_1, \cdots \) subject to the defining relations
\[
a_i = [a_i, a_{i+1}] \quad (i = 0, \pm 1, \cdots).
\]
Thus
\[
a_i^2 = a_{i+1}^{-1}a_i a_{i+1} \quad (i = 0, \pm 1, \cdots).
\]
Now suppose that \( K \) is a normal subgroup of \( G \) of finite index. Put
\[
x = aK, y = a^b K.
\]
We shall show that \( x = 1 \) which implies \( N(\leq G') \leq K \) as desired. For
suppose \( x \neq 1 \). Then \( x \) and \( y \) are of order \( n > 1 \), say. Since \( x^y = x^2 \) we find
\[
x = x^1 = x^n = x^{2n}.
\]
This implies \( x^{2n-1} = 1 \) and \( n \) divides \( 2^n - 1 \). But it is easy to see that the
smallest prime divisor of \( n \) is less than the smallest prime divisor of \( 2^n - 1 \)
(G. Higman [3]). This completes the proof.
References


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