A NOTE ON COMMUTATIVE L-GROUPS

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Introduction

Let $G$ be a commutative lattice ordered group. Theorem 1 gives necessary and sufficient conditions under which $a^\perp$ with $a \in G$ is a maximal $I$-ideal. A wide family of $L$-groups $G$ having the property that the orthogonal complement of each atom is a maximal $L$-ideal is described. Conditionally $\sigma$-complete and hence conditionally complete vector lattices belong to the family. It follows immediately that if $a$ is an atom in a conditionally complete vector lattice then $a^\perp$ is a maximal vector lattice ideal. This theorem has been proved in [7] by Yamamuro. Theorem 2 generalizes another result contained in [7]. Namely we prove that if $M$ is a closed maximal $L$-ideal of an archimedean $L$-group $G$ then there exists an atom $a \in G$ such that $M = a^\perp$.

1. Notations and supplementary results

In a commutative $L$-group $G$ with $a \in G$, we write $G^+$ for the set of positive elements; $(a)$ for the $L$-ideal generated by $a$, i.e. $(a) = \{g \in G : |g| \leq n|a| \text{ for some } n\}$; $(A)$ will denote the $L$-ideal generated by a subset $A$ of $G$. Two elements $g_1, g_2 \in G$ are said to be disjoint (written $g_1 \perp g_2$) if $|g_1| \wedge |g_2| = 0$. We put $a^\perp = \{g \in G : |g| \wedge |a| = 0\}$. It is well known that $a^\perp$ is an $L$-ideal. It follows easily that $(a) \cap a^\perp = \{0\}$. Further if $A$ is a subset of $G$ then $A^\perp$ is defined by $A^\perp = \{a^\perp : a \in A\}$ and $A^\perp\perp$ means $(A^\perp)^\perp$. In case $G = A^\perp \oplus A^\perp\perp$ the following properties of projections $p_1$ and $p_2$ onto $A^\perp$ and $A^\perp\perp$ respectively are easily proved:

(i) if $g \geq 0$ then $p_1(g) \geq 0$ and $p_2(g) \geq 0$,

(ii) $p_i(a+b) = p_i(a)+p_i(b)$ for $i = 1, 2$

and

(iii) $p_i(na) = np_i(a)$, $i = 1, 2$.

To obtain these results it is sufficient to bear in mind that any $L$-group is a distributive lattice and that $g_1 + g_2 = g_1 \vee g_2 + g_1 \wedge g_2$. For other concepts used and not defined we refer to Birkhoff [1].
2. Discrete archimedean elements and maximal $l$-ideals

**Definition 1.** An element $a \in G$ is said to be discrete [6] if the conditions $0 \leq g_1 \leq |a|$, $0 \leq g_2 \leq |a|$, $g_1 \perp g_2$ imply that at least one of the elements $g_1$ and $g_2$ equals zero.

**Definition 2.** A non-zero element $a \in G$ is said to be archimedean if for any $0 \neq g \in G^+$ there exist natural numbers $n_1$ and $n_2$ (depending on $g$) such that $n_1 g \not< |a|$ ($n_1 g$ is not less than $|a|$) and $n_2 |a| \not< g$.

**Remark.** It is quite obvious that an $l$-group $G$ is archimedean if and only if each of its non-zero elements is archimedean.

**Lemma 1.** The following statements are equivalent:
(i) $a \in G$ is discrete,
(ii) if $g_1 \perp g_2$ then at least one of them belongs to $a^\perp$.

**Proof.** Let $a$ be discrete and let $g_1 \perp g_2$. In this case $b_1 = |a| \wedge |g_1|$ and $b_2 = |a| \wedge |g_2|$ are disjoint positive elements dominated by $|a|$. Thus, by definition 1, at least one of them equals zero.

Conversely, suppose that $0 \leq g_1 \leq |a|$, $0 \leq g_2 \leq |a|$ and that $g_1 \perp g_2$. According to (ii), we may assume that e.g. $g_1 \in a^\perp$, i.e. $g_1 \wedge |a| = 0$. But $g_1 \wedge |a| = g_1$ since $g_1 \leq |a|$. Thus $g_1 = 0$.

**Lemma 2.** If $a \in G$ is discrete then the $l$-ideal $(a)$ generated by $a$ is totally ordered.

**Proof.** If $g \in (a)$ then $g^+$ and $g^-$ belong also to $(a)$. But $g^+ \perp g^-$ and so, by lemma 1, at least one of the elements $g^+$ and $g^-$ belongs to $a^\perp$. If e.g. $g^- \in a^\perp$, then $g^- \in (a) \cap a^\perp$ and hence $g^- = 0$. Thus, in this case $g = g^+ - g^- = g^+ \geq 0$.

**Lemma 3.** If $a \in G$ is a discrete archimedean element then $(a)$ is generated by any of its non-zero elements.

**Proof.** Let $g \in (a)$ and $g \neq 0$. Since $a$ is archimedean and $|g| > 0$, there exists $n$ such that $n|g| \not< |a|$. Since $n|g| \in (a)$ and $(a)$ is totally ordered, by lemma 2, $|a| \leq n|g|$. So $(a) \subseteq (g) \subseteq (a)$ and thus $(g) = (a)$.

**Lemma 4.** If $a \in G$ is archimedean and discrete then

\[ G = (a) \oplus a^\perp. \]

**Proof.** Since $a$ is archimedean, for any $g \in G^+$ there exists $n$ such that $n|a| \not< g$. Consider the elements $b_1 = (n|a| - g)^+$ and $b_2 = (n|a| - g)^-$. Since $n|a| \not< g$, it follows that $b_1 > 0$. On the other hand $b_1 \leq n|a|$ and hence $b_1 \in (a)$. Now $b_1 \in (a)$ and $b_1 \neq 0$ imply that $b_1 \notin a^\perp$. Taking into account that $b_2 \perp b_1$ and that $a$ is discrete, by lemma 1, we infer that $b_2 \in a^\perp$. Thus
For an arbitrary \( g \in G \) we have \( g = g^+ - g^- \) with \( g^+, g^- \in (a) \oplus a^\perp \). Thus \( g \in (a) \oplus a^\perp \) and so \((a) \oplus a^\perp = G\).

**Theorem 1.** For an element \( a \) belonging to a commutative \( l \)-group \( G \) the following statements are equivalent:

(i) \( a \) is archimedean and discrete,

(ii) \( a^\perp \) is a maximal \( l \)-ideal.

**Proof of (i) \( \Rightarrow \) (ii).** Since \( a^\perp \) is maximal and thus a proper \( l \)-ideal, it follows immediately that \( a \neq 0 \). Assume that \( g_1 \perp g_2 \), \( 0 < g_1 \leq |a| \) and \( 0 \leq g_2 \leq |a| \).

In this case the \( l \)-ideal \( J = (g_2, a^\perp) \) generated by \( g_2 \) and \( a^\perp \) is proper because \( g_1 \notin J \). Since \( a^\perp \) is maximal and \( a^\perp \subseteq J \), it follows that \( J = a^\perp \). Consequently, \( g_2 \in a^\perp \). Hence \( g_2 \in (a) \cap a^\perp \) and so \( g_2 = 0 \). Thus \( a \) is discrete whenever \( a^\perp \) is maximal.

Let us assume now that there exists an element \( 0 < g \in G^+ \) such that \( ng < |a| \) for each natural \( n \). It is easy to see that the ideal \( J = (g, a^\perp) \) generated by \( g \) and \( a^\perp \) is a proper ideal \((a \notin J)\) properly containing \( a^\perp \) \((g \in J, \text{ but } g \notin a^\perp)\). This is impossible, since \( a^\perp \) is maximal.

Finally, suppose that there exists \( g \in G^+ \) such that \( n|a| < g \) for all natural \( n \). In this case again we obtain a contradiction because the ideal \((a, a^\perp)\) generated by \( a \) and \( a^\perp \) is a proper ideal properly containing \( a^\perp \). Hence \( a \) is archimedean whenever \( a^\perp \) is maximal.

**3. Applications**

**Definition 3.** A commutative \( l \)-group \( G \) is said to be Stone if \( G = g^\perp \oplus g^{\perp\perp} \) for any \( g \in G \).

**Definition 4.** An element \( a \in G \) is said to be an atom \([7]\) if the conditions: \(|a| = g_1 + g_2, \ g_1 \perp g_2, \ g_1, g_2 \in G^+ \) imply that one of elements \( g_1, g_2 \) equals zero.
REMARK (i). Observe that the element 0 satisfies both the definitions of a discrete element and of an atom – this seems unnecessary, but we do not wish to cause confusion by deviating from the definitions in [6] and [7].

REMARK (ii). Comparing definitions 1 and 4 we conclude that every discrete element \( a \in G \) is an atom. The converse is in general not true (see example 1 in the last part of the paper). Nevertheless if \( G \) is Stone then the following holds:

**Lemma 5.** An element \( a \) of a Stone \( l \)-group \( G \) is an atom if and only if \( a \) is discrete.

**Proof.** According to the preceding remark it suffices to prove that if \( a \) is atomic and \( G \) is Stone then \( a \) is discrete. Since \( a \) is discrete whenever \( |a| \) is discrete, we may restrict ourselves to the case when \( a > 0 \).

Suppose that \( g_1, g_2 \in G^+, g_1 \perp g_2, g_1 > 0 \) and both are dominated by an atomic element \( a \). \( G \) is Stone, and so, by definition 3, \( G = g_1^\perp \oplus g_2^\perp \). Let \( p_1 \) and \( p_2 \) denote the projections on \( g_1^\perp \) and \( g_2^\perp \) respectively. We have then \( a = p_1(a) + p_2(a) \) with \( p_1(a) \perp p_2(a) \) and since \( a > 0 \), \( p_1(a), p_2(a) \in G^+ \). Thus definition 4 implies that either \( p_1(a) = 0 \) or \( p_2(a) = 0 \). But \( 0 < g_1 \leq a \) and thus, by the properties of projections, \( 0 < g_1 = p_1(g_1) \leq p_1(a) \). Therefore \( p_2(a) = 0 \). On the other hand in view of \( g_2 \perp g_1 \) we obtain \( 0 \leq g_2 = p_2(g_2) \leq p_2(a) = 0 \). So \( g_2 = 0 \). Consequently, \( a \) is a discrete element as required.

As a consequence of lemma 5 and theorem 1 we obtain

**Theorem 2.** If \( a \) is a non-zero archimedean atom of a Stone \( l \)-group \( G \) then \( a^\perp \) is a maximal \( l \)-ideal of \( G \).

**Theorem 3.** Every \( \sigma \)-complete (and a fortiori every complete) \( l \)-group \( G \) is archimedean Stone \( l \)-group.

**Proof.** A direct proof of Theorem 3 will be given soon in [5]. It can also be easily deduced from known results.

Combining theorems 2 and 3 we obtain

**Corollary.** If \( a \) is a non-zero atom of a complete vector lattice \( E \) then \( a^\perp \) is a maximal \( l \)-ideal.

This proposition has been proved by S. Yamamuro in [7]. Lemma 3 of the same paper states that if \( M \) is a closed maximal ideal of a complete vector lattice \( E \), then there exists an atomic element \( a \in E \) such that \( M = a^\perp \). This statement may be essentially generalised. Namely we are able to prove:

**Theorem 4.** If \( M \) is a closed maximal \( l \)-ideal of an archimedean \( l \)-group \( G \) then there exists an atom \( a \in G \) such that \( M = a^\perp \).

**Proof.** The fact that \( M \) is closed \( l \)-ideal in an archimedean \( l \)-group implies, by Johnson and Kist [3] (see also Conrad and McAllister [2]) that \( M = M^\perp \).
Thus $M^\perp \neq \{0\}$ and there is an $a > 0$ in $M^\perp$. For this $a$ we have $a^\perp \supseteq M^\perp = M$ with $a \not\in a^\perp$. Thus the maximality of $M$ implies $a^\perp = M$. So, by Theorem 1 $a$ is discrete and hence $a$ is an atom.

Repeating the reason from [7], we obtain

**Corollary.** If $G$ is an archimedean Stone $l$-group then $G$ is atomic (the set of atoms is dense in $G$) if and only if the intersection of all closed maximal $l$-ideals of $G$ equals zero, and $G$ is non-atomic (there exist no atoms in $G$) if and only if there exist no closed maximal $l$-ideals in $G$.

4. **Examples**

1. Let $E = C[0, 1]$. The function $a \in E$:

$$a(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq \frac{1}{2}, \\ t - \frac{1}{2} & \text{for } \frac{1}{2} < t \leq 1 \end{cases}$$

is an atom but it is not a discrete element. Thus, according to theorem 1, $a^\perp$ is not maximal. Since $C[0, 1]$ is archimedean, theorem 2 implies that $C[0, 1]$ is not Stone.

2. Consider $R^2$ ‘lexicographically’ ordered, e.g. $(x, y) \geq 0$ iff (i) $x > 0$ or (ii) $x = 0$, $y \geq 0$. This space is totally ordered and hence every element $a \in R^2$ is an atom. On the other hand for any $0 \neq a \in R^2$ we have $a^\perp = \{0\}$ and thus for no atom $a$ of $R^2$ is $a^\perp$ maximal. This is so since no $a \in R^2$ is archimedean. The space in question is a Stone (non-archimedean) $l$-group. The ideal $M = \{(x, y) \in R^2 : x = 0\}$ is a maximal closed $l$-ideal, but as it was mentioned there exists no atom $a \in R^2$ such that $M = a^\perp$. This example shows that the condition that $G$ is archimedean is essential in theorem 4.

3. Let $E = C[0, 1] \times R \times R^2$ with $R^2$ ordered as in example 2. An element $(x, y, z) \in E$ (with $x \in C[0, 1]$, $y \in R$ and $z \in R^2$) is said to be positive iff $x \geq 0$, $y \geq 0$, $z \geq 0$. $E$ is non-Stone and non-archimedean vector lattice. Nevertheless the element $(0, a, 0)$ with $a > 0$ is an atom of $E$ and $(0, a, 0)^\perp = \{(x, 0, z) : x \in C[0, 1], z \in R^2\}$ is a maximal $l$-ideal of $E$. This is so because $(0, a, 0)$ is a discrete archimedean element of $E$.

4. Let $S$ be the $l$-group (in fact vector lattice) of all equivalence classes of simple functions defined on a totally $\sigma$-finite measure space $(X, \mathcal{F}, \mu)$. Then results of Masterson [4] pp. 469–470 imply that $S$ is an archimedean Stone $l$-group which is not $\sigma$-complete.

This example shows that theorem 2 is an essential generalization of theorem 2 in [7].
References


