A NOTE ON COMMUTATIVE $l$-GROUPS

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Introduction

Let $G$ be a commutative lattice ordered group. Theorem 1 gives necessary and sufficient conditions under which $a^\perp$ with $a \in G$ is a maximal $l$-ideal. A wide family of $l$-groups $G$ having the property that the orthogonal complement of each atom is a maximal $l$-ideal is described. Conditionally $\sigma$-complete and hence conditionally complete vector lattices belong to the family. It follows immediately that if $a$ is an atom in a conditionally complete vector lattice then $a^\perp$ is a maximal vector lattice ideal. This theorem has been proved in [7] by Yamamuro. Theorem 2 generalizes another result contained in [7]. Namely we prove that if $M$ is a closed maximal $l$-ideal of an archimedean $l$-group $G$ then there exists an atom $a \in G$ such that $M = a^\perp$.

1. Notations and supplementary results

In a commutative $l$-group $G$ with $a \in G$, we write $G^+$ for the set of positive elements; $(a)$ for the $l$-ideal generated by $a$, i.e. $(a) = \{ g \in G : |g| \leq n|a| \text { for some } n \}$; $(A)$ will denote the $l$-ideal generated by a subset $A$ of $G$. Two elements $g_1, g_2 \in G$ are said to be disjoint (written $g_1 \perp g_2$) if $|g_1| \wedge |g_2| = 0$. We put $a^\perp = \{ g \in G : |g| \wedge |a| = 0 \}$. It is well known that $a^\perp$ is an $l$-ideal. It follows easily that $(a) \cap a^\perp = \{ 0 \}$. Further if $A$ is a subset of $G$ then $A^\perp$ is defined by $A^\perp = \cap \{ a^\perp : a \in A \}$ and $A^{\perp\perp}$ means $(A^\perp)^\perp$. In case $G = A^{\perp} \oplus A^{\perp\perp}$ the following properties of projections $p_1$ and $p_2$ onto $A^\perp$ and $A^{\perp\perp}$ respectively are easily proved:

(i) if $g \geq 0$ then $p_1(g) \geq 0$ and $p_2(g) \geq 0$,

(ii) $p_i(a+b) = p_i(a)+p_i(b)$ for $i = 1, 2$

and

(iii) $p_i(na) = np_i(a)$, $i = 1, 2$.

To obtain these results it is sufficient to bear in mind that any $l$-group is a distributive lattice and that $g_1 + g_2 = g_1 \vee g_2 + g_1 \wedge g_2$. For other concepts used and not defined we refer to Birkhoff [1].
2. Discrete archimedean elements and maximal \( l \)-ideals

**Definition 1.** An element \( a \in G \) is said to be discrete [6] if the conditions \( 0 \leq g_1 \leq |a|, 0 \leq g_2 \leq |a|, g_1 \perp g_2 \) imply that at least one of the elements \( g_1 \) and \( g_2 \) equals zero.

**Definition 2.** A non-zero element \( a \in G \) is said to be archimedean if for any \( 0 \neq g \in G^+ \) there exist natural numbers \( n_1 \) and \( n_2 \) (depending on \( g \)) such that \( n_1 g < |a| \) (\( n_1 g \) is not less than \( |a| \)) and \( n_2 |a| < g \).

**Remark.** It is quite obvious that an \( l \)-group \( G \) is archimedean if and only if each of its non-zero elements is archimedean.

**Lemma 1.** The following statements are equivalent:

(i) \( a \in G \) is discrete,

(ii) if \( g_1 \perp g_2 \) then at least one of them belongs to \( a^\perp \).

**Proof.** Let \( a \) be discrete and let \( g_1 \perp g_2 \). In this case \( b_1 = |a| \wedge |g_1| \) and \( b_2 = |a| \wedge |g_2| \) are disjoint positive elements dominated by \( |a| \). Thus, by definition 1, at least one of them equals zero.

Conversely, suppose that \( 0 \leq g_1 \leq |a|, 0 \leq g_2 \leq |a| \) and that \( g_1 \perp g_2 \). According to (ii), we may assume that e.g. \( g_1 \in a^\perp \), i.e. \( g_1 \wedge |a| = 0 \). But \( g_1 \wedge |a| = g_1 \) since \( g_1 \leq |a| \). Thus \( g_1 = 0 \).

**Lemma 2.** If \( a \in G \) is discrete then the \( l \)-ideal \((a)\) generated by \( a \) is totally ordered.

**Proof.** If \( g \in (a) \) then \( g^+ \) and \( g^- \) belong also to \((a)\). But \( g^+ \perp g^- \) and so, by lemma 1, at least one of the elements \( g^+ \) and \( g^- \) belongs to \( a^\perp \). If e.g. \( g^- \in a^\perp \), then \( g^- \in (a) \cap a^\perp \) and hence \( g^- = 0 \). Thus, in this case \( g = g^+ - g^- = g^+ \geq 0 \).

**Lemma 3.** If \( a \in G \) is a discrete archimedean element then \((a)\) is generated by any of its non-zero elements.

**Proof.** Let \( g \in (a) \) and \( g \neq 0 \). Since \( a \) is archimedean and \( |g| > 0 \), there exists \( n \) such that \( n|g| < |a| \). Since \( n|g| \in (a) \) and \((a)\) is totally ordered, by lemma 2, \(|a| \leq n|g| \). So \((a) \subseteq (g) \subseteq (a)\) and thus \((g) = (a)\).

**Lemma 4.** If \( a \in G \) is archimedean and discrete then

\[ G = (a) \oplus a^\perp. \]

**Proof.** Since \( a \) is archimedean, for any \( g \in G^+ \) there exists \( n \) such that \( n|a| \leq g \). Consider the elements \( b_1 = (n|a| - g)^+ \) and \( b_2 = (n|a| - g)^- \). Since \( n|a| \leq g \), it follows that \( b_1 > 0 \). On the other hand \( b_1 \leq n|a| \) and hence \( b_1 \in (a) \).

Now \( b_1 \in (a) \) and \( b_1 \neq 0 \) imply that \( b_1 \notin a^\perp \). Taking into account that \( b_2 \perp b_1 \) and that \( a \) is discrete, by lemma 1, we infer that \( b_2 \in a^\perp \). Thus
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\[ n|a|g = b_1 - b_2 \in (a) \oplus a^\perp. \]

But
\[ n|a|\in (a) \subseteq (a) \oplus a^\perp, \]
and so
\[ g \in (a) \oplus a^\perp. \]

For an arbitrary $g \in G$ we have $g = g^+ - g^-$ with $g^+, g^- \in (a) \oplus a^\perp$. Thus $g \in (a) \oplus a^\perp$ and so $(a) \oplus a^\perp = G$.

**Theorem 1.** For an element $a$ belonging to a commutative $l$-group $G$ the following statements are equivalent:

(i) $a$ is archimedean and discrete,

(ii) $a^\perp$ is a maximal $l$-ideal.

**Proof of (i) $\Rightarrow$ (ii).** $a \neq 0$, by definition 2, so $a \notin a^\perp$ and hence $a^\perp$ is a proper $l$-ideal. Suppose that $M$ is an $l$-ideal of $G$ properly containing $a^\perp$. Let $b \in M \setminus a^\perp$. Then $b_1 = |b| \in M \setminus a^\perp$. Since $b_1 \notin a^\perp$, $c = b_1 \land |a| > 0$. So, $0 < c \leq |a|$ and thus, by lemma 3 and 4,
\[ G = (a) \oplus a^\perp = (c) \oplus a^\perp \subseteq M. \]

Consequently, $M = G$ and therefore $a^\perp$ is maximal.

**Proof of (ii) $\Rightarrow$ (i).** Since $a^\perp$ is maximal and thus a proper ideal, it follows immediately that $a \neq 0$. Assume that $g_1 \perp g_2$, $0 < g_1 \leq |a|$ and $0 \leq g_2 \leq |a|$. In this case the $l$-ideal $J = (g_2, a^\perp)$ generated by $g_2$ and $a^\perp$ is proper because $g_1 \notin J$. Since $a^\perp$ is maximal and $a^\perp \subseteq J$, it follows that $J = a^\perp$. Consequently, $g_2 \in a^\perp$. Hence $g_2 \in (a) \cap a^\perp$ and so $g_2 = 0$. Thus $a$ is discrete whenever $a^\perp$ is maximal.

Let us assume now that there exists an element $0 < g \in G^+$ such that $ng < |a|$ for each natural $n$. It is easy to see that the ideal $J = (g, a^\perp)$ generated by $g$ and $a^\perp$ is a proper ideal ($a \notin J$) properly containing $a^\perp$ ($g \in J$, but $g \notin a^\perp$). This is impossible, since $a^\perp$ is maximal.

Finally, suppose that there exists $g \in G^+$ such that $n|a| < g$ for all natural $n$. In this case again we obtain a contradiction because the ideal $(a, a^\perp)$ generated by $a$ and $a^\perp$ is a proper ideal properly containing $a^\perp$. Hence $a$ is archimedean whenever $a^\perp$ is maximal.

3. Applications

**Definition 3.** A commutative $l$-group $G$ is said to be Stone if $G = g^\perp \oplus g^{\perp\perp}$ for any $g \in G$.

**Definition 4.** An element $a \in G$ is said to be an atom [7] if the conditions:
\[ |a| = g_1 + g_2, \quad g_1 \perp g_2, \quad g_1, g_2 \in G^+ \]
imply that one of elements $g_1, g_2$ equals zero.
REMARK (i). Observe that the element 0 satisfies both the definitions of a discrete element and of an atom – this seems unnecessary, but we do not wish to cause confusion by deviating from the definitions in [6] and [7].

REMARK (ii). Comparing definitions 1 and 4 we conclude that every discrete element $a \in G$ is an atom. The converse is in general not true (see example 1 in the last part of the paper). Nevertheless if $G$ is Stone then the following holds:

LEMMA 5. An element $a$ of a Stone $l$-group $G$ is an atom if and only if $a$ is discrete.

PROOF. According to the preceding remark it suffices to prove that if $a$ is atomic and $G$ is Stone then $a$ is discrete. Since $a$ is discrete whenever $|a|$ is discrete, we may restrict ourselves to the case when $a > 0$.

Suppose that $g_1, g_2 \in G^+$, $g_1 \perp g_2$, $g_1 > 0$ and both are dominated by an atomic element $a$. $G$ is Stone, and so, by definition 3, $G = g_1^+ \oplus g_2^+$. Let $p_1$ and $p_2$ denote the projections on $g_1^+$ and $g_2^+$ respectively. We have then $a = p_1(a) + p_2(a)$ with $p_1(a) \perp p_2(a)$ and since $a > 0$, $p_1(a), p_2(a) \in G^+$. Thus definition 4 implies that either $p_1(a) = 0$ or $p_2(a) = 0$. But $0 < g_1 \leq a$ and thus, by the properties of projections, $0 < g_1 = p_1(g_1) \leq p_1(a)$. Therefore $p_2(a) = 0$. On the other hand in view of $g_2 \perp g_1$ we obtain $0 \leq g_2 = p_2(g_2) \leq p_2(a) = 0$. So $g_2 = 0$. Consequently, $a$ is a discrete element as required.

As a consequence of lemma 5 and theorem 1 we obtain

THEOREM 2. If $a$ is a non-zero archimedean atom of a Stone $l$-group $G$ then $a^\perp$ is a maximal $l$-ideal of $G$.

THEOREM 3. Every $\sigma$-complete (and a fortiori every complete) $l$-group $G$ is archimedean Stone $l$-group.

PROOF. A direct proof of Theorem 3 will be given soon in [5]. It can also be easily deduced from known results.

Combining theorems 2 and 3 we obtain

COROLLARY. If $a$ is a non-zero atom of a complete vector lattice $E$ then $a^\perp$ is a maximal $l$-ideal.

This proposition has been proved by S. Yamamuro in [7]. Lemma 3 of the same paper states that if $M$ is a closed maximal ideal of a complete vector lattice $E$, then there exists an atomic element $a \in E$ such that $M = a^\perp$. This statement may be essentially generalised. Namely we are able to prove:

THEOREM 4. If $M$ is a closed maximal $l$-ideal of an archimedean $l$-group $G$ then there exists an atom $a \in G$ such that $M = a^\perp$.

PROOF. The fact that $M$ is closed $l$-ideal in an archimedean $l$-group implies, by Johnson and Kist [3] (see also Conrad and McAllister [2]) that $M = M^\perp$. 

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Thus \( M^\perp \neq \{0\} \) and there is an \( a > 0 \) in \( M^\perp \). For this \( a \) we have \( a^\perp \supseteq M^{\perp \perp} = M \) with \( a \notin a^\perp \). Thus the maximality of \( M \) implies \( a^\perp = M \). So, by Theorem 1 \( a \) is discrete and hence \( a \) is an atom. Repeating the reason from [7], we obtain

**Corollary.** If \( G \) is an archimedean Stone \( l \)-group then \( G \) is atomic (the set of atoms is dense in \( G \)) if and only if the intersection of all closed maximal \( l \)-ideals of \( G \) equals zero, and \( G \) is non-atomic (there exist no atoms in \( G \)) if and only if there exist no closed maximal \( l \)-ideals in \( G \).

### 4. Examples

1. Let \( E = C[0, 1] \). The function \( a \in E \):

\[
a(t) = \begin{cases} 
0 & \text{for } 0 \leq t \leq \frac{1}{2}, \\
\frac{t - \frac{1}{2}}{2} & \text{for } \frac{1}{2} < t \leq 1
\end{cases}
\]

is an atom but it is not a discrete element. Thus, according to theorem 1, \( a^\perp \) is not maximal. Since \( C[0, 1] \) is archimedean, theorem 2 implies that \( C[0, 1] \) is not Stone.

2. Consider \( R^2 \) ‘lexicographically’ ordered, e.g. \( (x, y) \geq 0 \) iff (i) \( x > 0 \) or (ii) \( x = 0, y \geq 0 \). This space is totally ordered and hence every element \( a \in R^2 \) is an atom. On the other hand for any \( 0 \neq (a, 0) \in R^2 \) we have \( a^\perp = \{0\} \) and thus for no atom \( a \) of \( R^2 \) is \( a^\perp \) maximal. This is so since no \( a \in R^2 \) is archimedean. The space in question is a Stone (non-archimedean) \( l \)-group. The ideal \( M = \{(x, y) \in R^2 : x = 0 \} \) is a maximal \( l \)-ideal, but as it was mentioned there exists no atom \( a \in R^2 \) such that \( M = a^\perp \). This example shows that the condition that \( G \) is archimedean is essential in theorem 4.

3. Let \( E = C[0, 1] \times R \times R^2 \) with \( R^2 \) ordered as in example 2. An element \( (x, y, z) \in E \) (with \( x \in C[0, 1], y \in R \) and \( z \in R^2 \)) is said to be positive iff \( x \geq 0, y \geq 0, z \geq 0 \). \( E \) is non-Stone and non-archimedean vector lattice. Nevertheless the element \( (0, a, 0) \) with \( a > 0 \) is an atom of \( E \) and \( (0, a, 0)^\perp = \{(x, 0, z) : x \in C[0, 1], z \in R^2 \} \) is a maximal \( l \)-ideal of \( E \). This is so because \( (0, a, 0) \) is a discrete archimedean element of \( E \).

4. Let \( S \) be the \( l \)-group (in fact vector lattice) of all equivalence classes of simple functions defined on a totally \( \sigma \)-finite measure space \( (X, \mathcal{F}, \mu) \). Then results of Masterson [4] pp. 469–470 imply that \( S \) is an archimedean Stone \( l \)-group which is not \( \sigma \)-complete.

This example shows that theorem 2 is an essential generalization of theorem 2 in [7].
References