ON THE MATHIEU GROUP $M_{23}$

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1. Introduction

Until 1965, when Janko [7] established the existence of his finite simple group $J_1$, the five Mathieu groups were the only known examples of isolated finite simple groups. In 1951, R. G. Stanton [10] showed that $M_{12}$ and $M_{24}$ were determined uniquely by their order. Recent characterizations of $M_{22}$ and $M_{23}$ by Janko [8], $M_{22}$ by D. Held [6], and $M_{11}$ by W. J. Wong [12], have facilitated the unique determination of the three remaining Mathieu groups by their orders. D. Parrott [9] has so characterized $M_{22}$ and $M_{11}$, while this paper is an outline of the characterization of $M_{23}$ in terms of its order.

MAIN THEOREM. Let $G$ be a non-abelian simple group of order 10,200,960. Then $G$ is isomorphic to $M_{23}$.

2. Some known results

1. The results used in the proof of the main theorem were obtained by R. Brauer [1], [2], [3], H. F. Tuan [4] and applied by R. G. Stanton [10], D. Parrott [9] and S. K. Wong [11]. Some of the important theorems are given here without proof.

2. If $G$ is a group of order $|G|$ containing $k$ classes $K_1, \ldots, K_k$ of conjugate elements, then there exists exactly $k$ distinct irreducible characters $\zeta_1(\eta), \ldots, \zeta_k(\eta)$ where $\eta$ denotes a variable element of $G$. Let $p$ be a prime which divides $|G|$, then the $k$ characters are distributed into a certain number of $p$-blocks $B_1(p), B_2(p), \ldots$. The principal $p$-block $B_1(p)$ is always taken as the block containing the 1-character $\zeta_1(\eta) = 1$ for all $\eta \in G$. Suppose $p^2 \nmid |G|$; if for all characters $\zeta_\mu$ of $B_\mu(p)$ the degrees $z_\mu$ of $\zeta_\mu$ is divisible by $p^\sigma$ while at least one of the degrees $z_\mu$ is not divisible by $p^{\sigma+1}$ then $B_\mu(p)$ is a block of defect $(\gamma-\alpha)$, or type $\alpha$. In particular if $p \nmid |G|$ a $p$-block $B_\sigma(p)$ is of defect 0 (highest type) or of defect 1 (lowest type).

An element $g$ is $p$-regular if its order is prime to $p$, otherwise $g$ is called $p$-singular.
3. We assume in this section that \( p \nmid |G| \). Let \( G_p \) be a Sylow \( p \)-subgroup of \( G \). Then \( C_G(G_p) = G_p \times V_p \). If \( V_p \) has \( l \) conjugate classes in the group \( N_G(G_p) \) then \( G \) has \( l \) blocks of defect 1. Let \( t \) denote the number of conjugate classes of elements of order \( p \) in \( G \). To each of the \( l \) \( p \)-blocks \( B_r(p) \) of defect 1 there corresponds a certain multiple \( t_o \) of \( t \), where \( t_o | p - 1 \), such that \( B_o(p) \) has \((p - 1)/t_o \) characters \( \zeta_\mu \) which are \( p \)-conjugate only to themselves and one exceptional family of \( t_o \) \( p \)-conjugate characters.

**Theorem 2.1** ([1]. Theorem 11). For the block \( B_1(p) \), we have \( t_1 = t \). The degrees \( z_\mu \) of the characters \( \zeta_\mu \) of \( B_1(p) \) satisfy:

\[
(2.1) \quad z_\mu \equiv \delta_\mu \equiv \pm 1 \pmod{p}, \quad 1 \leq \mu \leq \omega = (p - 1)/t
\]

\[
(2.2) \quad tz_{\omega + 1} \equiv \delta_{\omega + 1} \equiv \pm 1 \pmod{p},
\]

where \( z_{\omega + 1} \) is the degree of a representative of the exceptional family.

\[
(2.3) \quad \sum_{\mu = 1}^{\omega + 1} \delta_\mu z_\mu = 0 \quad (\delta_1 = z_1 = 1).
\]

Moreover, for \( p \)-singular elements \( P \) of \( G \) we have

\[
\zeta_\mu(P) = \delta_\mu \quad (1 \leq \mu \leq \omega).
\]

**Corollary 1.** Let \( G \) be a group of order \( pq^bq^* \) where \( p \) and \( q \) are distinct primes, \( b \) and \( q^* \) positive integers and \((pq, q^*) = 1\). Suppose that \( G \) has an element of order \( pq \), then \( q^* \) cannot divide the degree of any irreducible character \( \zeta_\mu \) in \( B_1(p) \).

We shall say a character \( \zeta \) of \( B_1(p) \) is of type 0 for the prime \( p \) if \( \zeta(1) \equiv 1 \pmod{p} \) or if \( \zeta \) belongs to the exceptional family of \( B_1(p) \) and \( \zeta(1) \equiv -(p - 1)/t \pmod{p} \); \( \zeta \) is of type 1 if \( \zeta(1) \equiv -1 \pmod{p} \) or if \( \chi \) belongs to the exceptional family and \( \zeta(1) \equiv +(p - 1)/t \pmod{p} \).

**Theorem 2.2** ([10] Lemma 6). Let \( G \) be a group of order \( |G| \). Assume \( p \) and \( p' \) are distinct primes which divide \( |G| \) to the first power only and that \( G \) has no elements of order \( pp' \). Let \( a_{ij} \) be the number of characters in \( B_1(p) \cap B_1(p') \) which are of type \( i \) for \( p \) and type \( j \) for \( p' \), the indices \( i \) and \( j \) being 0 or 1 as described above. Then

\[
a_{00} + a_{11} = a_{01} + a_{10}.
\]

It is clear that a character \( \zeta \) in \( B_1(p) \cap B_1(p') \) cannot be exceptional for both primes \( p \) and \( p' \).

**Theorem 2.3** ([4], Lemma 1). Let \( G \) be a finite group which is identical with its commutator group \( G' \), and assume that the principal \( p \)-block \( B_1(p) \) contains an irreducible faithful character \( \zeta \) of degree \( z < 2p \). Then the order of the centralizer \( C_G(G_p) \) of a Sylow \( p \)-subgroup \( G_p \) of \( G \) is a power of \( p \).
3. The Sylow 23-normalizer of $G$

We assume from now on, that $G$ is an non-abelian finite simple group of order $10,200,960 = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$.

Let $S_{23}$ be a Sylow 23-subgroup of $G$ and let $n_{23} = |G : N_G(S_{23})|$. Then $n_{23}$ has the following possibilities: (1) $2^7 \cdot 3^2 \cdot 5 \cdot 7$, (2) $2^6 \cdot 5 \cdot 11$, (3) $2^6 \cdot 3$, (4) $2^4 \cdot 3 \cdot 5 \cdot 7$, (5) $2^3 \cdot 3^2 \cdot 7 \cdot 11$, (6) $2^3 \cdot 3$, (7) $2 \cdot 3^2 \cdot 5 \cdot 11$, (8) $2 \cdot 5 \cdot 7$, (9) $3 \cdot 7 \cdot 11$.

We know that $G$ has either 1, 2, or 11 classes of elements of order 23 according as $t$ for prime 23 (written as $t_{23}$) is 1, 2, or 11. Using equations (2.1), (2.2), and (2.3), and Theorem 2.3 $t_{23} = 11$ is ruled out, consequently $|N_G(S_{23})/(C_G(S_{23}))| = 11$ or 22. Hence cases (2), (5), (7), and (9) above, for $n_{23}$ are not possible. The impossibility of cases (4) and (8) follows almost as quickly, because otherwise $G$ has no elements of order $5 \cdot 23$, $7 \cdot 23$, or $11 \cdot 23$ thus facilitating the use of Stanton’s block intersection theorem (Theorem 2.2). Suppose $n_{23} = 2^3 \cdot 3$, case (6). Then $|N_G(S_{23})| = 2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$. $G$ then contains elements of order $2 \cdot 23$, $3 \cdot 23$, $5 \cdot 23$, and $7 \cdot 23$. From this it follows that 528 is the only possible degree of a nonexceptional character and 264 the only possible exceptional degree. But both of these degrees are even, and for $(2 \cdot 3)$ to be satisfied $B_1(23)$ must contain a character of odd degree. Case (3) is ruled out similarly. Hence we have proved

Lemma 3.1. The Sylow 23-normalizer $N_G(S_{23})$ is a Frobenius group of order $23 \cdot 11$.

Corollary 3.1. The principal 23-block $B_1(23)$ is the only 23-block of defect 1, and consists of 11 non-exceptional characters and a family of 2 exceptional characters. All other characters of $G$ have degrees divisible by 23.

4. The Sylow 11-normalizer of $G$

Let $S_{11}$ be a Sylow 11-subgroup of $G$ and $n_{11} = |G : N_G(S_{11})|$. Lemma 3.1 reduces the possible values for $n_{11}$ to the following: (1) $3^2 \cdot 5 \cdot 23$, (2) $2 \cdot 3 \cdot 5 \cdot 7 \cdot 23$, (3) $2^2 \cdot 3 \cdot 23$, (4) $2^2 \cdot 7 \cdot 23$, (5) $2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 23$, (6) $2^4 \cdot 3^2 \cdot 23$, (7) $2^5 \cdot 3 \cdot 7 \cdot 23$, (8) $2^6 \cdot 5 \cdot 23$, (9) $2^7 \cdot 3^2 \cdot 7 \cdot 23$.

Using the same methods as for the prime 23, one proves quickly that $t_{11} \neq 5$ and so $|N_G(S_{11})/(C_G(S_{11}))| = 5$ or 10. This in turn eliminates cases (1), (2), (5) and (8), from the above list for $n_{11}$.

Suppose $|N_G(S_{11})| = 2^5 \cdot 3 \cdot 5 \cdot 7 \cdot 11$, case (3). Then $|C_G(S_{11})| = 2^5 \cdot 3 \cdot 7 \cdot 11$ or $2^4 \cdot 3 \cdot 7 \cdot 11$.

If $|C_G(S_{11})| = 2^5 \cdot 3 \cdot 7 \cdot 11$, then $t_{11} = 2$ and $B_1(11)$ consists of 5 nonexceptional characters $\chi_1, \chi_2, \chi_3, \chi_4$ and $\chi_5$ and a family of 2 exceptional charac-
ters with representative \( \chi_6 \). Since \( G \) has elements of order \( 2 \cdot 11, 3 \cdot 11 \) and \( 7 \cdot 11 \), the possible degrees for the non-exceptional characters are

\[
\begin{array}{ccc}
1, & 23, & 276 \\
230, & 736, & 2760 \\
\end{array}
\equiv +1 \pmod{11}
\equiv -1 \pmod{11}
\]

while the possible degrees for \( \chi_6 \) are

\[
\begin{array}{ccc}
115, & 368, & 1380 \\
138, & 160, & 1920 \\
\end{array}
\equiv +5 \pmod{11}
\equiv -5 \pmod{11}
\]

Then the degrees in \( B_1(23) \cap B_1(11) \) are 1 and 160, and so \( \chi_6(1) = 160 \). Applying theorem 2.2 to \( B_1(11) \cap B_1(5) \) we see that only degrees 1 and 736 lie in this intersection. Let \( \chi_2(1) = 736 \). Substitute the values 1, 160 and 736 in the degree equation (2.3). Then

\[
\delta_3 z_3 + \delta_4 z_4 + \delta_5 z_5 = -(1-736+160) = 575
\]

and so \( z_3 = 23, z_4 = z_5 = 276 \). The characters \( 1_G, \chi_2, \chi_3 \) and \( \chi_6 \) are real on 11-regular elements, but this implies that in the tree for \( B_1(11) \), two characters having the same sign \( \delta = +1 \) are joined by one edge contrary to a result of Brauer ([2], Theorem 5).

Thus \( |C_G(S_{11})| = 2^4 \cdot 3 \cdot 7 \cdot 11 \), and so \( t_{11} = 1 \) and \( B_1(11) \) consists of 10 non-exceptional characters whose possible degrees are given by Table I. But then the only character which could lie in the principal 23-block and the principal 11-block is the principal character which is impossible.

Using similar arguments cases (4), (6) and (8) are removed and so we have

**Lemma 4.1.** The Sylow 11-normalizer \( N_G(S_{11}) \) is a Frobenius group of order \( 5 \cdot 11 \).

**Corollary 4.1.** The principal 11-block \( B_1(11) \) is the only 11-block of defect 1. All other characters of \( G \) have degrees divisible by 11, and lie in 11-blocks of defect 0.

**5. The determination of degrees and blocks of characters of \( G \)**

We know now that \( G \) has no elements of order \( 23 \cdot 11, 23 \cdot 7, 23 \cdot 5, 23 \cdot 3, 11 \cdot 7, 11 \cdot 5 \) or \( 11 \cdot 3 \). Applying Theorem 2.2 to the intersection of \( B_1(23) \) and \( B_1(5) \) we see that both blocks contain a character of degree 896. This character is then the exceptional character for \( B_1(11) \) and using the degree equation (2.3) together with Theorem 2.2, we have
**Lemma 5.1.** The principal 11-block $B_1(11)$ contains only characters with the following degrees $1$, $45$, $45$, $1035$, $230$, $896$. All other characters of $G$ have degrees which are divisible by 11.

Since a character of degree $896 = 2^7 \cdot 7$ lies in $B_1(5)$ then $G$ has no elements of order $7 \cdot 5$, or $2 \cdot 5$. As shown earlier, $G$ has no elements of order $23 \cdot 5$ or $11 \cdot 5$ and so a Sylow 5-subgroup $S_5$ of $G$ can be centralized only by elements of order 3 or 9. Further $|N_G(S_5)/C_G(S_5)| \leq 4$, whence $|N_G(S_5)| = 2 \cdot 5$ or $2^2 \cdot 3 \cdot 5$. But in $B_1(5)$ we have already 3 non-exceptional characters and so $|N_G(S_5)| = 2^2 \cdot 3 \cdot 5$. Hence $t_5 = 1$ and $B_1(5)$ contains exactly 5 characters. These are found easily using equation (2.3).

**Lemma 5.2.** $|N_G(S_5)| = 2^2 \cdot 3 \cdot 5$. $B_1(5)$ consists of 5 characters with the following degrees: 1, 896, 896, 231, 2024.

Using the same methods we have

**Lemma 5.3.** The principal 23-block $B_1(23)$ contains only characters with the following degrees: 1, 22, 45, 45, 231, 2024, 896, 896, 990, 990 and 770. All other degrees of characters of $G$ are divisible by 23.

**Lemma 5.4.** $|N_G(S_7)/C_G(S_7)| = 3$. The principal 7-block $B_1(7)$ contains only characters with the following degrees: 1, 2024, 1035 and 990.

We have determined 16 characters of $G$, the sum of squares of degrees is $(10200960-64009)$. Further, the degrees of the remaining characters must be divisible by both 23 and 11. However $(11 \cdot 23)^2 = 64009$, so $G$ has only one more character and that is of degree $253 = 11 \cdot 23$.

**Lemma 5.5.** $G$ has 17 characters with the following degrees: 1, 22, 45, 45, 230, 231, 231, 231, 231, 253, 770, 770, 896, 896, 990, 990, 1035 and 2024.

It is thus clear there are two 7-blocks of defect 1, and hence two conjugate classes of 7-regular elements of $C_G(S_7)$ in $N_G(S_7)$. Further since $|N_G(S_7)/C_G(S_7)| = 3$, $|N_G(S_7)|$ has the following possible orders, $27 \cdot 3 \cdot 7$, $2^4 \cdot 3 \cdot 7$ and $2 \cdot 3 \cdot 7$, but only when $|N_G(S_7)| = 2 \cdot 3 \cdot 7$, are there the required two classes of 7-regular elements. Finally, there is only one 3-block of defect 2 and so a Sylow 3-subgroup is self centralizing.

### 6. Conclusion

The group $G$ has 17 conjugate classes and we have so far determined 16 of them, as is shown in the table below.

<table>
<thead>
<tr>
<th>Order of element</th>
<th>1</th>
<th>23</th>
<th>11</th>
<th>7</th>
<th>14</th>
<th>5</th>
<th>15</th>
<th>6</th>
<th>4</th>
<th>3</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of classes</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
There is at least one class of involutions, and at least one class of elements of order 3 with one class to be determined.

By Sylow theorems, the order of the normaliser of a Sylow 3-subgroup of $G$ is either $2^23^2$ or $2^4 \cdot 3^2$, and consequently a Sylow 3-subgroup is elementary abelian. Suppose $G$ has two classes of elements of order 3. Let $R$ be a Sylow 3-subgroup of $G$. We know that $R$ is self centralising and that $|N_G(R)| = 2^2 \cdot 3^2$, and so $N_G(R)/R$ is cyclic of order 4. Let $Q$ be a subgroup of order 3 in $R$ and $C_G(Q)$ the centraliser of $Q$ in $G$. Then since $N_{C_G(Q)}(R) = R$, we have by Burnside’s result ([5], p. 252) that $C_G(Q)$ has a normal 3-complement, say $N$. Let $\bar{Q}$ be the subgroup of order 3 of $R$ which is centralised by an element of order 5.

Then $C_G(\bar{Q}) = R\bar{N}$ where $\bar{N}$ is the normal 3-complement in $C_G(\bar{Q})$ and $5||\bar{N}|$. But then by the Frattini argument ([5], p. 12), $9||N_G(G_5)||$ where $G_5$ is a Sylow 5-subgroup of $G$, which is false. Hence $G$ has only one class of elements of order 3 and so we have proved

**Lemma 6.1.** The group $G$ has one class of elements of order 3. A Sylow 3-subgroup is normalised by a semi-dihedral group of order 16, and so $G$ has only one class of involutions and one class of elements of order 8.

Let $t$ be the involution in the normaliser of a Sylow 7-subgroup $G_7$ of $G$, and consider the centraliser of $t$ in $G$, $C_G(t)$. It follows immediately that $N_G(G_7) \subset C_G(t)$. Since $G$ has no elements of order $2 \cdot 23$, $2 \cdot 11$, or $2 \cdot 5$, then $C_G(t)$ has order $2^\alpha \cdot 3^\beta \cdot 7$, where $\alpha \leq 7$ and $\beta \leq 2$. We know that $G$ has only one class of involutions, and because $|C_G(t) : N_G(G_7)| \equiv 1(\text{mod } 7)$, the order of $C_G(t)$ is $2^7 \cdot 3 \cdot 7$.

Suppose the group $C_G(t)$ is soluble. Let $G_2$ be a Sylow 2-subgroup of $G$ which is contained in $C = C_G(t)$. Let $O_2(C)$ be the maximal normal subgroup of 2-power order in $C$. Then the factor group $C/O_2(C)$ is soluble. Let $\bar{N}$ be a minimal normal subgroup of $C/O_2(C)$. Then $\bar{N}$ has order 7 and so $O_2(C) = G_2$. But then $C_G(t)$ is 2-closed and so by a result of Suzuki ([5], p. 466), $G$ is one of known list of finite simple groups. However, none of these have the order 10, 200, 960, a contradiction.

Hence we conclude that $C_G(t) = C$ is insoluble. Write $E = O_2(C)$. Because we must have $|C/E : N_{C/E}(G_7)| \equiv 1(\text{mod } 7)$ where $G_7$ is a Sylow 7-subgroup in $C/E$, we have $|E| = 2$ or 16.

Suppose we have $|E| = 2$. Since $2^6 \cdot 3 \cdot 7$ is not the order of any simple group, $C/E$ contains a normal subgroup. Let $\bar{N}$ be a minimal normal subgroup of $C/E$, then $\bar{N}$ is either elementary abelian or a direct product of isomorphic simple groups. Clearly $\bar{N}$ cannot be an elementary abelian 2-group. Further, $\bar{N}$ cannot be of order 3 for then $G$ would have elements of order 21, and $\bar{N}$ cannot be of order 7 for this would imply that $|N_G(G_7)| > 2 \cdot 3 \cdot 7$. So we conclude that $|\bar{N}| = 2^3 \cdot 3 \cdot 7$, and $\bar{N} \simeq PSL(2, 7)$. Write $N = O_2(C)\bar{N}$, then we have $N < C = C_G(t)$. Let $N_7$
be a Sylow 7-subgroup of $N$. By the Frattini argument $C = NN_C(N_7)$ and so $C/N \cong N_C(N_7)/N_N(N_7)$. But then order of the normaliser of a Sylow 7-subgroup is greater than $2 \cdot 3 \cdot 7$, which is a contradiction.

Thus we conclude that $|O_2(C)| = 16$. Since $C_G(t)$ is insoluble, $C_G(t)$ is an extension of $E = O_2(C)$ of order 16 by $PSL(2, 7)$. Suppose that $E = O_2(C)$ is non-abelian. Let $Z(E)$ be the centre of $E$. It follows that $|Z(E)| \neq 4$ for otherwise the order of the centraliser of a Sylow 7-subgroup in $C$ is $4 \cdot 7$. Hence $Z(E) = \langle t \rangle$. Let $\Phi(E)$ be the Frattini subgroup of $E$, then $\Phi(E)$ has order 4 or 2. If $|\Phi(E)| = 4$ then $\Phi(E) \triangleleft C_G(t)$ and again we have that a Sylow 7-subgroup of $C$ has a normalizer of order 4.7. So $\Phi(E) = Z(E) = E' = \langle t \rangle$ and hence $E$ is an extra special 2-group, but this is impossible as $|E| = 2^4$. So $E$ is abelian.

By a result of Suzuki ([5], p. 177) a Sylow 7-subgroup $H$ of $C$ acts as an automorphism group of $E$, and so $E = \langle t \rangle Z$ where $\langle t \rangle \cap Z = \langle 1 \rangle$ and $Z$ is an $H$-admissible subgroup of $E$. The group $Z$ is then of order 8 and so is elementary abelian. Hence $E$ is elementary abelian.

Let $T$ be a Sylow 2-subgroup of $C_G(t)$. Clearly the centre of $T$, $Z(T)$, is contained in $E$. If $Z(T)$ is of order 8, then at least two involutions say $z$ and $z'$ in $Z(T) \setminus \langle t \rangle$ are conjugated in $C$ by an element of order 7. But this contradicts the result of Burnside ([5], p. 240) since they are not conjugate in $N_C(T) = T$. Suppose $Z(T)$ is of order 4 and let $z$ be an element in $E \setminus \langle t \rangle$. Since $z$ has 7 conjugates in $C$, $C_C(z)$ has order $2^7 \cdot 3$. Let $Q$ be a Sylow 3-subgroup of $C_C(z)$ and let $\tilde{T}$ be a Sylow 2-subgroup of $C_C(z)$. It is clear that $\tilde{T}$ is also a Sylow 2-subgroup of $G$. We have $E \lhd \tilde{T}$ and so $\langle t, z \rangle = Z(\tilde{T}) = C_E(Q)$. Further we have $|C_E(Q)| = 2^2 \cdot 3$ and hence $N_C(Q)$ has order $2^2 \cdot 3$.

Let $F^*$ be a Sylow 2-subgroup of $C_G(Q)$ which contains $\langle t, z \rangle$ and suppose by way of contradiction that $\langle t, z \rangle < F^*$ has a subgroup $F_1$ which contains $\langle t, z \rangle$ properly and $|F_1 : \langle t, z \rangle| = 2$. Since $F_1$ does not lie in $C$, $F_1$ is contained in $C_G(z)$ or in $C_G(tz)$ and so $|C_{C_E}(Q)| > 2^2 \cdot 3$ or $|C_{C_E}(Q)| > 2^2 \cdot 3$. But $G$ has only one class of involutions and so this is impossible. Hence $C_G(Q)$ has order $2^2 \cdot 3^2$. 5. By a result of Gaschütz ([5], p. 26) $Q$ splits in $C_G(Q)$ and so we may write $C_G(Q) = Q \times L$ where $L$ is a group of order 60. From the order of the normalizer of a Sylow 5-subgroup of $G$ (lemma 5.6) it follows that $L$ is insoluble, and so $L$ is simple. But then $L \cong A_5$ where $A_5$ is the alternating group on 5 letters. By a result of Gaschütz we may write $N_G(Q) = QK$ where $|K| = 2^3 \cdot 3 \cdot 5$, and so $L \lhd K$, where $L \cong A_5$ and $L \subseteq C_G(Q)$.

Let $F$ be the Sylow 2-subgroup of $N_G(Q)$, then $F$ must be Abelian since a dihedral group of order 8 cannot normalize a group of order 3. Consequently $K = L \times S$ where $S$ is a group of order 2. But then $G$ has elements of order 10, which is impossible. Hence a Sylow 2-subgroup of $G$ has cyclic centre of order 2. We have proved:

**Lemma 6.2.** The centralizer $C$ of an involution $t$ in the centre of a Sylow
2-subgroup $T$ of $G$ is an extension of an elementary abelian group $E$ of order 16 by a group $H$, $H \cong PSL(2, 7)$. Further the centre of $T$ is cyclic.

It now follows from a result of Janko [8] that $G \cong M_{23}$.

References


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