ON DOUBLY TRANSITIVE PERMUTATION GROUPS OF DEGREE PRIME SQUARED PLUS ONE

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(Received 5 January 1978)

Communicated by M. F. Newman

Abstract

Groups with the property of the title were considered by Chillag (1977); this paper completes his results by showing that, with known exceptions, they are triply transitive.


THEOREM A. Let $G$ be a 2-transitive permutation group of degree $p^2+1$, where $p$ is prime. Then one of the following occurs:

(a) $G$ is 3-transitive;
(b) $PSL(2,p^2) \leq G \leq PGL(2,p^2)$;
(c) $G$ is the Frobenius group of order 20, with $p = 2$.

Thus, conclusion (d) of Chillag's Corollary (asserting that the stabilizer of two points has orbit lengths 1, 1, $2(p-1)$, $(p-1)^2$) does not occur; or, more precisely, groups in case (d) occur already in case (b). This is a consequence of the following result.

THEOREM B. Let $G$ be a 2-transitive permutation group on $X$, of degree $n^2 + 1$ ($n > 1$). Suppose that, for $x, y \in X$, $x \neq y$, $G_{xy}$ has orbit lengths 1, 1, $2(n-1)$, $(n-1)^2$. Then $n = 3$, $G = PSL(2,9)$ or $P\Sigma L(2,9)$.

Proof. The results of Higman (1970) imply that $G_z$ is a subgroup of $S_n \wr S_2$; so $G_z$ has an imprimitive subgroup $N(x)$ of index 2. For $y \neq x$, $K = N(x) \cap G_y$ has orbit lengths 1, 1, $n-1$, $n-1$, $(n-1)^2$, and the orbit of length $(n-1)^2$ is isomorphic (as $K$-space) to the direct product of the two orbits of length $n-1$.
Now $N(y) \cap G_x = K'$ is a subgroup of index 2 in $G_{xy}$ with the same orbit lengths as $K$. If $K \neq K'$, then $K \cap K'$ has four orbits of length $\frac{1}{4}(n-1)$, so the $K$-orbit of length $(n-1)^2$ splits into four orbits of length $\frac{1}{4}(n-1)^2$ of $K \cap K'$. This is impossible since $|K : K \cap K'| = 2$. We conclude that

$$N(y) \cap G_x = N(x) \cap G_y \leq N(x),$$

whence $N(x)$ is strongly closed in $G_x$ with respect to $G$.

By the “two-graph transfer theorem” (see Taylor (1977), Theorem 6.1), either $G$ has a subgroup $N$ of index 2 with $N \cap G_x = N(x)$, or $G$ is an automorphism group of a non-trivial regular two-graph. In the first case, if $B$ is either orbit of length $n-1$ of $N_{xy}$, then $B \cup \{y\}$ is a block of imprimitivity for $N_x$, and so the setwise stabilizer $L$ of $B \cup \{x, y\}$ acts 2-transitively on it. Then

$$|N : L| = \frac{(n^2 + 1)n^2}{(n+1)n};$$

but this is not an integer for $n > 1$.

In the second case we use the fact that, if $H$ is a rank 3 group whose parameters (in Higman’s (1970) sense) are $k, l, \lambda, \mu$, and $G$ is a transitive extension of $H$ which acts on a regular two-graph, then $k = 2\mu$. (See Taylor (1977), Proposition 2.3.) Here $k = 2(n-1)$, $\mu = 2$; so we have $n = 3$. The rest of the theorem is clear.

References


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