CONCERNING NON-MEASURABLE SUBSETS OF A GIVEN MEASURABLE SET

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Let \( R, \mu \) and \( M_\mu \) denote the set of real numbers, Lebesgue outer measure and the class of Lebesgue measurable subsets of \( R \) respectively. It is easy to prove that the complement \( E^c \) of \( E \in M_\mu \) is a set of Lebesgue measure zero if the inequality \( \mu(E \cap I) \geq \delta \mu(I) \) holds for some \( \delta > 0 \) and all intervals \( I \) of \( R \). However, in [1], Hewitt raised a problem whether the result is still true if \( E \) is not a priori measurable set. In this paper, a negative answer to this question is given through a counter-example. Also, it is proved that for a given set \( E \in M_\mu \) with \( \mu(E) > 0 \) there is a non-measurable subset \( A \) of \( E \) satisfying \( \mu(A) = \mu(E) \).

**Lemma 1.** Let \( E \in M_\mu \) with \( \mu(E) < \infty \) and \( A \subset E \). Then \( A \in M_\mu \) if and only if \( \mu(E) = \mu(A) + \mu(E - A) \).

For the proof, the reader is referred to [2].

**Lemma 2.** If \( \{E_i\} \) is a sequence of pairwise disjoint sets of \( M_\mu \) each having positive measure and \( \{A_i\} \) is a sequence of non-measurable sets such that \( A_i \subset E_i \) for each \( i \), then \( \bigcup_{i=1}^{\infty} A_i \) is non-measurable and \( \mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i) \).

**Proof.** The non-measurability for \( \bigcup_{i=1}^{\infty} A_i \) is obvious. We need only prove

\[
\mu(\bigcup_{i=1}^{\infty} A_i) \geq \sum_{i=1}^{n} \mu(A_i)
\]

for every \( n \), from which \( \mu(\bigcup_{i=1}^{\infty} A_i) \geq \sum_{i=1}^{\infty} \mu(A_i) \) follows, and the conclusion is obtained in view of subadditivity of \( \mu \). By monotoneity of \( \mu \),

\[
\mu(\bigcup_{i=1}^{\infty} A_i) \geq \mu(\bigcup_{i=1}^{n} A_i)
\]

for all \( n \). We shall show that \( \mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i) \) by induction. The equality is trivial for \( n = 1 \). Assume that it holds for \( n = k \). Since \( A_{k+1} \subset E_{k+1} \) and \( \bigcup_{i=1}^{k} A_i \subset E_k \), measurability of \( E_{k+1} \) implies that

\[
\mu(\bigcup_{i=1}^{k+1} A_i) = \mu(\bigcup_{i=1}^{k} A_i) + \mu(A_{k+1}) = \sum_{i=1}^{k+1} \mu(A_i)
\]

The last equality follows by inductive hypothesis. The proof is now completed.
Lemma 3. If $E \in M_\mu$ with $\mu(E) > 0$, then there is a non-measurable subset $A$ of $E$ such that $\mu(A) \geq \frac{1}{2} \mu(E)$.

Proof. The existence of a non-measurable subset $Q$ of $E$ is well-known.

If $0 < \mu(E) < \infty$, then by lemma 1, $\mu(E) < \mu(Q) + \mu(E - Q)$. Thus we have $\mu(Q) > \frac{1}{2} \mu(E)$ or $\mu(E - Q) > \frac{1}{2} \mu(E)$. Since $Q \notin M_\mu$, $E - Q \notin M_\mu$. The conclusion follows.

If $\mu(E) = \infty$, then by $\sigma$-finiteness of $\mu$, there is a sequence of pairwise disjoint sets $\{E_i\}$ of $M_\mu$ such that $E = \bigcup_{i=1}^{\infty} E_i$ and $\mu(E_i) < \infty$ for each $i$ (we may assume $0 < \mu(E_i) < \infty$). By what we have just shown, there is a non-measurable subset $A_i$ of $E_i$ for each $i$ such that $\mu(A_i) > \frac{1}{2} \mu(E_i)$. Let $A = \bigcup_{i=1}^{\infty} A_i$. By lemma 2, $A \notin M_\mu$ and

$$\mu(A) = \sum_{i=1}^{\infty} \mu(A_i) \geq \frac{1}{2} \sum_{i=1}^{\infty} \mu(E_i) = \frac{1}{2} \mu(E).$$

Theorem. If $E \in M_\mu$ with $\mu(E) > 0$, then there is a non-measurable subset $A$ of $E$ such that $\mu(A) = \mu(E)$.

Proof. Case 1. $0 < \mu(E) < \infty$. We define $r_0 = \mu(E)$ and $B_0 = \emptyset$.

By lemma 3, there is $A_1 \subseteq E$ such that $A_1 \notin M_\mu$ and $\mu(A_1) \geq \frac{r_0}{2}$. Also, there is $B_1 \subseteq E$ such that $E - B_0 \supseteq B_1 \supseteq A_1$ and $\mu(B_1) = \mu(A_1)$. Let $r_1 = \mu(E - B_1)$. Clearly $0 \leq r_1 \leq \frac{r_0}{2}$.

If $r_1 = 0$, then we are through. Assume $r_1 > 0$. By the same reason, there are $A_2 \notin M_\mu$ and $B_2 \subseteq E$ such that $A_2 \subseteq B_2 \subseteq E - \bigcup_{k=0}^{r_1} B_k$ and $\mu(B_2) = \mu(A_2) \geq \frac{r_1}{2}$. Let $r_2 = \mu(E - \bigcup_{k=0}^{r_2} B_k)$, then $0 \leq r_2 \leq \frac{r_1}{2} \leq \frac{r_0}{2^2}$.

Suppose $\{A_j\}_{j=1}^{\infty}$, $\{B_j\}_{j=0}^{\infty}$ and $\{r_j\}_{j=0}^{\infty}$ have been defined such that $A_j \notin M_\mu$, $B_j \in M_\mu$, $A_j \subseteq B_j \subseteq E - \bigcup_{k=0}^{r_j} B_k$,

$$\mu(B_j) = \mu(A_j) \geq r_{j-1}/2, r_j = \mu(E - \bigcup_{k=0}^{j} B_k) \leq \frac{r_0}{2^j} \text{ for } j = 1, 2, \cdots n.$$ 

Clearly $\{B_j\}_{j=1}^{n}$ is pairwise disjoint. By lemma 2, $\bigcup_{j=1}^{n} A_j \notin M_\mu$ and

$$\mu(\bigcup_{j=1}^{n} A_j) = \sum_{j=1}^{n} \mu(A_j) = \sum_{j=1}^{n} \mu(B_j) = \mu(\bigcup_{j=1}^{n} B_j) = \mu(E) - r_n.$$ 

If $r_n = 0$, we may take $A = \bigcup_{j=1}^{n} A_j$. Otherwise, $\mu(E - \bigcup_{k=0}^{n} B_k) = r_n > 0$ and there are $A_{n+1} \notin M_\mu$, $B_{n+1} \in M_\mu$ such that $A_{n+1} \subseteq B_{n+1} \subseteq E - \bigcup_{k=0}^{n} B_k$,

$$\mu(B_{n+1}) = \mu(A_{n+1}) \geq r_n/2, r_{n+1} = \mu(E - \bigcup_{k=0}^{n+1} B_k) \leq \frac{r_0}{2^{n+1}}.$$ 

If this process does not terminate, we obtain infinite sequences $\{A_i\}$, $\{B_i\}$ and $\{r_i\}$. Let $A = \bigcup_{i=1}^{\infty} A_i$. By lemma 2 again, $A \notin M_\mu$ and

$$\mu(A) = \sum_{i=1}^{\infty} \mu(A_i) = \sum_{i=1}^{\infty} \mu(B_i) = \mu(\bigcup_{i=1}^{\infty} B_i).$$
Thus
\[ \mu(E) \geq \mu(A) \geq \mu\left( \bigcup_{i=1}^{n} B_i \right) = \mu(E) - r_n \geq \mu(E)(1 - 1/2^n) \]
for all \( n \). It follows that \( \mu(A) = \mu(E) \).

Case 2. \( \mu(E) = \infty \). By \( \sigma \)-finiteness of \( \mu \), there is a sequence of pairwise disjoint sets \( \{E_i\} \) of \( M_{\mu} \) such that \( E = \bigcup E_i \), \( 0 < \mu(E_i) < \infty \) for each \( i \). The conclusion follows easily from case 1 and lemma 2.

Finally, we proceed to the construction of a counter-example. Let \( E = [0, 1] \).

By the above theorem, there is a \( Q \subset E \) such that \( Q \notin M_{\mu} \) and \( \mu(Q) = \mu(E) = 1 \).

Let \( A = (-\infty, 0) \cup Q \cup (1, \infty) \).

Obviously \( A \notin M_{\mu} \), and therefore \( \mu(A^c) \neq 0 \). We assert that \( \mu(A \cap I) = \mu(I) \) for every interval \( I \) of \( R \).

Case 1. \( I \in [0, 1] \).

1.1. \( 0 \in I \) or \( 1 \in I \): There is a subinterval \( J \) of \( [0, 1] \) such that \( I \cap J = \emptyset \), \( I \cup J = [0, 1] \), where \( J \) may be empty or a singleton. Thus
\[ 1 = \mu(Q) = \mu(Q \cap I) + \mu(Q \cap I^c) = \mu(Q \cap I) + \mu(Q \cap J). \]

If \( \mu(Q \cap I) < \mu(I) \), then we would have
\[ 1 = \mu(Q) = \mu(Q \cap I) + \mu(Q \cap J) < \mu(I) + \mu(J) = 1. \]

This leads to a contradiction. Thus \( \mu(Q \cap I) = \mu(I) \) and hence \( \mu(A \cap I) = \mu(Q \cap I) = \mu(I) \).

1.2. \( 0 \notin I \) and \( 1 \notin I \): There are two subintervals \( J_1, J_2 \) of \( [0, 1] \) such that \( J_1, J_2 \) are pairwise disjoint and \( J_1 \cup I \cup J_2 = [0, 1] \). Since \( J_1 \cup I \in M_{\mu}, J_1 \in M_{\mu} \), we have
\[ 1 = \mu(Q) = \mu(Q \cap (J_1 \cup I)) + \mu(Q \cap J_2) = \mu(Q \cap J_1) + \mu(Q \cap I) + \mu(Q \cap J_2). \]

If \( \mu(Q \cap I) < \mu(I) \), then we would have
\[ 1 = \mu(Q) < \mu(J_1) + \mu(I) + \mu(J_2) = 1. \]

This leads to a contradiction too. Thus \( \mu(A \cap I) = \mu(Q \cap I) = \mu(I) \).

Case 2. \( I \notin [0, 1] \). Let \( I_1 = I \cap (-\infty, 0) \), \( I_2 = I \cap [0, 1] \) and \( I_3 = I \cap (1, \infty) \) (some of them may be empty). Since \( I_1, I_2 \in M_{\mu}, I_3 \in M_{\mu} \),
\[ \mu(A \cap I) = \mu(A \cap I_1) + \mu(A \cap (I_2 \cup I_3)) = \mu(A \cap I_1) + \mu(A \cap I_2) + \mu(A \cap I_3) = \mu(I_1) + \mu(A \cap I_2) + \mu(I_3). \]
By case 1, we have

$$\mu(A \cap I) = \mu(I_1) + \mu(A \cap I_2) + \mu(I_3) = \mu(I_1) + \mu(I_2) + \mu(I_3) = \mu(I).$$

References


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