APPORXIMATION OF FUNCTIONS
BY A NEW CLASS OF LINEAR OPERATORS

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1. Introduction

Various extensions and generalizations of Bernstein polynomials have been considered among others by Szasz [13], Meyer-Konig and Zeller [8], Cheney and Sharma [1], Jakimovski and Leviatan [4], Stancu [12], Pethe and Jain [11]. Bernstein polynomials are based on binomial and negative binomial distributions. Szasz and Mirakyan [9] have defined another operator with the help of the Poisson distribution. The operator has approximation properties similar to those of Bernstein operators. Meir and Sharma [7] and Jain and Pethe [3] deal with generalizations of Szasz-Mirakyan operator. As another generalization, we define in this paper a new operator with the help of a Poisson type distribution, consider its convergence properties and give its degree of approximation. The results for the Szasz-Mirakyan operator can easily be obtained from our operator as a particular case.

2. The operator and its convergence

The operator and its convergence are based on the following two lemmas:

LEMMA 1. For $0 < \alpha < \infty$, $|\beta| < 1$, let

$$(2.1) \quad \omega_p(k, \alpha) = \frac{\alpha^{k-1} e^{-\alpha + k\beta}}{k!}; \quad k = 0, 1, 2, \cdots$$

then

$$(2.2) \quad \sum_{k=0}^{\infty} \omega_p(k, \alpha) = 1.$$ 

It may be mentioned that (2.1) is a Poisson-type distribution which has been considered by Consul and Jain [2].

The proof of the lemma may be based upon results given by Jensen [5]. If we start with Lagrange’s formula
(2.3) \[ \phi(z) = \phi(0) + \sum_{k=1}^{\infty} \frac{1}{k!} \left[ \frac{d^{k-1}}{dz^{k-1}} ((f(z))^k) \phi'(z) \right]_{z=0} \left( \frac{z}{f(z)} \right)^k \]

and proceed by setting

\[ \phi(z) = e^{az} \text{ and } f(z) = e^{\beta z} \]

we shall get

(2.4) \[ e^{az} = \sum_{k=0}^{\infty} a(z + k\beta)^{k-1} u^k / k!, \quad u = ze^{-\beta z}, \]

where \( z \) and \( u \) are sufficiently small such that \( |\beta u| < e^{-1} \) and \( |\beta z| < 1 \).

By taking \( z = 1 \), the lemma in (2.2) is obvious.

**Lemma 2.** Let

(2.5) \[ S(r, \alpha, \beta) = \sum_{k=0}^{\infty} (r + \beta k)^{k+1} e^{-(r + \beta k)} \alpha ! / k!, \quad r = 0, 1, 2, \ldots \]

and

(2.6) \[ \alpha S(0, \alpha, \beta) = 1. \]

Then

(2.7) \[ S(r, \alpha, \beta) = \sum_{k=0}^{\infty} \beta^k (\alpha + k\beta) S(r-1, \alpha + k\beta, \beta). \]

**Proof.** It can easily be seen that the functions \( S(r, \alpha, \beta) \) satisfy the reduction formula

(2.8) \[ S(r, \alpha, \beta) = \alpha S(r-1, \alpha, \beta) + \beta S(r, \alpha + \beta, \beta). \]

By a repeated use of (2.8), the proof of the lemma is straightforward.

From (2.7) and (2.6) when \( \beta < 1 \) we get

(2.9) \[ S(1, \alpha, \beta) = \sum_{k=0}^{\infty} \beta^k = 1 / (1 - \beta), \]

and

(2.10) \[ S(2, \alpha, \beta) = \sum_{k=0}^{\infty} \beta^k (\alpha + k\beta) / (1 - \beta) = \frac{\alpha}{(1 - \beta)^2} + \frac{\beta^2}{(1 - \beta)^3}. \]

We may now define the operator as

(2.11) \[ P_n^{[\beta]}(f; x) = \sum_{k=0}^{\infty} \omega_\beta(k, nx)f(k/n), \]

where \( 1 > \beta \geq 0 \) and \( \omega_\beta(k, nx) \) has been defined in (2.1).

The parameter \( \beta \) may depend on the natural number \( n \). It is easy to see that for \( \beta = 0 \), (2.11) reduces to Szasz--Mirakyan operator [9].

The convergence property of the operator \( P_n^{[\beta]}(f; x) \) is proved in the following theorem:
THEOREM (2.1). If \( f \in C[0, \infty) \) and \( \beta \to 0 \) as \( n \to \infty \) then the sequence \( \{ P_n^{[\beta]}(f; x) \} \) converges uniformly to \( f(x) \) in \([a, b]\), where \( 0 \leq a < b < \infty \).

PROOF. Since \( P_n^{[\beta]}(f; x) \) is a positive linear operator for \( 1 > \beta \geq 0 \), it is sufficient, by Korovkin’s result, to verify the uniform convergence for test functions \( f(t) = 1, t \) and \( t^2 \).

It is clear from (2.2) that

\[(2.12) \quad P_n^{[\beta]}(1; x) = 1.\]

Going on to \( f(t) = t \) and using (2.9) we have

\[(2.13) \quad P_n^{[\beta]}(t; x) = nx \sum_{k=0}^{\infty} \frac{(nx+\beta k)^{k-1}}{k!} e^{-(nx+\beta k)} \left( \frac{k}{n} \right) = x S(1, nx+\beta, \beta) = \frac{x}{1-\beta}.

Proceeding to the function \( f(t) = t^2 \), it can easily be shown that

\[(2.14) \quad P_n^{[\beta]}(t^2; x) = nx \sum_{k=0}^{\infty} \frac{(nx+\beta k)^{k-1}}{k!} e^{-(nx+\beta k)} \frac{k^2}{n^2}.

and a use of (2.9) and (2.10) yields

\[(2.14) \quad P_n^{[\beta]}(t^2; x) = \frac{x^2}{(1-\beta)^2} + \frac{x}{n(1-\beta)^3}.

Thus combining the results of (2.12), (2.13) and (2.14) we have

\[\lim_{n \to \infty} P_n^{[\beta]}(t^2; x) = x^r, \quad r = 0, 1, 2, \text{ as } \beta \to 0\]

and hence by Korovkin’s theorem the proof of theorem (2.1) is complete.

3. Order of approximation

THEOREM (3.1). If \( f \in C[0, \lambda] \) and \( 1 > \beta'/n \geq \beta \geq 0 \) then

\[|f(x)-P_n^{[\beta]}(f; x)| \leq [1 + \lambda^4 (1 + \lambda \beta')^3] \cdot \omega(1/n^3),\]

where \( \omega(\delta) = \sup |f(x'')-f(x')|; x', x'' \in [0, \lambda], \delta \) being a positive number such that \( |x''-x'| < \delta \).

PROOF. By using the properties of modulus of continuity

\[(3.1) \quad |f(x'')-f(x')| \leq \omega(|x''-x'|);\]
and noting the fact that
\[
\sum_{k=0}^{\infty} \omega_p(k, nx) = 1 \quad \text{and} \quad \omega_p(k, nx) \geq 0, \forall n, k
\]
it can easily be seen, by the application of Cauchy's inequality, that

\[
|f(x)-P_n^{[\beta]}(f; x)| \leq \left\{ 1 + \frac{1}{\delta} \sum_{k=0}^{\infty} \frac{n(\beta+k)^k}{k!} e^{-(\beta+k)} \right\} \omega(\delta)
\]

Now by linearity of the operator and by using (2.12), (2.13) and (2.14) we have

\[
\sum_{k=0}^{\infty} \frac{n(\beta+k)^k}{k!} e^{-(\beta+k)} \left( x - \frac{k}{n} \right)^2 = x^2 P_n^{[\beta]}(1; x) - 2x P_n^{[\beta]}(t; x) + P_n^{[\beta]}(t^2; x)
\]

Hence using (3.4) in (3.3) and choosing \( \delta = 1/\sqrt{n} \) we prove

\[
|f(x)-P_n^{[\beta]}(f; x)| \leq \left[ 1 + \lambda \sqrt{1+\lambda} \right] \omega(\delta)/\sqrt{n}.
\]

For \( \beta = 0 \), the expression (3.5) reduces to an inequality for the Szasz-Mirakyan operator obtained earlier by Müller.

**Theorem 3.2.** If \( f \in C'[0, 1], 1 > \beta'/n \geq \beta \geq 0 \), then the following inequality holds

\[
|f(x)-P_n^{[\beta]}(f; x)| \leq \lambda \sqrt{1+\lambda} \sqrt{1+\lambda \beta'} \omega(1/\sqrt{n})/\sqrt{n},
\]

where \( w_1(\delta) \) is the modulus of continuity of \( f' \).

**Proof.** For definiteness, we prove the theorem for \( f'(x) \geq 0 \) but it also applies to \( f'(x) < 0 \). By the mean value theorem of differential calculus, it is known that

\[
f(x) - f\left( \frac{k}{n} \right) = \left( x - \frac{k}{n} \right) f'(...),
\]

where \( \xi \equiv \zeta_{n,k}(x) \) is an interior point of the interval determined by \( x \) and \( k/n \). Now

\[
f(x) - f\left( \frac{k}{n} \right) \leq \left( x - \frac{k}{n} \right) [f'(\xi) - f'(x)] + \left( x - \frac{k}{n} \right) f'(x).
\]
Multiplying both sides of the inequality by \( x(n + \beta k)^{k-1} e^{-(nx + \beta k)/k!} \), summing over \( k \) and using (2.13) we get

\[
|f(x) - P_n^{(\beta)}(f; x)| \leq \sum_{k=0}^{\infty} \left| x - \frac{k}{n} \right| \frac{n(n + \beta k)^{k-1} e^{-(nx + \beta k)}}{k!} |f'(\xi) - f'(x)|.
\]

But by (3.1) and (3.2)

\[
|f'(\xi) - f'(x)| \leq \omega_1(|\xi - x|) \leq \left( 1 + \frac{1}{\delta} |\xi - x| \right) \omega_1(\delta) \leq \left( 1 + \frac{1}{\delta} \left| \frac{k}{n} - x \right| \right) \omega_1(\delta);
\]

where \( \delta \) is a positive number not depending on \( k \).

A use of this in (3.6) gives

\[
|f(x) - P_n^{(\beta)}(f; x)| \leq \left\{ \sum_{k=0}^{\infty} \left| x - \frac{k}{n} \right| \frac{n(n + \beta k)^{k-1} e^{-(nx + \beta k)}}{k!} \right. + \left. \frac{1}{\delta} \sum_{k=0}^{\infty} \left( x - \frac{k}{n} \right)^2 \frac{n(n + \beta k)^{k-1} e^{-(nx + \beta k)}}{k!} \right\} \omega_1(\delta).
\]

Hence by applications of Cauchy’s inequality and (3.4)

\[
|f(x) - P_n^{(\beta)}(f; x)| \leq \frac{\lambda^{1/2}(1 + \lambda \beta \gamma)^{1/2}}{n^{1/2}} \left[ 1 + \frac{\lambda^{1/2}(1 + \lambda \beta \gamma)^{1/2}}{\delta n^{1/2}} \right] \omega_1(\delta).
\]

Choosing \( \delta = 1/\sqrt{n} \), theorem (3.2) is proved.

We may put \( \beta = 0, \delta = 1/\sqrt{n} \) in (3.7) to get the expression for Szasz-Mirakyan operator. The substitutions reduce (3.7) to

\[
|f(x) - P_n^{(\beta)}(f; x)| \leq \frac{1}{\sqrt{n}} (\lambda + \sqrt{\lambda}) \omega_1(1/\sqrt{n}); \quad x \in [0, \lambda]
\]

in agreement with Stancu [12].

References


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