

STRONG CARMICHAEL NUMBERS

Dedicated to George Szekeres on his 65th birthday

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A composite number N is called a pseudoprime for the base a in case

$$(1) \quad a^{N-1} \equiv 1 \pmod{N}.$$

An odd pseudoprime N is called strong for the base a in case

$$(2) \quad a^{(N-1)/2} \equiv \left(\frac{a}{N}\right) \pmod{N}$$

where the symbol on the right is that of Jacobi. To explain the terminology, experiments show that (2), which implies (1), holds only rarely among the ordinary pseudoprimes. Hence (2) makes a good hypothesis item in a test for primality. $N = 561$, which is a pseudoprime for every base prime to 561, is a strong pseudoprime for the base 2 but not for the base 5 since

$$5^{280} \equiv 67 \pmod{561}.$$

A pseudoprime, like 561, for which (1) holds for all bases a prime to N is called a universal pseudoprime or Carmichael number. The first 23 such numbers are

561 = 3 · 11 · 17	15841 = 7 · 31 · 73	101101 = 7 · 11 · 13 · 2101
1105 = 5 · 13 · 17	29341 = 13 · 37 · 61	115921 = 13 · 37 · 241
1729 = 7 · 13 · 19	41041 = 7 · 11 · 13 · 41	126217 = 7 · 13 · 19 · 73
2465 = 5 · 17 · 29	46657 = 13 · 37 · 97	162401 = 17 · 41 · 233
2821 = 7 · 13 · 31	52633 = 7 · 73 · 103	172081 = 7 · 13 · 31 · 61
6601 = 7 · 23 · 41	62745 = 3 · 5 · 47 · 89	188461 = 7 · 13 · 19 · 109
8911 = 7 · 19 · 67	63973 = 7 · 13 · 19 · 37	252601 = 41 · 61 · 101
10585 = 5 · 29 · 73	75361 = 11 · 13 · 17 · 31	

A strong Carmichael number N would be such that (2) holds for all bases a prime to N . In this note we show that such numbers do not exist.

THEOREM. *No Carmichael number is strong.*

PROOF. In 1912 Carmichael (1912) showed that every Carmichael number N is the product of distinct primes. Therefore we can write

$$N = p_1 \cdot p_2 \cdots p_t \quad (p_1 > 2, t > 1)$$

and we see that

$$a^{N-1} \equiv 1 \pmod{p_i} \quad (i = 1(1)t)$$

holds for every a prime to N and in particular for $a = g$ any common primitive root of all the p 's. Therefore

$$N - 1 \equiv 0 \pmod{p_i - 1}.$$

Now the t primes p_i can be of two types:

- Type 1 those p for which $(N - 1)/2 \equiv 0 \pmod{p - 1}$
- Type 2 those p for which $(N - 1)/2 \equiv (p - 1)/2 \pmod{p - 1}$.

Thus we have

$$(3) \quad a^{(N-1)/2} \equiv \begin{cases} 1 \pmod{p} & \text{if } p \text{ is of Type 1} \\ \left(\frac{a}{p}\right) \pmod{p} & \text{if } p \text{ is of Type 2.} \end{cases}$$

We now choose a to be a quadratic nonresidue of p_1 and a residue of all the other p 's. First suppose there is a prime of Type 1 which we may take to be p_1 . If N were strong (2) and (3) would give us

$$-1 = \left(\frac{a}{N}\right) \equiv a^{(N-1)/2} \equiv 1 \pmod{p_1}.$$

This contradiction shows that all the p 's are of Type 2.

Hence (3) gives

$$a^{(N-1)/2} \equiv 1 \pmod{p_2}.$$

But, by (2)

$$a^{(N-1)/2} \equiv \left(\frac{a}{N}\right) \pmod{p_1 p_2},$$

that is,

$$a^{(N-1)/2} \equiv -1 \pmod{p_2}.$$

This contradiction completes the proof.

Reference

R. D. Carmichael (1912), *Amer. Math. Monthly* **19**, 22–27.

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