A CHARACTERISATION OF ERGODIC MEASURES

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Consider a set $X$ together with a $\sigma$-algebra $\mathcal{B}$ of subsets of $X$. Let $G$ be a family of $\mathcal{B}$-measurable transformations on $X$, let $p(X)$ be the convex set of all probability measures on $\mathcal{B}$ and let $I$ be the convex set of all $G$-invariant probability measures in $p(X)$. For $\mu \in p(X)$ we define $\mathcal{B}_\mu = \{ A \in \mathcal{B} : \mu(gA \Delta A) = 0 \text{ for all } g \in G \}$ and we define $\mathcal{B}_0 = \{ A \in \mathcal{B} : gA = A \text{ for all } g \in G \}$. Then $\mathcal{B}_0 \subseteq \mathcal{B}_\mu$ and both are $\sigma$-subalgebras of $\mathcal{B}$. $G$ is said to act transitively on $X$ if for $x \in X, y \in X$, $gx = y$ for some $g \in G$.

Consider the following conditions on an element $\mu \in I$:

(a) $\mu$ is an extreme point of $I$,

(b) $\mu(\mathcal{B}_\mu) = \{0, 1\}$,

(c) $\mu(\mathcal{B}_0) = \{0, 1\}$.

Each of these conditions has been considered in the literature as a definition of ergodicity of $\mu$. Feldman has shown that (a) and (b) are equivalent (1966; page 81). Under certain conditions (b) and (c) are known to be equivalent (see Feldman (1966; page 84) for a discussion) and the result of this paper is one of this type. Our result was provided in the case where $G$ is a separable topological group by Varadarajan (1963).

**Theorem.** Let $G$ be a Hausdorff locally compact $\sigma$-compact topological group of $\mathcal{B}$-measurable transformations on $X$ such that the associated mapping $(g, x) \rightarrow gx$ on $G \times X$ to $X$ is jointly measurable when $G$ is equipped with the $\sigma$-algebra of Borel sets. Let $\mu \in I$. Then $\mu \in exI$ if and only if $\mu(\mathcal{B}_0) = \{0, 1\}$. If $G$ acts transitively on $X$, there is at most one $G$-invariant measure in $p(X)$.

Before proving this theorem we make some definitions. A fixed left invariant Haar measure on $G$ will be denoted by $d\lambda$. For a function $\phi$ on $G$ and $g \in G$,
$l_g\phi$ is defined on $G$ by: $l_g\phi(h) = \phi(gh)$ for $h \in G$. If $\phi \in L^1(G)$ and $f$ is a bounded real valued $\mathcal{B}$-measurable function on $X$, $\phi * f$ is defined by:

$$\phi * f(x) = \int_G \phi(g)f(g^{-1}x)d\lambda(g).$$

Then define $P(\phi,f) = \{x \in X: \phi * f(x) = 1\}$ and $Q(\phi,f) = \bigcap_{g \in G} P(l_g\phi,f)$. It follows from the definition of $Q(\phi,f)$ that $Q(\phi,f) = g(Q(\phi,f))$ for all $g \in G$.

**Lemma.** Let $G$ be $\sigma$-compact, let $\phi$ be a continuous real valued function on $G$ having compact support and let $f$ be a bounded real valued, $\mathcal{B}$-measurable function on $X$. Then $Q(\phi,f) \in \mathcal{B}_0$.

**Proof.** $\phi$ is uniformly continuous on $G$, and a sequence $(g_n)$ can be chosen in $G$ so that $\{l_{g_n}\phi: i = 1, 2, \ldots\}$ is uniformly dense in $\{l_g\phi: g \in G\}$. Let $K$ be the support of $\phi$, $g \in G$, $\varepsilon > 0$ and $x \in \bigcap_{i}^\infty P(l_{g_i}\phi,f)$. We may assume $\lambda(K) > 0$. Choose $i$ so that

$$\|l_g\phi - l_{g_i}\phi\| < \frac{\varepsilon}{2\lambda(K)} \|f\|^{-1}.$$

Then

$$\left| (l_g\phi * f)(x) - 1 \right| = \left| (l_g\phi * f)(x) - (l_{g_i}\phi * f)(x) \right|$$

$$\leq \left( \int_G \left| \phi(gp) - \phi(g_ip) \right| d\lambda(p) \right) \cdot \|f\|$$

$$= \left( \int_{g_i^{-1}K \cup g_i^{-1}K} \left| \phi(gp) - \phi(g_ip) \right| d\lambda(p) \right) \cdot \|f\|$$

$$\leq (\lambda(g^{-1}K) + \lambda(g_i^{-1}K)) \cdot \|l_g\phi - l_{g_i}\phi\| \cdot \|f\|$$

$$\leq \varepsilon, \text{ true for all } \varepsilon > 0, \text{ all } g \in G.$$

Hence $x \in \bigcap_{g \in G} P(l_g\phi,f)$ so that $Q(\phi,f) = \bigcap_{i}^\infty P(l_{g_i}\phi,f)$. Since each $P(l_{g_i}\phi,f) \in \mathcal{B}$, $Q(\phi,f) \in \mathcal{B}_0$.

**Proof of Theorem:** There is a characterization of $exI$ due to Feldman (1966: page 81) which says: $\mu \in exI$ if and only if $\mu(\mathcal{B}) = \{0,1\}$.

Hence if $\mu \in exI$, $\mu(\mathcal{B}) = \{0,1\}$ since $\mathcal{B}_0 \subseteq \mathcal{B}$, and both $X$ and the void set are in $\mathcal{B}_0$. Conversely, if $\mu(\mathcal{B}) = \{0,1\}$ let $A \in \mathcal{B}$. Let $\chi_A$ be the characteristic function of $A$. Let $\phi$ be a continuous real valued function on $G$, having compact support and such that $\int_G \phi(g)d\lambda(g) = 1$. Let $A_0 = Q(\phi, \chi_A) \in \mathcal{B}_0$ by the Lemma. It now follows, by an adaptation of the argument of Varadarajan [5] p. 1, that $\mu(A) = \mu(A_0) \in \{0,1\}$. Hence $\mu(\mathcal{B}) = \{0,1\}$.

If $G$ acts transitively on $X$, then $\mathcal{B}_0$ is the trivial $\sigma$-algebra and $\mu(\mathcal{B}_0) = \{0,1\}$ for all $\mu \in p(X)$. In this case $I \subseteq exI \subseteq I$ so that $I$ has at most one element.
References


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