ON HOMOMORPHISMS OF AN ORTHOGONALLY
DECOMPOSABLE HILBERT SPACE II

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Abstract

We present two more characterizations of maps which preserve orthogonal decompositions defined on Hilbert spaces ordered by natural cones.


Let $M$ be a von Neumann algebra on a Hilbert space $H$. We shall assume that there is a cyclic and separating vector $\xi_0 \in H$ for $M$. Then, by the Tomita-Takesaki theory, there are the conjugation operator $J$ and the modular operator $\Delta$ associated with $\xi_0$ such that

$$H^+ = \left\{ xj(x)\xi_0 : x \in M \right\} = \left\{ \Delta^{1/4}x\xi_0 : x \in M^+ \right\}$$

defines the "natural" positive cone of $H$, where $j(x) = JxJ$ and $M^+$ is the set of all positive elements of $M$. Then, every element $\xi$ of $H$ such that $\xi = J|\xi$ admits a unique orthogonal decomposition: $\xi = \xi^+ - \xi^-$, $\xi^+ \in H^+$, $\xi^- \in H^+$ and $(\xi^+, \xi^-) = 0$. For the details of these facts, see [1] and [2]. A continuous linear operator $\varphi : H \to H$ is called an o.d. homomorphism if $\varphi\xi = \varphi\xi^+ - \varphi\xi^-$ is also an orthogonal decomposition. This is equivalent to that $\varphi(H^+) \subset H^+$ and $(\varphi\xi, \varphi\eta) = 0$ whenever $\xi \in H^+$, $\eta \in H^+$ and $(\xi, \eta) = 0$. The following fact has been proved in [3].

**Theorem 1.** Let $\varphi : H \to H$ be a continuous linear operator. Then, $\varphi$ is an o.d. homomorphism if and only if $\varphi(H^+) \subset H^+$ and $\varphi^*\varphi \in M \cap M'$ (the center of $M$).
The aim of this note is to add two more characterizations of o.d. homomorphisms.

**Theorem 2.** Let $\phi: H \to H$ be a continuous linear operator such that $\phi(H^+) \subset H^+$. The following conditions are equivalent.

1. $\phi$ is an o.d. homomorphism.
2. $\phi^* x \phi \in M \cap M'$ for every $x \in M \cap M'$.

**Proof.** (1) $\Rightarrow$ (2). Let $x \in (M \cap M')^+$. When $\xi \in H^+$, $\eta \in H^+$ and $(\xi, \eta) = 0$, it follows from the condition (1) that $\phi \xi \in H^+$, $\phi \eta \in H^+$ and $(\phi \xi, \phi \eta) = 0$. Furthermore,

\[
(x^{1/2} \phi \xi, x^{1/2} \phi \eta) = (x^{1/2} p_{\phi \xi} \phi \xi, x^{1/2} p_{\phi \eta} \phi \eta) = (p_{\phi \xi} x^{1/2} \phi \xi, p_{\phi \eta} x^{1/2} \phi \eta) = 0,
\]

where $p_{\phi \xi} = [M' \phi \xi]$ and $p_{\phi \eta} = [M' \phi \eta]$ are cyclic projections. Since $x^{1/2}(H^+) \subset H^+$, this implies that $x^{1/2}$ is an o.d. homomorphism. Hence, by Theorem 1,

\[
\phi^* x \phi = (\phi^* x^{1/2})(x^{1/2} \phi) = (x^{1/2} \phi)^*(x^{1/2} \phi) \in M \cap M'.
\]

(2) $\Rightarrow$ (1). For $x = 1$, the identity of $M$, we have $\phi^* \phi \in M \cap M'$. Hence, by Theorem 1, $\phi$ is an o.d. homomorphism.

**Corollary 3.** Let $\phi: H \to H$ be a continuous linear operator such that $\phi(H^+) \subset H^+$. Then, if $\phi^* x \phi \in M$ for every $x \in M$, $\phi$ is an o.d. homomorphism.

**Proof.** By the Tomita-Takesaki theory, we have $J^* = J$ and $M' = j(M)$. Since $\phi(H^+) \subset H^+$, we have $\phi J = J \phi$. Hence, for any $x' \in M'$, we can take $x \in M$ such that $x' = j(x)$ and

\[
\phi^* x' \phi = \phi^* J x J \phi = J(\phi^* x \phi) J = j(M) = M'.
\]

It then follows from the assumption that $\phi^*(M \cap M') \phi \subset M \cap M'$. Hence, $\phi$ is an o.d. homomorphism by Theorem 2.

The second characterization has its origin in the following lemma which, when $\phi$ is a unitary operator, is due to [2].

**Lemma 4.** Let $\phi: H \to H$ be a continuous linear bijection such that $\phi(H^+) = H^+$. Then, for any cyclic and separating vector $\xi \in H^+$ for $M$, there is a unital Jordan $^*$-isomorphism $\alpha_{\phi, \xi}$ of $M$ such that

\[
\Delta_{\phi, \xi}^{1/4} \alpha_{\phi, \xi}(x) \phi \xi = \phi \left( \Delta_{\xi}^{1/4} x \xi \right) \quad \text{for all } x \in M,
\]

where $\Delta_{\xi}$ and $\Delta_{\phi, \xi}$ are the modular operators associated with the cyclic and separating vectors $\xi$ and $\phi \xi$ respectively.
Proof. Let \( H_\xi = \{ \eta \in H: -\lambda \xi \leq \eta \leq \lambda \xi \text{ for some } \lambda > 0 \} \), \( H_{\phi \xi} = \{ \eta \in H: -\phi \xi \leq \eta \leq \phi \xi \text{ for some } \lambda > 0 \} \) and \( M_h \) be the set of all self-adjoint elements of \( M \). Since \( \phi \) is bijective and \( \phi(H^+) = H^+ \), \( \phi \) maps \( H_\xi \) onto \( H_{\phi \xi} \) bijectively. On the other hand, by [1], Lemma 2.5.40, and [2], Proposition 1.2, there are bijective order isomorphisms

\[
\Delta^4_{\phi \xi} : M_h \to H_{\phi \xi} \quad \text{and} \quad \Delta^4_{\xi} : M_h \to H_\xi
\]
defined by

\[
\Delta^4_{\phi \xi}(x) = \phi(\Delta^4_\xi x \phi \xi) \quad \text{and} \quad \Delta^4_{\xi}(x) = \Delta^4_\xi x \xi
\]
for all \( x \in M_h \). Hence, we can define a bijection \( \alpha_{\phi, \xi} : M_h \to M_h \) by

\[
\Delta^4_{\phi \xi} \alpha_{\phi, \xi}(x) = \phi(\Delta^4_\xi x \xi) \quad \text{for all } x \in M_h.
\]
By the linearity, \( \alpha_{\phi, \xi}(x) \) is defined for all \( x \in M \). It satisfies \( \alpha_{\phi, \xi}(1) = 1 \) and \( \alpha_{\phi, \xi}(M^+) = M^+ \). Therefore, by a theorem of Kadison [4] (see also [1], Theorem 3.2.3), \( \alpha_{\phi, \xi} \) is a Jordan *-isomorphism.

We shall prove that a continuous linear operator \( \phi : H \to H \) such that \( \phi(H^+) = H^+ \) is an o.d. homomorphism if and only if \( \alpha_{\phi, \xi} = \alpha_{\phi, \xi} \) for every cyclic and separating vector \( \xi \in H^+ \) for \( M \), where \( \alpha_{\phi, \xi} = \alpha_{\phi, \xi} \). It is known that the equality \( \alpha_{\phi, \xi} = \alpha_{\phi, \xi} \) holds for a special class of o.d. homomorphisms. For example, it has been shown in [2], Theorem 3.2 (see also [1], Theorem 3.2.15) that, when \( u \) is a unitary operator such that \( u(H^+) = H^+ \), we have the equality \( \alpha_{\phi, \xi} = \alpha_{\phi, \xi} \) for every cyclic and separating vector \( \xi \in H^+ \) for \( M \), and, conversely, for any unital Jordan *-isomorphism \( \alpha_{\phi} : M \to M \), there is a unique unitary operator \( u_{\phi} \) such that

\[
u_{\phi}(\Delta^4_\xi x \xi) = \Delta^4_{\phi \xi} \phi_{\phi}(x) u_{\phi} \xi \quad \text{for all } x \in M
\]
for all cyclic and separating vector \( \xi \in H^+ \) for \( M \). These facts and the symbol \( u_{\phi} \) will be used in the following discussion.

By definition, an o.d. isomorphism is a continuous linear bijection \( \phi : H \to H \) such that \( \phi \) and \( \phi^{-1} \) are both o.d. homomorphisms. It has been proved in [3], (3.1), that bijective o.d. homomorphisms are o.d. isomorphisms. Obviously, a unitary operator \( u \) is an o.d. isomorphism if and only if \( u(H^+) = H^+ \).

Theorem 5. Let \( \phi : H \to H \) be a continuous linear bijection such that \( \phi(H^+) = H^+ \). The following conditions are equivalent.

1. \( \phi \) is an o.d. isomorphism.
2. For the polar decomposition \( \phi = u|\phi| \).
(i) $|\phi|$ is an o.d. isomorphism and $\alpha_{|\phi|, \xi} = 1$ for all cyclic and separating vector $\xi \in H^+$ for $M$.

(ii) $u$ is an o.d. isomorphism, $\alpha_u = a_u$ and $u = u_a$.

(3) $\alpha_u = \alpha_{|\phi|, \xi}$ for every cyclic and separating vector $\xi \in H^+$.

(4) $\|\phi^{-1}\|^{-1} u_{a}\xi \leqslant \phi \xi \leqslant \|\phi\| u_{a}\xi$ for all $\xi \in H^+$.

**Proof.** (1) $\Rightarrow$ (2). (i). Since $|\phi| \in (M \cap M')^+$ by Theorem 1, we have $|\phi|(H^+) = |\phi|^{1/2}((\phi|^{1/2})(H^+)) \subset H^+$. Hence, it follows from Theorem 1 that $|\phi|$ is an o.d. homomorphism. Since it is bijective, it is, in fact, an o.d. isomorphism. Now, let $\xi \in H^+$ be a cyclic and separating vector for $M$. Then, $|\phi|\xi$ is also a cyclic and separating vector in $H^+$ and, since $|\phi|$ is an invertible element of $(M \cap M')^+$, we have $\Delta_{|\phi|\xi}^{1/4} = \Delta_{\xi}^{1/4}$. Hence,

$$
|\phi| \left( \phi^{1/4}_\xi x \xi \right) = \Delta_{|\phi|\xi}^{1/4} x |\phi| \xi \quad \text{for all } x \in M.
$$

This is equivalent to $\alpha_{|\phi|, \xi}(x) = x$ for all $x \in M$. To prove (ii), we first note that $u = \phi|\phi|^{-1}$ is an o.d. isomorphism because $\phi$ and $|\phi|^{-1}$ are. Then, by ($\#$),

$$
\Delta_{|\phi|\xi_0}^{1/4} \alpha_{\phi}(x) \phi \xi_0 = \phi \left( \Delta_{\xi_0}^{1/4} x \xi_0 \right) = u|\phi| \left( \Delta_{\xi_0}^{1/4} x \xi_0 \right)
$$

$$
= u \left( \Delta_{|\phi|\xi_0}^{1/4} x |\phi| \xi_0 \right) = \Delta_{u|\phi|\xi_0}^{1/4} \alpha_u(x) \phi \xi_0
$$

for all $x \in M$. Therefore, $\alpha_u = \alpha_{\phi}$. Furthermore, for every $x \in M$,

$$
u_a \left( \Delta_{\xi_0}^{1/4} x \xi_0 \right) = \Delta_{u_a \xi_0}^{1/4} \alpha_u(x) u_{a}\xi_0 = \Delta_{u_a \xi_0}^{1/4} \alpha_u(x) u_{a}\xi_0
$$

$$
= u \left( \Delta_{u_a \xi_0}^{1/4} x u^* u_{a}\xi_0 \right),
$$

that is,

$$
u_a \left( \Delta_{\xi_0}^{1/4} x \xi_0 \right) = \Delta_{u_a \xi_0}^{1/4} x u^* u_{a}\xi_0,
$$

where $u^* u_{a}$ is a unitary operator such that $u^* u_{a}(H^+) = H^+$. This equation shows that the unital Jordan *-isomorphism determined by $u^* u_{a}$ is the identity map. Hence, $u^* u_{a} = 1$, or, $u = u_{a}$.

(2) $\Rightarrow$ (3). Let $\xi \in H^+$ be a cyclic and separating vector for $M$. Then, since $\alpha_{|\phi|, \xi} = 1$,

$$
\Delta_{|\phi|\xi}^{1/4} \alpha_{\phi}(x) \phi \xi = \phi \left( \Delta_{\xi}^{1/4} x \xi \right) = u|\phi| \left( \Delta_{\xi}^{1/4} x \xi \right)
$$

$$
= u \left( \Delta_{|\phi|\xi}^{1/4} x \xi \right) = \Delta_{\xi}^{1/4} \alpha_u(x) \phi \xi
$$

for all $x \in M$. This implies $\alpha_{\phi, \xi} = \alpha_u = \alpha_{\phi}$.

(3) $\Rightarrow$ (4). For any cyclic and separating vector $\xi \in H^+$,

$$
\phi \left( \Delta_{\xi}^{1/4} x \xi \right) = \Delta_{\phi \xi}^{1/4} \alpha_{\phi}(x) \phi \xi = u_a \left( \Delta_{u_a \phi \xi}^{1/4} x u^* \phi \xi \right)
$$
for every \( x \in M \). Therefore,
\[
\left\| \Delta_x^{1/4} x u_\alpha^* \phi \xi \right\| \leq \| \phi \| \left\| \Delta_x^{1/4} \xi \right\| \quad \text{for every} \ x \in M.
\]

By [2], Lemma 3.13, this inequality is equivalent to
\[
u_\alpha^* \phi \xi \leq \| \phi \| \xi.
\]

Since this inequality holds for every cyclic and separating vector \( \xi \in H^+ \) and such vectors are dense in \( H^+ \), we have
\[
u_\alpha^* \phi \xi \leq \| \phi \| \xi \quad \text{for every} \ \xi \in H^+.
\]

Since \( u_\alpha(H^+) = H^+ \), this is equivalent to
\[
\phi \xi \leq \| \phi \| u_\alpha \xi \quad \text{for every} \ \xi \in H^+.
\]

Starting with \( \phi^{-1} \) instead of \( \phi \), we arrive at
\[
\phi^{-1} \xi \leq \| \phi^{-1} \| u_\alpha^* \xi \quad \text{for every} \ \xi \in H^+.
\]

(4) \( \Rightarrow \) (1). We only need to show that \( (\phi \xi, \phi \eta) = 0 \) whenever \( \xi \in H^+, \ \eta \in H^+ \) and \( (\xi, \eta) = 0 \). However, this is obvious because \( u_\alpha \) satisfies this condition.

References


