THE SECOND DUAL OF $C_0(S, A)$

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Abstract

The second dual of the vector-valued function space $C_0(S, A)$ is characterized in terms of generalized functions in the case where $A^*$ and $A^{**}$ have the Radon-Nikodým property. As an application we present a simple proof that $C_0(S, A)$ is Arens regular if and only if $A$ is Arens regular in this case. A representation theorem of the measure $\mu h$ is given, where $\mu \in C_0^*(S, A)$, $h \in L_\infty(|\mu|, A^{**})$ and $\mu h$ is defined by the Arens product.


1. Introduction

Let $A$ be a Banach space and let $S$ be a locally compact Hausdorff topological space, $\mathcal{B}(S)$ be the $\sigma$-algebra of all the Borel sets of $S$. The space of continuous functions from $S$ to $A$ vanishing at infinity, endowed with the uniform norm, is denoted by $C_0(S, A)$. The second dual of $C_0(S, A)$ is considered in the case where $S$ is compact and the dual $A^*$ has the Radon-Nikodým property in [3], and for the case where $S$ is locally compact and $A$ is a Banach algebra with a positive cone satisfying certain conditions in [5]. When $A = \mathbb{C}$, the complex numbers, the second dual of $C_0(S, A)$ is characterized by means of generalized functions in [12].

In Section 2, a characterization of $C_{0}^{**}(S, A)$ by means of generalized functions is given, in the case where $A^*$ and $A^{**}$ have the Radon-Nikodým property.

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Recall that both $A^*$ and $A^{**}$ have the Radon-Nikodým property if any one of the following conditions is satisfied:

(i) $A$ is reflexive,
(ii) $A^{**}$ is separable,
(iii) $A^{**}/A$ is separable (see [6, p. 219], for example).

Let $A$ be a Banach algebra and $F, G$ be in $C_0^*(S, A)$. Denote by $FG$ and $F.G$ the left and right Arens products of $F$ and $G$ in $C_0^*(S, A)$ respectively. A Banach algebra is called Arens regular if the two Arens products coincide. It has been shown recently in [11] that if $S$ is compact and $A$ is Arens regular, then $C(S, A)$ is Arens regular. Since the technique used in [11] is quite complicated and $C_0(S, A)$ is a very important Banach algebra, as an application of the results in Section 2, we present a simpler proof of the Arens regularity of $C_0(S, A)$ in Section 3, assuming of course $A^*$ and $A^{**}$ both have the Radon-Nikodým property.

Let $T : C_0(S, A) \to \mathbb{C}$ be a bounded linear operator. The representing measure $m : \mathcal{B}(S) \to A^*$ is a weakly compact measure and it is shown in [8, p. 54] that the total variation $|m|$ and the semivariation $\tilde{m}$ of $m$ are the same in this case. Let $MW(S, A^*)$ and $D(S, A^*)$ be the collection of all weakly compact measures and dominated measures respectively. Then

$$C_0^*(S, A) = MW(S, A^*) = D(S, A^*)$$

(see [2] and [5], for example). Since $|m|(S) = \tilde{m}(S) = ||T||$ is finite, we see that $|m| \in M(S)$, the space of all bounded regular Borel measures on $S$ [2, Theorem 2.8].

If $\mu \in MW(S, A^*)$ and $h \in L_\infty(|\mu|, A^{**})$, then $h$ can be viewed as an element in $C_0^*(S, A)$ (see Lemma 3) and $\mu h$ defined by the right Arens product is an element in $C_0^*(S, A)$. A representation theorem for $\mu h$, when considered as a measure, is given in Section 4.

Throughout this paper we assume that $A^*$ and $A^{**}$ have the Radon-Nikodým property and we follow the standard notation of Arens product used in Duncan and Hosseinium [9]. The bilinear integration theory used in this paper is developed in Dinculeanu [8].

2. The dual of $MW(S, A^*)$

Let $L_1(|m|, A^*)$ be the space of all the equivalence classes of $A^*$-valued Bochner integrable functions defined on $S$. Then the dual

$$L_1(|m|, A^*)^* = L_\infty(|m|, A^{**})$$
if and only if $A^{**}$ has the Radon-Nikodým property (see, for example, Diestel and Uhl [6], where $L^\infty(|m|, A^{**})$ stands for the space of equivalence classes of $A^{**}$-valued Bochner integrable functions defined on $S$ that are $|m|$-essentially bounded, that is, such that

$$
\|f\|_{m, \infty} = \inf \{\sup \{\|f(x)\| : x \notin N\} : |m|(N) = 0\} \\
= \inf \{c > 0 : |m|\{x \in S : \|f(x)\| > c\}) = 0\} < \infty.
$$

Consider the product linear space $\prod\{L^\infty(|m|, A^{**}) : m \in MW(S, A^*)\}$. An element $f = (f_m)_{m \in MW(S, A^*)}$ in this product is called a generalized function on $S$ provided

(i) $\|f\| = \sup \|f_m\|_{m, \infty} : m \in MW(S, A^*) < \infty$,

(ii) if $\mu, \nu \in MW(S, A^*)$ are such that $|\mu| \ll |\nu|$, then $f_\mu = f_\nu \ |\mu|$-a.e.

Here $|\mu| \ll |\nu|$ means $|\mu|$ is absolutely continuous with respect to $|\nu|$. It is easy to see that condition (ii) above is meaningful for the equivalence classes of functions.

Let $GL(S, A^{**})$ denote the linear subspace of $A^{**}$-valued generalized functions on $S$. It is easy to verify that $GL(S, A^{**})$ is a Banach space with norm $\|f\| = \sup\{\|f_\mu\|_{\mu, \infty} : \mu \in MW(S, A^*)\}$.

**DEFINITION.** Let $\mu, \nu : \mathcal{B}(S) \rightarrow A^*$ be $\in MW(S, A^*)$. If $\lim_{|\mu|(E) \rightarrow 0} \nu(E) = 0$, then $\nu$ is called $|\mu|$-continuous and is denoted by $\nu \ll |\mu|$.

**THEOREM 1.** for each bounded linear functional $F \in MW(S, A^*)^*$, there is a unique generalized function $f \in GL(S, A^*)$ such that

$$
F(\mu) = \int f_\mu \, d\mu \quad (\mu \in MW(S, A^*))
$$

and $\|F\| = \|f\|$.

**PROOF.** For $\mu \in MW(S, A^*)$, $F$ induces a bounded linear functional $F_\mu$ on

$$
L_1(|\mu|, A^*) = \{\nu \in MW(S, A^*) : \nu \ll |\mu|\},
$$

since $A^*$ has the Radon-Nikodým property. Let $\nu \leftrightarrow f_\nu$ be the corresponding mapping. Now, since $A^{**}$ has the Radon-Nikodým property, we see that $L_1(|\mu|, A^*)^* = L^\infty(|\mu|, A^{**})$ and that there is $f_\mu \in L^\infty(|\mu|, A^{**})$ such that

$$
F_\mu(\nu) = F(\nu) = \int (f_\nu, f_\mu) \, d|\mu|,
$$

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for any $v \in L_1(|\mu|, A^*)$ [6, pages 98–99]. Since

$$v(E) = \int_E f^v \, d|\mu|,$$

and the countably valued functions in $L_\infty(|\mu|, A^{**})$ are dense in $L_\infty(|\mu|, A^{**})$ (see [6, p. 97], for example), we see that

$$\int (f^v, f_\mu) \, d|\mu| = \int f_\mu \, dv.$$ 

In particular $F(\mu) = \int f_\mu \, d\mu.$

We shall show that $f = (f_\mu)_{\mu \in MW(S, A^*)}$ is a generalized function. Let $\mu, \nu \in MW(S, A^*)$ such that $|\mu| \ll |\nu|$. For $\gamma \in L_1(|\mu|, A^*)$, we have $\gamma \ll |\mu|, \nu$. Hence

$$\int f_\mu \, d\gamma = F_\mu(\gamma) = F(\gamma) = F_\nu(\gamma) = \int f_\nu \, d\gamma.$$ 

Thus $f_\mu = f_\nu |\mu|-a.e.$ Also, for $\mu \in MW(S, A^*)$,

$$\|f_\mu\|_{\mu, \infty} = \|F_\mu\| = \sup\{|F_\mu(\nu)| : \nu \in L_1(|\mu|, A^*), \|\nu\| \leq 1\} \leq \|F\|.$$ 

On the other hand,

$$\|F\| = \sup\{|F(\nu)| : \nu \in L_1(|\mu|, A^*), \|\nu\| \leq 1\}$$

$$= \sup\{|\int f_\nu \, d\nu| : \nu \in L_1(|\mu|, A^*), \|\nu\| \leq 1\}$$

$$\leq \sup\{\|f_\nu\|_{\nu, \infty} \cdot \|\nu\| \leq \|f\|.$$ 

Hence $f \in GL(S, A^{**})$ and $\|F\| = \|f\|.$

Let $i$ be the identity of $A$. It is shown in [4, Lemma 2.4] that $(1_S \otimes i)$ is the identity of $C_0^*(S, A)$. Let $I = (I_\mu)_{\mu \in MW(S, A^*)}$ be defined by $I_\mu = i$ for all $\mu \in MW(S, A^*)$. Then $I$ is the identity in $GL(S, A^{**}).$

**Theorem 2.** Let $T : GL(S, A^{**}) \rightarrow MW(S, A^*)^*$ be defined by

$$Tf(\mu) = \int f_\mu \, d\mu \quad (\mu \in MW(S, A^*), \ f \in GL(S, A^{**})).$$

Then $T$ is an isometric isomorphism of $GL(S, A^{**})$ onto $MW(S, A^*)^*$ and $TI = (1_S \otimes i).$
PROOF. Let \( f \in GL(S, A^{**}) \). For \( \mu, \nu \in MW(S, A^*) \), we see \(|\mu|, |\nu| \in M(S)\). Let \( x^* \) be any element in \( A^* \) with \( ||x^*|| = 1 \). Set \( \tau = (|\mu| + |\nu|)x^* \). Then \( \tau \in MW(S, A^*) \). Clearly \(|\mu + \nu| \ll |\tau|, |\mu| \ll |\tau|, |\nu| \ll |\tau|\). Hence

\[
Tf(\mu + \nu) = \int f_{\mu + \nu} d(\mu + \nu) = \int f_\tau d(\mu + \nu)
= \int f_\tau d\mu + \int f_\tau d\nu
= \int f_\mu d\mu + \int f_\nu d\nu
= Tf(\mu) + Tf(\nu)
\]

and, for \( \alpha \in \mathbb{C} \),

\[
Tf(\alpha \mu) = \int f_{\alpha \mu} d(\alpha \mu) = \alpha \int f_{\alpha \mu} d\mu
= \alpha \int f_\mu d\mu = \alpha Tf(\mu),
\]

since \(|\mu| \ll |\alpha \mu|\). Thus \( Tf \) is linear. Now

\[
||Tf|| = \sup\{|Tf(\mu)| : \mu \in MW(S, A^*), ||\mu|| \leq 1\}
\leq \sup\{||f_\mu||_{\mu, \infty} \cdot ||\mu|| : ||\mu|| \leq 1\} \leq ||f||.
\]

Thus \( Tf \) is a bounded linear functional on \( MW(S, A^*) \). Since \( T \) is onto, Theorem 1 shows that it is an isometric isomorphism.

Furthermore, for \( \mu \in MW(S, A^*) \),

\[
(TI)(\mu) = \int I_\mu d\mu = i(\mu(S)) = \mu(S)i
= \mu^**(1_S \otimes i) = (1_S \otimes i)(\mu)
\]

(the above first equality is implicitly implied in the proof of [2, Theorem 2.2]). Thus \( TI = 1_S \otimes i \).

3. Arens regularity of \( C_0(S, A) \)

Although \( L_\infty(|\mu|, A^{**}) \) consists of equivalence classes of functions, the next lemma shows that we can view each element in \( L_\infty(|\mu|, A^{**}) \) as an element in \( C_0^{**}(S, A) \).
LEMMA 3. Every element $f$ in $L_\infty(|\mu|, A^{**})$ can be considered as an element $f$ in $C_0^{**}(S, A)$. If $F, G \in C_0^{**}(S, A)$, then $\mu^{**}(F) = \mu^{**}(G)$ if and only if $F$ and $G$ agree on $L_1(|\mu|, A^*)$, where $\mu^{**}$ is the second adjoint of $\mu$, as a linear functional on $C_0^{**}(S, A)$.

PROOF. Since $A^*$ and $A^{**}$ have the Radon-Nikodym property,

$$
L_\infty(|\mu|, A^{**}) = L_1(|\mu|, A^*)^*,
$$

$$
L_1(|\mu|, A^*) = \{v \in MW(S, A^*) : v \ll |\mu| \}
$$

$$
\subseteq MW(S, A) = C_0^*(S, A).
$$

Thus by the Hahn-Banach theorem, each $f \in L_\infty(|\mu|, A^{**})$ can be extended to an element in $C_0^{**}(S, A)$. For simplicity we again use $f$ to denote the extension. Note that the extension is not unique and that all these extensions agree on $L_1(|\mu|, A^*)$.

Suppose now that $F, G \in C_0^{**}(S, A)$ agree on $L_1(|\mu|, A^*)$. Then, for any complex number $\alpha$,

$$
\mu^{**}(F)(\alpha) = F(\mu^{*}(\alpha)), \quad \mu^{**}(G)(\alpha) = G(\mu^{*}(\alpha)).
$$

Now for every $h \in C_0(S, A)$, $\mu^{*}(\alpha)(h) = \alpha \mu(h)$. We conclude that $\mu^{*}(\alpha) \ll |\mu|$, and so

$$
\mu^{**}(F)(\alpha) = F(\mu^{*}(\alpha)) = G(\mu^{*}(\alpha)) = \mu^{**}(G)(\alpha).
$$

Thus $\mu^{**}(F) = \mu^{**}(G)$.

On the other hand, suppose now that $\mu^{**}(F) = \mu^{**}(G)$. Let $f, g \in GL(S, A^{**})$ such that $Tf = F, Tg = G$. Then,

$$
Tf(\mu) = \int f_\mu d\mu, \quad Tg(\mu) = \int g_\mu d\mu.
$$

Let $v \in L_1(|\mu|, A^*)$. Then $v \ll |\mu|$ and so

$$
F(v) = Tf(v) = \int f_v dv
$$

$$
= \int f_\mu dv \quad (|v| \ll |\mu|),
$$

$$
G(v) = Tg(v) = \int g_v dv
$$

$$
= \int g_\mu dv \quad (|v| \ll |\mu|).$$
Since \( \mu^{**}(F) = \mu^{**}(G) \), we see that \( f_\mu = g_\mu |\mu|-\text{a.e.} \). Now \( \nu \in L_1(|\mu|, A^*) \) implies that \( f_\mu = g_\mu |v|-\text{a.e.} \). Thus

\[
F(\nu) = \int f_\mu \, d\nu = \int g_\mu \, d\nu = G(\nu),
\]
completing the proof.

**REMARKS 4.**

1. If \( f \) and \( g \) belong to the same equivalence class in \( L_\infty(|\mu|, A^{**}) \), they must agree on \( L_1(|\mu|, A^*) \).
2. For \( \mu \in MW(S, A^*) \), \( f \in GL(S, A^{**}) \), Theorems 1 and 2 imply, in view of Lemma 3, that

\[
\mu^{**}(f_\mu) = f_\mu(\mu) = \int f_\mu \, d\mu = L_\infty(|\mu|, A^{**}) = L_1(|\mu|, A^*)^*
\]

Conversely, if \( h \in C_0^*(S, A) \) and \( f \in GL(S, A^{**}) \) are such that \( T\mu = h \), then

\[
\mu^{**}(h) = h(\mu) = Tf(\mu) = \int f_\mu \, d\mu = \mu^{**}(f_\mu).
\]

From Lemma 3, we see that \( f_\mu = h \) on \( L_1(|\mu|, A^*) \) for each \( \mu \in MW(S, A^*) \).

**DEFINITION.** For \( F, G \in C_0^{**}(S, A) \), there are \( f, g \in GL(S, A^{**}) \) such that \( T\mu = F, \ Tg = G \) by Theorem 2. For \( \mu \in MW(S, A^*) \), define \( F \times \mu \in C_0^{**}(S, A) \) by

\[
(F \times \mu)(\mu) = \mu^{**}(hf_\mu), \quad (h \in C_0^*(S, A)).
\]

Then \( F \times \mu \in C_0^*(S, A) \) in particular. Note that the above definition is independent of the extension of \( f_\mu \) to \( C_0^{**}(S, A) \). In fact, if \( F = F' \) and \( G = G' \) on \( L_1(|\mu|, A^*) \), then \( FG = F'G \) and \( FG = Fg' \) also on \( L_1(|\mu|, A^*) \) because \( G\mu \) is in \( L_1(|\mu|, A^*) \) and \( G\mu = G'\mu \) (since \( \mu f \in L_1(|\mu|, A^*) \) for \( f \in C_0(S, A) \)). Let \( F \times G \) be the element in \( MW(S, A^*) \) defined by

\[
(F \times G)(\mu) = F(G \times \mu), \quad (\mu \in MW(S, A^*)).
\]

**THEOREM 5.** Let \( F, G \in C_0^{**}(S, A) \). Then \( F \times G = FG \), where \( FG \) is the left Arens product in \( C_0^*(S, A) \).
PROOF. For $\mu \in MW(S, A^*)$, $(F \times G)(\mu) = F(G \times \mu)$ and $(FG)(\mu) = F(G\mu)$. For any $h \in C_0(S, A)$,

$$(G \times \mu)(h) = \mu^{**}(hg) = (hg)(\mu) = h(g\mu) = (g\mu)(h),$$

and

$$(G\mu)(h) = G(\mu h) = T g(\mu h) = \int g_{\mu h} d(\mu h)$$

$$= \int g_{\mu} d(\mu h) \quad (|\mu h| \ll |\mu|)$$

$$= (\mu h)^{**}(g_{\mu}) = g_{\mu}(\mu h) = (g\mu)(h). \quad \text{(Remark 4(2))}$$

Thus $F \times G = FG$.

Similarly, for $\mu \in MW(S, A^*)$ define $\mu \otimes F \in C_0^{**}(S, A)$ by

$$(h)(\mu \otimes F) = \mu^{**}(f_{\mu}h), \quad (h \in C_0^{**}(S, A)).$$

Then $\mu \otimes F \in C_0^*(S, A)$ in particular. Define $F \otimes G \in C_0^{**}(S, A)$ by $(\mu)(F \otimes G) = G(\mu \otimes F)$. Then $F \otimes G = F \cdot G$, the right Arens product in $C_0^{**}(S, A)$.

THEOREM 6. $C_0(S, A)$ is Arens regular if and only if $A$ is Arens regular.

PROOF. We shall show that $F \times G = F \otimes G$ for $F, G \in MW(S, A^*)^*$. Let $f, g \in GL(S, A^{**})$ be such that $Tf = F$, $Tg = G$. For $\mu \in MW(S, A^*)$,

$$(Tf \times Tg)(\mu) = Tf(Tg \times \mu) = \int f_{Tg \times \mu} d(Tg \times \mu)$$

$$= \int f_{\mu} d(Tg \times \mu) \quad (|Tg \times \mu| \ll |\mu|)$$

$$= (Tg \times \mu)^{**}(f_{\mu}) \quad \text{(Remark 4(2))}$$

$$= (Tg \times \mu)(f_{\mu}) = \mu^{**}(f_{\mu}g_{\mu}).$$

On the other hand, $T(fg)(\mu) = \mu^{**}(fg)$, by Remark 4(2). Thus $Tf \times Tg = T(fg)$.

Now we shall show that $Tf \otimes Tg = T(f \cdot g)$,

$$(\mu)(Tf \otimes Tg) = Tg(\mu \otimes Tf) = \int g_{\mu \otimes Tf} d(\mu \otimes Tf)$$

$$= \int g_{\mu} d(\mu \otimes Tf) \quad (|\mu \otimes Tf| \ll |\mu|)$$

$$= (g_{\mu})(\mu \otimes Tf) = \mu^{**}(fg).$$
On the other hand, \( T(f \cdot g)(\mu) = \mu^{**}(f \mu \cdot g_\mu) \) by Remark 4(2). Thus \( Tf \otimes Tg = T(f \cdot g) \).

Since \( A \) is Arens regular and \( f \cdot g \) take values in \( A^{**} \), \( (f \cdot g)(x) = (fg)(x) \). We conclude that \( C_0(S, A) \) is Arens regular if \( A \) is. The rest of the proof of the theorem is clear.

### 4. Representation theorems

It is easy to verify that if \( F_1, F_2 \in C_0^{**}(S, A) \) agree on \( L_1(|\mu|, A^*) \subset C_0^*(S, A) \), then the left and right Arens products satisfy \( F_1 \mu = F_2 \mu, \mu F_1 = \mu F_2 \) respectively. Thus if \( h \in L_\infty(|\mu|, A^{**}) \) and we consider it as an element in \( C_0^{**}(S, A) \) by Lemma 3, then the left and right Arens products, \( h \mu \) and \( \mu h \), are well-defined.

Similarly, if \( G_1, G_2 \in C_0^{**}(S, A) \) agree on \( L_1(|\mu|, A^*) \) then \( G_1 F_1(\mu) = G_2 F_2(\mu) \) and \( (\mu)G_1 \cdot F_1 = (\mu)G_2 \cdot F_2 \), respectively. Thus if \( f, g \in L_\infty(|\mu|, A^{**}) \), the values of the Arens products \((\mu)f \cdot g\) and \((fg)(\mu)\) are uniquely determined.

**DEFINITION.** For \( \mu \in MW(S, A^*), h \in L_\infty(|\mu|, A^{**}), \) define \( \mu h \in MW(S, A^*) \) by

\[
\int g \, d\mu h = \int hg \, d\mu \quad (g \in C_0(S, A))
\]

by the Riesz Representation Theorem.

**THEOREM 7.** Let \( h \in L_\infty(|\mu|, A^{**}). \) Then \( \mu h = \mu h \).

**PROOF.** Let \( g \in C_0(S, A) \) and let \( \pi \) be the canonical embedding of \( C_0(S, A) \) into \( C_0^{**}(S, A) \). Then

\[
(g)\mu h = (g\mu)h = h(\pi(g)\mu) = h\pi(g)(\mu) = \mu^{**}(h\pi(g)),
\]

since \( g\mu = \pi(g)\mu \).

On the other hand, we see from Remark 4(2) that \( \mu_h(g) = \int hg \, d\mu = \mu^{**}(hg) \). Thus \( \mu_h = \mu h \).

**THEOREM 8.** For \( g, h \in L_\infty(|\mu|, A^{**}), \int g \, d(\mu h) = \int h \cdot g \, d\mu, \) where \( h \cdot g(s) \) is defined by the right Arens product in \( A^{**}, \) that is, \( h \cdot g(s) = h(s)g(s) \) for \( s \in S. \)
PROOF. We see that from Remark 4(2) that
\[
\int g \, d(\mu h) = (\mu h)^{**}(g)
\]
\[
= (\mu h)g = (\mu) \cdot g
\]
\[
= \mu^{**}(h \cdot g) = \int h \cdot g \, d\mu.
\]

Similarly, let \( h\mu \in MW(S, A^*) \) be defined by
\[
\int g \, d(h\mu) = \int gh \, d\mu \quad (g \in C_0(S, A)).
\]

Then we have

THEOREM 9. Let \( \mu \in MW(S, A^*) \) and let \( g, h \in L_\infty(\mu, A^{**}) \). Then \( h\mu = \mu \) and
\[
\int g \, d(h\mu) = \int gh \, d\mu,
\]
where \( gh(s) \) is defined by the left Arens product in \( A^{**} \).

REMARKS 10.
(1) Since the \( A^{**} \)-valued simple functions are in \( L_\infty(\mu, A^{**}) \) it is not difficult to verify that Theorems 8 and 9 hold for \( g \in L_\infty(\mu, A^{**}), h \in L_1(\mu, A^{**}) \) or vice versa.
(2) If \( h \in C_0^{**}(S, A) \) and \( f \in GL(S, A^{**}) \) are such that \( h = Tf \), then \( h = f_\mu \) on \( L_1(\mu, A^*) \) by Remark 4(2). Thus \( \mu h = \mu f_\mu \) and \( h\mu = f_\mu \mu \) respectively. Hence Theorems 8 and 9 are valid for \( h \in C_0^{**}(S, A) \), that is,
\[
\int g \, d(\mu h) = \int f_\mu \cdot g \, d\mu, \quad \int g \, d(h\mu) = \int gf_\mu \, d\mu.
\]
(3) Theorems 8 and 9 are not valid for general \( g, h \in L_1(\mu, A^{**}) \), since \( hg \) need not be in \( L_1(\mu, A^{**}) \) even for \( A = \mathbb{C} \).

References
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