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TORSION BIRATIONAL MOTIVES OF SURFACES AND UNRAMIFIED COHOMOLOGY

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In memory of Noriyuki Suwa

Abstract Let S and T be smooth projective varieties over an algebraically closed field k. Suppose that S is a surface admitting a decomposition of the diagonal. We show that, away from the characteristic of k, if an algebraic correspondence $T \to S$ acts trivially on the unramified cohomology, then it acts trivially on any normalized, birational and motivic functor. This generalizes Kahn's result on the torsion order of S. We also exhibit an example of S over $\mathbb{C}$ for which $S \times S$ violates the integral Hodge conjecture.

1. Introduction

Let k be an algebraically closed field, and let $\mathbf{Chow}^{\mathrm{eff}}_{\mathbb{Z}}$ be the covariant category of effective Chow motives over k with $\mathbb{Z}$-coefficients. Until §1.3 we assume the characteristic p of k is zero for simplicity, although most results remain valid away from p if $p > 0$.

1.1. Main exact sequence

Recall that a smooth projective variety X over k is said to admit a *decomposition of the diagonal* if the degree map induces an isomorphism $\mathrm{CH}_0(X_{k(X)}) \otimes \mathbb{Q} \cong \mathbb{Q}$, where $k(X)$

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denotes the total ring of fractions of X. This condition implies that X is connected, and $H^0(X,\Omega^1_{X/k}) = H^0(X,\Omega^2_{X/k}) = 0$. If $\dim X = 2$, Bloch's conjecture predicts the converse (see §2.6 for details).

Let S be a projective smooth surface over k which admits a decomposition of the diagonal. In his paper [29], Kahn introduced a new category $\mathbf{Chow}^{\mathrm{nor}}_{\mathbb{Z}}$, the category of normalized birational motives, which is defined as a quotient category of $\mathbf{Chow}^{\mathrm{eff}}_{\mathbb{Z}}$ and has the property that there is a canonical isomorphism

$$\mathbf{Chow}^{\mathrm{nor}}_{\mathbb{Z}}(T,S) \cong \mathrm{CH}_0(S_{k(T)})_{\mathrm{Tor}} \qquad \text{(cf. (6.2))}$$

for any smooth projective variety T over k. By this isomorphism for $T = S$, the motive of S is a torsion object in $\mathbf{Chow}^{\mathrm{nor}}_{\mathbb{Z}}$ (cf. Definition 2.13). To compute its order, he established an exact sequence

$$0 \to \mathbf{Chow}^{\mathrm{nor}}_{\mathbb{Z}}(S,S) \to \mathrm{Tor}(H^1_{\mathrm{ur}}(S), H^2_{\mathrm{ur}}(S))^{\oplus 2} \to H^3_{\mathrm{ur}}(S \times S) \to 0 \tag{1.1}$$

in [29, Corollary 6.4(a)]; cf. Example 7.6 below. Here, for a smooth scheme X over k and $i \in \mathbb{Z}_{>0}$, $H^i_{\mathrm{ur}}(X)$ is the *unramified cohomology* of X, defined as follows:

$$H^i_{\mathrm{ur}}(X) := H^0_{\mathrm{Zar}}(X, \mathscr{H}^i), \tag{1.2}$$

where $\mathscr{H}^i$ is the Zariski sheaf on X associated to the presheaf $U \mapsto H^i_{\text{ét}}(U, \mathbb{Q}/\mathbb{Z}(i-1))$. As is well known, we have $H^1_{\mathrm{ur}}(X) \cong H^1_{\text{ét}}(X, \mathbb{Q}/\mathbb{Z})$ and $H^2_{\mathrm{ur}}(X) \cong \mathrm{Br}(X)$, the Brauer group of X (see §2.5 for details).

Kahn deduced (1.1) by applying $T = S$ to a complicated result [29, Theorem 6.3] that involves $\mathbf{Chow}^{\mathrm{nor}}_{\mathbb{Z}}(T,S)$ for a general smooth projective variety T over k. Attempting to foster a better understanding of it, we found the following simple statement. (See Remark 7.2 below for more discussion.)

Theorem 1.1 (Theorem 7.1). *Let k and S be as above, and let T be a smooth projective variety over k. Then there is an exact sequence*

$$0 \to \mathbf{Chow}^{\mathrm{nor}}_{\mathbb{Z}}(T,S) \to \bigoplus_{i=1,2} \mathrm{Hom}(H^i_{\mathrm{ur}}(S), H^i_{\mathrm{ur}}(T)) \to H^3_{\mathrm{ur}}(S \times T) \to 0. \tag{1.3}$$

We shall prove the exactness of (1.3) by computing the image of the cycle class map

$$\mathrm{CH}_0(S_{k(T)})_{\mathrm{Tor}} \longrightarrow H^4_{\text{ét}}(S_{k(T)}, \mu_m^{\otimes 2})$$

for a sufficiently large m, using Vishik's method [44], which gives an alternative proof of (1.1).

1.2. Motivic, birational and normalized functors

Recall from [29] that a contravariant functor F defined on the category of smooth projective varieties over k and with values in the category of abelian groups is called

- *motivic* if F factors through an additive functor on $\mathbf{Chow}^{\mathrm{eff}}_{\mathbb{Z}}$,
- *birational* if $F(f)$ is an isomorphism for any birational morphism f, and
- *normalized* if $F(\operatorname{Spec} k) = 0$.

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A normalized, birational and motivic functor is equivalent to a functor which factors through an additive functor on $\mathbf{Chow}^{\mathrm{nor}}_{\mathbb{Z}}$. See §2.4 for details. Fundamental examples of such functors include $H^0(-,\Omega^i_{-/k})$ for $i > 0$ and the unramified cohomology (1.2). We deduce the following result from the injectivity of the first map in (1.3):

Theorem 1.2 (Theorem 7.3). *Let S and T be smooth projective varieties over k. Suppose that S admits a decomposition of the diagonal and $\dim S = 2$. Let $f : T \to S$ be an algebraic correspondence such that $H^i_{\mathrm{ur}}(f) : H^i_{\mathrm{ur}}(S) \to H^i_{\mathrm{ur}}(T)$ vanishes for $i = 1,2$. Then $F(f) : F(S) \to F(T)$ vanishes for any normalized, birational and motivic functor F.*

Theorem 1.2 will be applied to the K3 cover $f : T \to S$ of an Enriques surface S over $\mathbb{C}$ to interpret Beauville's result [4] in Example 7.5 below.

1.3. Explicit computation of $\mathrm{CH}_0(S_{k(S)})_{\mathrm{Tor}}$ and $H^3_{\mathrm{ur}}(S \times S)$

The groups appearing in (1.3) attracted some attention. Kahn [29, p. 840, footnote] raised a question asking the structure of $\mathrm{CH}_0(S_{k(S)})_{\mathrm{Tor}}$ for an Enriques surface S. The group $H^3_{\mathrm{ur}}(X)$ for a smooth projective variety X over $\mathbb{C}$ is studied by many authors since it gives an obstruction to the integral Hodge conjecture by a theorem of Colliot-Thélène and Voisin [14] (see Theorem 7.12). Therefore, there is some interest in making each term in (1.3) explicit. In this direction, we obtain the following result.

Theorem 1.3 (Theorem 7.8). *Let S be a smooth projective surface over k having a decomposition of the diagonal. Suppose moreover that $H^1_{\mathrm{ur}}(S)$ is a cyclic group of prime order ℓ. Then we have*

$$|\mathrm{CH}_0(S_{k(S)})_{\mathrm{Tor}}| = |H^3_{\mathrm{ur}}(S \times S)| = \ell.$$

This applies to an Enriques surface S (with $\ell = 2$), thereby answering Kahn's question. (See Example 7.13 for this point and for more examples.) It also provides us with counterexamples for the integral Hodge conjecture (see Corollary 7.11).

1.4. A remark on the p-part in characteristic $p > 0$

Suppose now that k has characteristic $p > 0$. As alluded to in the beginning of the introduction, most of our proof works over k for the non-p-primary torsion part, with the help of an isomorphism $\mathbb{Z}/m\mathbb{Z} \cong \mu_m$ for $m \in \mathbb{Z}_{>0}$ invertible in k.

To pursue a p-primary analogue of our arguments, one may consider a p-adic counterpart of the unramified cohomology, which is defined, for $i,j \in \mathbb{Z}_{\geq 0}$ and a smooth k-scheme X, as

$$H^{i,j}_{\mathrm{ur}}(X)\{p\} := \varinjlim_{n \geq 1} H^0_{\mathrm{Zar}}(X, \mathscr{H}^{i,j}_{p^n}).$$

Here, $\mathscr{H}^{i,j}_{p^n}$ is the Zariski sheaf on X_{Zar} associated to the presheaf $U \mapsto H^{i-j}_{\text{ét}}(U, W_n\Omega^j_{U,\log})$, and $W_n\Omega^j_{U,\log}$ is the étale subsheaf of the logarithmic part of the Hodge-Witt sheaf $W_n\Omega^j_U$ (see [28]). The functors $H^{i,j}_{\mathrm{ur}}(-)\{p\}$ are birational and motivic by [32, Proposition 1.3] and Proposition 9.1 below, and normalized for $(i,j) \neq (0,0)$. However, the groups $H^{i,j}_{\mathrm{ur}}(S)\{p\}$ do *not* necessarily detect the p-primary torsion part $\mathrm{CH}_0(S_{k(T)})_{p\text{-Tor}}$. In fact, when S is

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a supersingular Enriques surface over k with $\mathrm{ch}(k) = 2$, we have $H^{i,j}_{\mathrm{ur}}(S)\{2\} = 0$ for all $(i,j) \neq (0,0)$, but $\mathrm{CH}_0(S_{k(S)})_{2\text{-Tor}}$ is nonzero. We will discuss this example in detail, later in Remark 3.9 (2) below.

Organization of the paper

§2 is a recollection on the Chow motives and birational motives. We then study a torsion direct summand of the Chow motive of a surface admitting a decomposition of the diagonal in §3. A key result is Proposition 3.6. §4 is devoted to a preliminary computation of cohomology of torsion motive of a surface. In §5, we employ the method of Vishik [44] to study the motivic cohomology of a torsion motive constructed in §3. This result is then applied to deduce an exact sequence in §6, which relates the Chow group $\mathrm{CH}_0(S_{k(S)})_{\mathrm{Tor}}$ appearing in Theorems 1.1 and 1.3 with the unramified cohomology. The main results (Theorems 7.1, 7.3, 7.8) are proved in §7, which also contains a discussion of examples and related topics. §8 is an appendix where we prove elementary results on homological algebra that are used in the body of the paper. Another appendix §9 contains a proof of the proposition saying that a $\mathbb{P}^1$-invariant Nisnevich sheaf with transfer is a motivic and birational functor.

Notations and conventions

We use the following notations throughout this paper.

- k is a field, which will be assumed to be algebraically closed from §3 onward.
- p is the characteristic of k if it is positive, and $p := 1$ otherwise.
- Λ is either $\mathbb{Z}, \mathbb{Z}[1/p]$ or $\mathbb{Q}$. From §3 onward, we assume $\Lambda = \mathbb{Z}[1/p]$.

Notations relative to k.

- **Fld** is the category of fields over k and k-homomorphisms. Denote by $\mathbf{Fld}^{\mathrm{fg}}$ (resp. $\mathbf{Fld}^{\mathrm{ac}}$) its full subcategory consisting of those which are finitely generated over k (resp. algebraically closed).
- **Sch** is the category of separated k-schemes of finite type and k-morphisms. Its full subcategory consisting of smooth (resp. smooth and projective) k-schemes is denoted by **Sm** (resp. **SmProj**). We write $\times$ for the product in **Sch** (i.e., the fiber product over $\operatorname{Spec} k$ in the category of all schemes).

Notations relative to $X \in \mathbf{Sch}$:

- $X_R := X \times_{\operatorname{Spec} k} \operatorname{Spec} R$ for a k-algebra R.
- $K(X)$ is the total ring of fractions of X_K for $K \in \mathbf{Fld}$.
- $X_{(i)}$ is the set of all points of X of dimension i for $i \in \mathbb{Z}$.
- $\mathrm{CH}_i(X)$ is the Chow group of dimension i cycles on X for $i \in \mathbb{Z}$.
- $\mathrm{Pic}(X)$ is the Picard group of X.
- $\mathrm{NS}(X)$ is the Néron-Severi group if $X \in \mathbf{Sm}$.

Additional general notations, where A is an abelian group:

- $A[m] := \{a \in A \mid ma = 0\}$ for $m \in \mathbb{Z}_{>0}$, $A_{\mathrm{Tor}} := \cup_{m \in \mathbb{Z}_{>0}} A[m]$, and $A_{\mathrm{fr}} := A/A_{\mathrm{Tor}}$.
- $\exp(A) := \inf\{m \in \mathbb{Z}_{>0} \mid mA = 0\} \in \mathbb{Z}_{>0} \cup \{\infty\}$.

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- $A_R := A \otimes_{\mathbb{Z}} R$ for a commutative ring R.
- The set of all morphisms from X to Y in a category $\mathscr{C}$ is written by $\mathscr{C}(X,Y)$.
- $\mathbf{Mod}_\Lambda$ is the category of all Λ-modules and Λ-homomorphisms.

2. Preliminaries

In this section, we recall some definitions and results from [12, 22, 29, 30, 43, 44] that will be used later.

2.1. Chow motives

We write $\mathbf{Chow}(k)_\Lambda$ for the *covariant* category of Chow motives over k with coefficients in Λ, defined, for example, in [29, §1.5, 1.6], [43, §4, p.2092]. (This is opposite of the more frequently used contravariant version; see, for example, [39].) It is a Λ-linear rigid symmetric monoidal pseudo-abelian category. Any object of $\mathbf{Chow}(k)_\Lambda$ can be written as (X,π,r) for some equidimensional $X \in \mathbf{SmProj}$, a projector π of X, and $r \in \mathbb{Z}$. (By a projector of X, we mean $\pi \in \mathrm{CH}_{\dim X}(X \times X)_\Lambda$ such that $\pi \circ \pi = \pi$, where $\circ$ denotes the composition of algebraic correspondences.) We have

$$\mathbf{Chow}(k)_\Lambda((X,\pi,r),(Y,\rho,s)) = \rho \circ \mathrm{CH}_{\dim X + r - s}(X \times Y)_\Lambda \circ \pi,$$

where $X,Y \in \mathbf{SmProj}$ (with X equidimensional), π,ρ projectors of X,Y, and $r,s \in \mathbb{Z}$. We write $\Lambda(r) := (\operatorname{Spec} k,\ \mathrm{id}_{\operatorname{Spec} k}, r)$ and $M(r) := M \otimes \Lambda(r)$ for $M \in \mathbf{Chow}(k)_\Lambda$. Thus, $\Lambda := \Lambda(0)$ is a unit object for the monoidal structure. We denote by $M^\vee$ the dual object of M.

The category of effective Chow motives $\mathbf{Chow}(k)^{\mathrm{eff}}_\Lambda$ is the full subcategory of $\mathbf{Chow}(k)_\Lambda$ consisting of all objects isomorphic to those of the form (X,π,r) with $r \geq 0$. There is a covariant functor

$$h^{\mathrm{eff}} : \mathbf{SmProj} \to \mathbf{Chow}(k)^{\mathrm{eff}}_\Lambda, \qquad h^{\mathrm{eff}}(X) = (X, \mathrm{id}_X, 0). \tag{2.1}$$

We have $h^{\mathrm{eff}}(X) = h^{\mathrm{eff}}(X)^\vee(d)$ if $X \in \mathbf{SmProj}$ is purely d-dimensional. For $M \in \mathbf{Chow}(k)_\Lambda$ and $r \in \mathbb{Z}$, we write $\mathrm{CH}_r(M)_\Lambda := \mathbf{Chow}(k)_\Lambda(\Lambda(r), M)$ so that we have $\mathrm{CH}_r(h^{\mathrm{eff}}(X))_\Lambda = \mathrm{CH}_r(X)_\Lambda$ for any $X \in \mathbf{SmProj}$.

We abbreviate $\mathbf{Chow}_\Lambda := \mathbf{Chow}(k)_\Lambda$ and $\mathbf{Chow}^{\mathrm{eff}}_\Lambda := \mathbf{Chow}(k)^{\mathrm{eff}}_\Lambda$. For any $K \in \mathbf{Fld}$, there is a base change functor $\mathbf{Chow}_\Lambda \to \mathbf{Chow}(K)_\Lambda$ written by $M \mapsto M_K$.

2.2. Torsion motives

Vishik [44, Definition 2.4] defines a torsion motive to be an object $M \in \mathbf{Chow}_\Lambda$ such that $m \cdot \mathrm{id}_M = 0$ for some $m \in \mathbb{Z}_{>0}$. Since we will need a similar notion considered in different categories, we introduce the following general terminology:

Definition 2.1. We say an object A of an additive category $\mathscr{C}$ is *torsion* if there exists $m \in \mathbb{Z}_{>0}$ such that $m \cdot \mathrm{id}_A = 0$ in $\mathscr{C}(A,A)$. This is equivalent to saying that $\mathscr{C}(A,B)$ (or $\mathscr{C}(B,A)$) is a torsion abelian group for any $B \in \mathscr{C}$.

The following is an obvious variant of a result of Gorchinskiy-Guletskii [22, Lemma 1] (compare [17, Proposition 2.1]).

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Lemma 2.2. *For $M \in \mathbf{Chow}_\Lambda$, the following conditions are equivalent.*

(1) *M is a torsion object of $\mathbf{Chow}_\Lambda$.*
(2) *$\mathrm{CH}_n(M_K)_\Lambda$ is torsion for any $n \in \mathbb{Z}$ and for any $K \in \mathbf{Fld}$.*
(3) *$\mathrm{CH}_n(M_K)_\Lambda$ is torsion for any $n \in \mathbb{Z}$ and for any $K \in \mathbf{Fld}^{\mathrm{ac}}$.*
(4) *$\mathrm{CH}_n(M_K)_\Lambda$ is torsion for any $n \in \mathbb{Z}$ and for any $K \in \mathbf{Fld}^{\mathrm{fg}}$.*

Proof. (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) are obvious. (3) $\Rightarrow$ (2) holds because $\ker(\mathrm{CH}_n(M_K)_\Lambda \to \mathrm{CH}_n(M_{\overline{K}})_\Lambda)$ is torsion, where $\overline{K}$ is an algebraic closure of $K \in \mathbf{Fld}$. (4) $\Rightarrow$ (2) is seen by taking colimit. We have shown the equivalence (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).

Let us show (1) $\Rightarrow$ (4). By the shown equivalence (3) $\Leftrightarrow$ (4), we are reduced to the case k is algebraically closed (in particular, k is perfect). Take $K \in \mathbf{Fld}^{\mathrm{fg}}$. By Nagata's compactification and de Jong's alteration (see [15, Theorem 4.1], [16, Theorem 4.1]), we can find an integral proper k-scheme $X \in \mathbf{Sch}$ with $K = k(X)$ and a proper surjective generically finite morphism $f : Y \to X$ with $Y \in \mathbf{SmProj}$ integral. We then have a sequence of induced maps

$$\mathrm{CH}_{n+d_Y}(M \otimes Y)_\Lambda \twoheadrightarrow \mathrm{CH}_n(M_{k(Y)})_\Lambda \xrightarrow{f_*} \mathrm{CH}_n(M_{k(X)})_\Lambda,$$

where $d_Y := \dim Y$. The first map is surjective, and the cokernel of the second map is annihilated by $[k(Y) : k(X)]$. Since $\mathrm{CH}_{n+d_Y}(M \otimes Y)_\Lambda = \mathbf{Chow}_\Lambda(\Lambda(n+d_Y), M \otimes Y)$ is torsion by the assumption (1), we conclude that $\mathrm{CH}_n(M_{k(X)})_\Lambda$ is torsion as well.

It remains to prove (2) $\Rightarrow$ (1), for which we follow [22, Lemma 1]. Write $M = (X, \pi, r) \in \mathbf{Chow}_\Lambda$ with X equidimensional and put $d_X := \dim X$. We take $N \in \mathbf{Chow}_\Lambda$ and show that $\mathbf{Chow}_\Lambda(M, N)$ is torsion. We may assume $N = h^{\mathrm{eff}}(Y)$ for connected $Y \in \mathbf{SmProj}$ (by replacing r if necessary). Given $Z \in \mathbf{Sch}$, we define $\mathrm{CH}_n(M \otimes Z)_\Lambda$ as the image of an idempotent operator

$$\mathrm{CH}_n(X \times Z)_\Lambda \to \mathrm{CH}_n(X \times Z)_\Lambda, \qquad \alpha \mapsto p_{23*}(p_{13}^*(\alpha) \cdot_{p_{12}} \pi),$$

where p_{ij} are respective projections on $X \times X \times Z$, and $\cdot_{p_{12}}$ is the global product along p_{12} defined in [18, §8.1]; this product exists since $X \times X$ is smooth. We show that $\mathrm{CH}_n(M \otimes Z)_\Lambda$ is torsion for any integral $Z \in \mathbf{Sch}$ and for any n by induction on $d_Z := \dim Z$. The case $d_Z = 0$ is immediate from the assumption (2). If $d_Z > 0$, from the localization sequence for $X \times Z$, we deduce an exact sequence

$$\bigoplus_W \mathrm{CH}_n(M \otimes W)_\Lambda \to \mathrm{CH}_n(M \otimes Z)_\Lambda \to \mathrm{CH}_{n-d_Z}(M_{k(Z)})_\Lambda \to 0,$$

where W runs through integral proper closed subschemes of Z. The claim now follows by induction. Applying this to $Z = Y$ and $n = d_X + r$, we conclude $\mathrm{CH}_{d_X+r}(M \otimes Y)_\Lambda = \mathbf{Chow}_\Lambda(M, N)$ is torsion. $\square$

2.3. Birational motives

We write $\mathbf{Chow}_\Lambda^{\mathrm{bir}}$ for the category of birational motives over k with coefficients in Λ from [30, Definition 2.3.6]. (This is denoted by $\mathbf{Chow}^\circ(k, \Lambda)$ in [30].) It comes equipped

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with a functor $\mathbf{Chow}^{\mathrm{eff}}_\Lambda \to \mathbf{Chow}^{\mathrm{bir}}_\Lambda$. We write the composition of it with h^{eff} by

$$h^{\mathrm{bir}} : \mathbf{SmProj} \to \mathbf{Chow}^{\mathrm{bir}}_\Lambda. \tag{2.2}$$

We then have

$$\mathbf{Chow}^{\mathrm{bir}}_\Lambda(h^{\mathrm{bir}}(X), h^{\mathrm{bir}}(Y)) = \mathrm{CH}_0(Y_{k(X)})_\Lambda$$

for any $X, Y \in \mathbf{SmProj}$ (see [30, Lemma 2.3.7]).

Remark 2.3. There are several variants of $\mathbf{Chow}^{\mathrm{bir}}_\Lambda$. We recall two of them.

(1) Denote by $\mathbf{Chow}^{\mathrm{bir},1}_\Lambda$ the pseudo-abelian envelope of the category obtained from $\mathbf{Chow}^{\mathrm{eff}}_\Lambda$ by inverting all birational morphisms.
(2) Denote by $\mathbf{Chow}^{\mathrm{bir},2}_\Lambda$ the pseudo-abelian envelope of $\mathbf{Chow}^{\mathrm{eff}}_\Lambda/\mathbb{L}$, where $\mathbb{L}$ is the ideal of $\mathbf{Chow}^{\mathrm{eff}}_\Lambda$ consisting of all morphisms which factor through an object of the form $M(1)$ with $M \in \mathbf{Chow}^{\mathrm{eff}}_\Lambda$.

There are functors

$$\mathbf{Chow}^{\mathrm{bir},2}_\Lambda \xrightarrow{\cong} \mathbf{Chow}^{\mathrm{bir},1}_\Lambda \longrightarrow \mathbf{Chow}^{\mathrm{bir}}_\Lambda.$$

The first one is always an equivalence, and so is the second at least if p is invertible in Λ (see [30, Proposition 2.2.9, Corollary 2.4.3]). As $\mathbf{Chow}^{\mathrm{eff}}_\Lambda \to \mathbf{Chow}^{\mathrm{bir}}_\Lambda$ factors through $\mathbf{Chow}^{\mathrm{bir},2}_\Lambda$, the image of $M(1)$ vanishes in $\mathbf{Chow}^{\mathrm{bir}}_\Lambda$ for any $M \in \mathbf{Chow}^{\mathrm{eff}}_\Lambda$.

Finally, we write $\mathbf{Chow}^{\mathrm{nor}}_\Lambda$ for the quotient category of $\mathbf{Chow}^{\mathrm{bir}}_\Lambda$ by the ideal consisting of all morphisms which factor through $\Lambda = h^{\mathrm{bir}}(\operatorname{Spec} k)$, introduced in [29, Definition 2.4]. Denote by

$$h^{\mathrm{nor}} : \mathbf{SmProj} \to \mathbf{Chow}^{\mathrm{nor}}_\Lambda \tag{2.3}$$

the composition of h^{bir} and the localization functor $\mathbf{Chow}^{\mathrm{bir}}_\Lambda \to \mathbf{Chow}^{\mathrm{nor}}_\Lambda$. We have

$$\mathbf{Chow}^{\mathrm{nor}}_\Lambda(h^{\mathrm{nor}}(X), h^{\mathrm{nor}}(Y)) = \mathrm{Coker}(\mathrm{CH}_0(Y)_\Lambda \to \mathrm{CH}_0(Y_{k(X)})_\Lambda) \tag{2.4}$$

for any $X, Y \in \mathbf{SmProj}$ (see loc. cit.).

Remark 2.4. If no confusion is likely, we abbreviate $h^{\mathrm{eff}}(X)$, $h^{\mathrm{bir}}(X)$, and $h^{\mathrm{nor}}(X)$ by X for $X \in \mathbf{SmProj}$. Similarly, for $M \in \mathbf{Chow}^{\mathrm{eff}}_\Lambda$, we use the same letter M to denote its images in $\mathbf{Chow}^{\mathrm{bir}}_\Lambda$ and $\mathbf{Chow}^{\mathrm{nor}}_\Lambda$. For instance, the left-hand side of (2.4) will be written by $\mathbf{Chow}^{\mathrm{nor}}_\Lambda(X, Y)$.

2.4. Motivic invariants

Denote by $\mathbf{Mod}_\Lambda$ the category of Λ-modules. Following [29, Definition 2.1], we introduce some definitions.

Definition 2.5. Let $F : \mathbf{SmProj}^{\mathrm{op}} \to \mathbf{Mod}_\Lambda$ be a functor.

(1) We say F is *birational* if $F(f)$ is an isomorphism for any birational morphism f.
(2) We say F is *motivic* if F factors through an additive functor $\mathbf{Chow}^{\mathrm{eff,op}}_\Lambda \to \mathbf{Mod}_\Lambda$.
(3) We say F is *normalized* if $F(\operatorname{Spec} k) = 0$.

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Lemma 2.6. *Suppose that p is invertible in Λ. A functor $F:\mathbf{SmProj}^{\mathrm{op}}\to\mathbf{Mod}_\Lambda$ is birational and motivic (resp. normalized, birational and motivic) if and only if F factors through an additive functor $\mathbf{Chow}_\Lambda^{\mathrm{bir,op}}\to\mathbf{Mod}_\Lambda$ (resp. $\mathbf{Chow}_\Lambda^{\mathrm{nor,op}}\to\mathbf{Mod}_\Lambda$).*

Proof. This is immediate from what we recalled in §2.3. □

Remark 2.7. Given a motivic (resp. birational and motivic, resp. normalized, birational and motivic) functor $F:\mathbf{SmProj}^{\mathrm{op}}\to\mathbf{Mod}_\Lambda$, its extension to $\mathbf{Chow}_\Lambda^{\mathrm{eff}}$ (resp. $\mathbf{Chow}_\Lambda^{\mathrm{bir}}$, resp. $\mathbf{Chow}_\Lambda^{\mathrm{nor}}$) is denoted by the same letter F.

Example 2.8.

(1) Suppose $p=1$ or $\Lambda=\mathbb{Z}$. It is a classical fact that $H^0(-,\Omega^i_{-/k})$ is birational and motivic for any $i\in\mathbb{Z}_{\geq 0}$; it is also normalized if $i>0$. It is less classical that the same is true of $H^i(-,\mathscr{O})$ if k is perfect (see [10]).

(2) It is obvious from the definition that the functor

$$\mathbf{Chow}_\Lambda^{\mathrm{nor}}(-,S):T\mapsto\mathbf{Chow}_\Lambda^{\mathrm{nor}}(T,S)=\mathrm{Coker}(\mathrm{CH}_0(S)_\Lambda\to\mathrm{CH}_0(S_{k(T)})_\Lambda) \tag{2.5}$$

is birational, motivic and normalized for any fixed $S\in\mathbf{SmProj}$.

(3) Let M be a cycle module in the sense of Rost [38]. Then its 0-th cycle cohomology $A^0(-,M_n)$ is birational and motivic by [30, Corollary 6.1.3]. We will only use a special case of unramified cohomology, which will be recalled in the next subsection.

(4) A $\mathbb{P}^1$-invariant Nisnevich sheaf with transfers is birational and motivic. We include a proof of this fact, due to Bruno Kahn, in an appendix (see Proposition 9.1 below). This recovers all examples discussed above, except $H^i(-,\mathscr{O})$.

2.5. Unramified cohomology

A general reference for this subsection is [12]. Let $K\in\mathbf{Fld}$ and $i\in\mathbb{Z}$. For $n\in\mathbb{Z}_{>0}$ invertible in k, the *unramified cohomology* of K/k is defined by

$$H^i_{\mathrm{ur},n}(K/k):=\ker\left(H^i_{\mathrm{Gal}}(K,\mu_n^{\otimes(i-1)})\to\bigoplus_v H^{i-1}_{\mathrm{Gal}}(F_v,\mu_n^{\otimes(i-2)})\right), \tag{2.6}$$

where v ranges over all discrete valuations of K that are trivial on k, and F_v is the residue field of v. The maps appearing in the definition are the residue maps (see [12, (3.6)]). We set

$$H^i_{\mathrm{ur}}(K/k):=\varinjlim_{(n,p)=1}H^i_{\mathrm{ur},n}(K/k), \tag{2.7}$$

where n ranges over all $n\in\mathbb{Z}_{>0}$ that is invertible in k. By Rost-Voevodsky's norm residue isomorphism theorem (which is the former Bloch-Kato conjecture and proved in [48, Theorem 6.16]), we may identify $H^i_{\mathrm{ur},n}(K/k)$ with the n-torsion part of $H^i_{\mathrm{ur}}(K/k)$:

$$H^i_{\mathrm{ur},n}(K/k)\cong H^i_{\mathrm{ur}}(K/k)[n]. \tag{2.8}$$

Let $X \in \mathbf{Sm}$ and $i \in \mathbb{Z}$. For $n \in \mathbb{Z}_{>0}$ invertible in k, the *unramified cohomology* of X is defined as

$$H^i_{\mathrm{ur},n}(X) := H^0_{\mathrm{Zar}}(X, \mathscr{H}^i_n), \qquad H^i_{\mathrm{ur}}(X) := \varinjlim_{(n,p)=1} H^i_{\mathrm{ur},n}(X), \tag{2.9}$$

where $\mathscr{H}^i_n$ is the Zariski sheaf on X associated to the presheaf $U \mapsto H^i_{\text{ét}}(U, \mu_n^{\otimes(i-1)})$, and the colimit in the second formula is taken in the same way as (2.7). We have canonical isomorphisms (see [12, Propositions 4.2.1, 4.2.3])

$$H^1_{\mathrm{ur},n}(X) \cong H^1_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}), \qquad H^2_{\mathrm{ur},n}(X) \cong \mathrm{Br}(X)[n], \tag{2.10}$$

where $\mathrm{Br}(X) := H^2_{\text{ét}}(X, \mathbb{G}_m)$ is the Brauer group of X. If further X is integral and proper over k, we also have (see [12, Theorem 4.1.1])

$$H^i_{\mathrm{ur},n}(X) \cong H^i_{\mathrm{ur},n}(k(X)/k), \qquad H^i_{\mathrm{ur}}(X) \cong H^i_{\mathrm{ur}}(k(X)/k). \tag{2.11}$$

The following well-known fact plays an essential role in this paper:

Proposition 2.9. *Let $i,n \in \mathbb{Z}$ and suppose that n is invertible in k. Then the functor $H^i_{\mathrm{ur},n} : \mathbf{SmProj} \to \mathbf{Mod}_{\mathbb{Z}[1/p]}$ is birational and motivic. The same is true for H^i_{ur}. They are also normalized if $i > 0$ and k is algebraically closed.*

Proof. The first statement follows from [12, Theorem 4.1.1] (see also [38, (2.5)]) and [30, Corollary 6.1.3], and the second from the first. The third statement is obvious from the definition. □

2.6. Varieties admitting a decomposition of the diagonal

Proposition 2.10. *The following conditions are equivalent for $X \in \mathbf{SmProj}$:*

(1) *The degree map induces an isomorphism $\mathrm{CH}_0(X_{k(X)})_{\mathbb{Q}} \cong \mathbb{Q}$.*

(2) *The class of the generic point of X in $\mathrm{CH}_0(X_{k(X)})_{\mathbb{Q}}$ belongs to*
$$\mathrm{Im}(\mathrm{CH}_0(X)_{\mathbb{Q}} \to \mathrm{CH}_0(X_{k(X)})_{\mathbb{Q}}).$$

(3) *The structure map induces an isomorphism $h^{\mathrm{bir}}(X) \cong \mathbb{Q}$ in $\mathbf{Chow}^{\mathrm{bir}}_{\mathbb{Q}}$*

(4) *The object $h^{\mathrm{nor}}(X)$ of $\mathbf{Chow}^{\mathrm{nor}}_{\mathbb{Z}}$ is torsion in the sense of Definition 2.1.*

Proof. See [30, Proposition 3.1.1] for (1)–(3). Equivalence of (2) and (4) is obvious from the definition and (2.4) (see also [29, §2.3]). □

Remark 2.11. If k is an algebraically closed field with infinite transcendental degree over its prime subfield, then these conditions are also equivalent to the following:

(1)' The degree map induces an isomorphism $\mathrm{CH}_0(X)_{\Lambda} \cong \Lambda$ for either $\Lambda = \mathbb{Z}$ or $\mathbb{Q}$.

(See [30, Proposition 3.1.1].)

Definition 2.12. We say $X \in \mathbf{SmProj}$ admits a *decomposition of the diagonal* if the conditions of Proposition 2.10 are satisfied.

This notion goes back to Bloch-Srinivas [8]. For such X, Kahn [29, Definition 2.5] and Chatzistamatiou-Levine [9, Definition 1.1] defined a numerical invariant called the *torsion order*, which can be written as $\mathrm{Tor}^{\mathrm{nor}}_{\mathbb{Z}}(X)$ in terms of the following definition:

Definition 2.13.

(1) Let A be an object of an additive category $\mathscr{C}$ that is torsion in the sense of Definition 2.1. The smallest $m \in \mathbb{Z}_{>0}$ such that $m \cdot \mathrm{id}_A = 0$ is called the *torsion order* of A.

(2) The torsion order of a torsion object M of $\mathbf{Chow}^{\mathrm{eff}}_\Lambda$ (resp. $\mathbf{Chow}^{\mathrm{bir}}_\Lambda$, resp. $\mathbf{Chow}^{\mathrm{nor}}_\Lambda$) is denoted by $\mathrm{Tor}^{\mathrm{eff}}_\Lambda(M)$ (resp. $\mathrm{Tor}^{\mathrm{bir}}_\Lambda(M)$, resp. $\mathrm{Tor}^{\mathrm{nor}}_\Lambda(M)$).

We write $b_i(X)$ and $\rho(X)$ for the Betti and Picard numbers of $X \in \mathbf{SmProj}$:

$$b_i(X) := \dim_{\mathbb{Q}_\ell} H^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell), \qquad \rho(X) := \mathrm{rank}_{\mathbb{Z}} \mathrm{NS}(X_{\overline{k}})/\mathrm{NS}(X_{\overline{k}})_{\mathrm{Tor}},$$

where $\overline{k}$ is an algebraic closure of k, and ℓ is any prime number different from p.

Proposition 2.14. *Suppose that $X \in \mathbf{SmProj}$ admits a decomposition of the diagonal.*

(1) *We have $b_1(X) = 0$, $b_2(X) = \rho(X)$ and $\mathrm{Pic}(X) = \mathrm{NS}(X)$.*

(2) *Suppose that k is algebraically closed. For any prime number ℓ invertible in k, we have canonical isomorphisms*

$$\begin{aligned} &H^1_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) \cong \mathrm{NS}(X)_{\mathrm{Tor}, \mathbb{Z}_\ell}, \\ &H^1_{\mathrm{ur}}(X)_{\mathbb{Z}_\ell} \cong H^1_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \cong H^2_{\text{ét}}(X, \mathbb{Z}_\ell)_{\mathrm{Tor}}, \\ &H^2_{\mathrm{ur}}(X)_{\mathbb{Z}_\ell} \cong \mathrm{Br}(X)_{\mathbb{Z}_\ell} \cong H^3_{\text{ét}}(X, \mathbb{Z}_\ell(1))_{\mathrm{Tor}}. \end{aligned}$$

(3) *Suppose that p is invertible in Λ, and put $m := \mathrm{Tor}^{\mathrm{nor}}_\Lambda(X)$. Then we have $mF(X) = 0$ for any normalized, birational and motivic functor $F : \mathbf{SmProj}^{\mathrm{op}} \to \mathbf{Mod}_\Lambda$.*

Proof. See [30, Proposition 3.1.4] for the proof of (1) and [29, Lemma 2.6] for (3). (2) follows from (1), (2.10) and the following Lemma. □

Lemma 2.15. *Suppose that k is algebraically closed. Let ℓ be a prime number invertible in k. For any $X \in \mathbf{SmProj}$, we have a canonical isomorphism*

$$H^1_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) \cong \mathrm{Pic}(X)_{\mathrm{Tor}, \mathbb{Z}_\ell} \tag{2.12}$$

and canonical surjective morphisms

$$H^1_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \twoheadrightarrow H^2_{\text{ét}}(X, \mathbb{Z}_\ell)_{\mathrm{Tor}}, \qquad \mathrm{Br}(X)_{\mathbb{Z}_\ell} \twoheadrightarrow H^3_{\text{ét}}(X, \mathbb{Z}_\ell(1))_{\mathrm{Tor}}. \tag{2.13}$$

Moreover, the first (resp. second) morphism in (2.13) is bijective if $b_1(X) = 0$ (resp. $b_2(X) = \rho(X)$).

Proof. For any $m, n \in \mathbb{Z}$ with $m, n > 0$, we have exact sequences of étale sheaves:

$$0 \to \mu_{\ell^m} \to \mu_{\ell^{m+n}} \to \mu_{\ell^n} \to 0, \qquad 0 \to \mu_{\ell^n} \to \mathbb{G}_m \to \mathbb{G}_m \to 0.$$

From the second sequence, we obtain an isomorphism $H^1_{\text{ét}}(X, \mu_{\ell^n}) \cong \mathrm{Pic}(X)[\ell^n]$, from which we deduce (2.12) by taking a colimit over n. The upper exact row in the following

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diagram is obtained in a similar way, while the lower row is obtained by taking a limit over m and a colimit over n of the long exact sequence deduced from the first sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pic}(X)\otimes\mathbb{Q}_\ell/\mathbb{Z}_\ell & \longrightarrow & H^2_{\text{ét}}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) & \longrightarrow & \mathrm{Br}(X)_{\mathbb{Z}_\ell} & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & H^2_{\text{ét}}(X,\mathbb{Z}_\ell(1))\otimes\mathbb{Q}_\ell/\mathbb{Z}_\ell & \longrightarrow & H^2_{\text{ét}}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) & \longrightarrow & H^3_{\text{ét}}(X,\mathbb{Z}_\ell(1))_{\mathrm{Tor}} & \longrightarrow & 0. \end{array}$$

(The limit preserves the exactness of the lower low since $H^i_{\text{ét}}(X,\mu_{\ell^m})$ is finite for each i,m.) The left and the right vertical maps are induced since the composition $\mathrm{Pic}(X)\otimes\mathbb{Q}/\mathbb{Z}\to H^3_{\text{ét}}(X,\mathbb{Z}_\ell(1))_{\mathrm{Tor}}$ vanishes (as the source is divisible and the target is finite). The second surjection in (2.13) is obtained as the right vertical map in this diagram, which is bijective if $b_2(X)=\rho(X)$ because so is the left vertical map under this hypothesis.

By a similar argument with different Tate twist, we get an exact sequence

$$0\to H^i_{\text{ét}}(X,\mathbb{Z}_\ell(r))\otimes\mathbb{Q}_\ell/\mathbb{Z}_\ell\to H^i_{\text{ét}}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(r))\to H^{i+1}_{\text{ét}}(X,\mathbb{Z}_\ell(r))_{\mathrm{Tor}}\to 0 \qquad (2.14)$$

for any $i,r\in\mathbb{Z}$. The first surjection in (2.13) is obtained as the second arrow in this sequence for $(i,r)=(1,0)$, which is bijective if $b_1(X)=0$ because the first term vanishes under this hypothesis. (We will use (2.14) for other (i,r) later.) □

Remark 2.16.

(1) If $S\in\mathbf{SmProj}$ is a surface such that $b_1(S)=0$ and $b_2(S)=\rho(S)$, then Bloch's conjecture predicts that S should admit a decomposition of the diagonal (see [30, Proposition 3.1.4]).

(2) It is obvious that $\mathrm{Tor}^{\mathrm{nor}}_\Lambda(M)\mid\mathrm{Tor}^{\mathrm{bir}}_\Lambda(M)\mid\mathrm{Tor}^{\mathrm{eff}}_\Lambda(M)$ for torsion $M\in\mathbf{Chow}^{\mathrm{eff}}_\Lambda$. The opposite divisibility does not hold in general. (For example, we have $\mathrm{Tor}^{\mathrm{eff}}_\Lambda(M)=\mathrm{Tor}^{\mathrm{eff}}_\Lambda(M(1))$, but the image of $M(1)$ vanishes in $\mathbf{Chow}^{\mathrm{bir}}_\Lambda$.) Yet, it can hold in some nontrivial cases, as seen in Proposition 3.6 below.

3. Torsion motives of surfaces

Setting 3.1. From now on, we suppose k is algebraically closed and $\Lambda=\mathbb{Z}[1/p]$. Fix $S\in\mathbf{SmProj}$ admitting a decomposition of the diagonal and such that $\dim S=2$.

3.1. Surfaces admitting a decomposition of the diagonal

Lemma 3.2. *For any prime number $\ell\neq p$, we have the following:*

(1) $b_0(S)=b_4(S)=1$, $b_2(S)=\rho(S)$, *and* $b_i(S)=0$ *for any* $i\neq 0,2,4$.

(2) $H^0_{\text{ét}}(S,\mathbb{Z}_\ell)=H^4_{\text{ét}}(S,\mathbb{Z}_\ell(2))=\mathbb{Z}_\ell$, $H^1_{\text{ét}}(S,\mathbb{Z}_\ell)=0$, *and* $H^3_{\text{ét}}(S,\mathbb{Z}_\ell(1))$ *is finite.*

(3) $\mathrm{Pic}(S)=\mathrm{NS}(S)$ *is a finitely generated* $\mathbb{Z}$*-module;* $\mathrm{NS}(S)_{\mathrm{Tor},\Lambda}$ *and* $\mathrm{Br}(S)_\Lambda$ *are finite abelian groups canonically dual to each other.*

(4) $\mathrm{CH}_1(S_K)\cong\mathrm{NS}(S)$ *for any* $K\in\mathbf{Fld}$ *and* $\mathrm{CH}_0(S_{\overline{K}})\cong\mathbb{Z}$ *for any* $\overline{K}\in\mathbf{Fld}^{\mathrm{ac}}$.

Proof. (1) Proposition 2.14 shows the statement for $i\le 2$. Then the Poincaré duality $b_{4-i}(X)=b_i(X)$ completes the proof for other i.

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(2) All assertions follow from (1), plus a fact $H^1_{\text{ét}}(S,\mathbb{Z}_\ell)_{\text{Tor}} = 0$ which is seen from (2.14).

(3) Proposition 2.14 shows the first statement. It also shows $\text{NS}(S)_{\text{Tor},\mathbb{Z}_\ell} \cong H^2_{\text{ét}}(S,\mathbb{Z}_\ell(1))_{\text{Tor}}$ and $\text{Br}(S)_\Lambda \cong H^3_{\text{ét}}(S,\mathbb{Z}_\ell(1))_{\text{Tor}}$; hence, they are dual to each other by the Poincaré duality.

(4) Proposition 2.14 shows the vanishing of the Picard variety of X, whence the first statement. Since this implies the vanishing of the Albanese variety Alb_S of S, the last statement of (4) follows from Roitman's theorem [37, p. 565, Consequence III] (which says $\text{CH}_0(S_{\overline{K}})[m] \cong \text{Alb}_S(\overline{K})[m]$ for any $m \in \mathbb{Z}$ invertible in k). □

Lemma 3.3. *Let $\rho := \rho(S)$ and take $e_1,\dots,e_\rho \in \text{NS}(S)$ such that their classes form a $\mathbb{Z}$-basis of $\text{NS}(S)/\text{NS}(S)_{\text{Tor}}$. Let $a_{ij} := \langle e_i,e_j\rangle \in \mathbb{Z}$, where $\langle\cdot,\cdot\rangle$ denotes the intersection form on S. Then $\delta := \det((a_{ij})_{i,j=1,\dots,\rho})$ is invertible in Λ.*

Proof. It suffices to show that $\delta \in \mathbb{Z}_\ell^\times$ for any prime number $\ell \neq p$. By Proposition 2.14 we have an isomorphism $\text{NS}(S)_{\mathbb{Z}_\ell} \cong H^2_{\text{ét}}(S,\mathbb{Z}_\ell(1))$ which is compatible with the intersection pairing and the cup product. Therefore, it suffices to show that the cup product induces an isomorphism

$$H^2_{\text{ét}}(S,\mathbb{Z}_\ell(1))_{\text{fr}} \xrightarrow{\cong} \text{Hom}_{\mathbb{Z}_\ell}(H^2_{\text{ét}}(S,\mathbb{Z}_\ell(1))_{\text{fr}},\mathbb{Z}_\ell),$$

where we put $M_{\text{fr}} := M/M_{\text{Tor}}$ for a $\mathbb{Z}_\ell$-module M. This follows [49, Corollary 1.3]. □

Proposition 3.4. *There exists a direct sum decomposition $h^{\text{eff}}(S) \cong L \oplus M \oplus N$ in $\mathbf{Chow}^{\text{eff}}_\Lambda$ satisfying the following conditions:*

1. *We have isomorphisms $L \cong \Lambda \oplus \Lambda(2)$ and $N \cong \Lambda(1)^{\rho(S)}$;*
2. *M is torsion in $\mathbf{Chow}^{\text{eff}}_\Lambda$ in the sense of Definition 2.1;*
3. *We have isomorphisms $L \cong L^\vee(2)$, $M \cong M^\vee(2)$ and $N \cong N^\vee(2)$ which are compatible with those in (1) and the Poincaré duality $h^{\text{eff}}(S) \cong h^{\text{eff}}(S)^\vee(2)$.*

Proof. The statement without the condition (3) is shown by Gorchinskiy-Orlov in (the proof of) [23, Proposition 2.3, Remark 2.5] when $k = \mathbb{C}$, and the full statement by Vishik in [44, Proposition 4.1] when S is the classical Godeaux surface. The same proof works without any essential change, but for the sake of completeness, we give a brief account.

Let $\rho := \rho(S)$ and take $e_1,\dots,e_\rho \in \text{NS}(S)$ such that their classes form a $\mathbb{Z}$-basis of $\text{NS}(S)/\text{NS}(S)_{\text{Tor}}$. Let a_{ij} be as in Lemma 3.3, and set $A := (a_{ij}) \in \text{GL}_\rho(\Lambda)$. Write $A^{-1} = (b_{ij}) \in \text{GL}_\rho(\Lambda)$. Take also a closed point $x_0 \in S_{(0)}$. We then define orthogonal projectors

$$\pi_L := [S \times x_0] + [x_0 \times S], \quad \pi_N := \sum_{i,j} b_{ij}[e_i \times e_j] \quad \in \mathbf{Chow}^{\text{eff}}_\Lambda(S,S) = \text{CH}_2(S \times S)_\Lambda.$$

Set $L := (S,\pi_L,0), N := (S,\pi_N,0), M := (S,1-\pi_L-\pi_N) \in \mathbf{Chow}^{\text{eff}}_\Lambda$. Then we have (1) and (3). Observe that (1) and Lemma 3.2 imply that for any $\overline{K} \in \mathbf{Fld}^{\text{ac}}$,

$$\text{CH}_1(M_{\overline{K}})_\Lambda = \text{NS}(S)_{\Lambda,\text{Tor}} \quad \text{and} \quad \text{CH}_i(M_{\overline{K}})_\Lambda = 0 \text{ for } i \neq 1. \tag{3.1}$$

It then follows by Lemma 2.2 that M satisfies (2) too. We are done. □

The summand M is not necessarily unique. We choose one and fix it.

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Setting 3.5. In what follows, we denote by $M \in \mathbf{Chow}^{\mathrm{eff}}_{\Lambda}$ a Chow motive constructed in Proposition 3.4. Observe that we have $S = M$ in $\mathbf{Chow}^{\mathrm{nor}}_{\Lambda}$, because $\Lambda(r)$ vanishes in $\mathbf{Chow}^{\mathrm{nor}}_{\Lambda}$ for any $r \geq 0$ by Remark 2.3.

3.2. Injectivity

The following proposition proves the injectivity of the first map in (1.3).

Proposition 3.6.

(1) *We take $T \in \mathbf{SmProj}$ and consider the maps*

$$\mathbf{Chow}^{\mathrm{eff}}_{\Lambda}(T,M) \xrightarrow{a} \mathbf{Chow}^{\mathrm{nor}}_{\Lambda}(T,M) \xrightarrow{b} \bigoplus_{i=1,2} \mathrm{Hom}(H^i_{\mathrm{ur}}(M), H^i_{\mathrm{ur}}(T)),$$

where a is induced by the functor $\mathbf{Chow}^{\mathrm{eff}}_{\Lambda} \to \mathbf{Chow}^{\mathrm{nor}}_{\Lambda}$, and b is induced by the functors H^i_{ur} for $i = 1,2$ using Lemma 2.6 and Proposition 2.9. Then a is bijective and b is injective.

(2) *We have*

$$\mathrm{Tor}^{\mathrm{eff}}_{\Lambda}(M) = \mathrm{Tor}^{\mathrm{nor}}_{\Lambda}(M) = \mathrm{Tor}^{\mathrm{nor}}_{\Lambda}(S) = \exp(\mathrm{NS}(S)_{\mathrm{Tor},\Lambda}) = \exp(\mathrm{Br}(S)_{\Lambda}), \qquad (3.2)$$

where $\exp(A) := \min\{m \in \mathbb{Z}_{>0} \mid mA = 0\}$ for an abelian group A.

Proof. (1) (Compare [23, Proposition 2.3].) We consider a commutative diagram

$$\begin{array}{ccc}
\mathbf{Chow}^{\mathrm{eff}}_{\Lambda}(T,M) & \xrightarrow{a} & \mathbf{Chow}^{\mathrm{nor}}_{\Lambda}(T,M) \\
{\scriptstyle c}\uparrow & \searrow^{e} & \downarrow{\scriptstyle b} \\
\mathbf{Chow}^{\mathrm{eff}}_{\Lambda}(T,S) & \xrightarrow{d} & \bigoplus_{i=1,2} \mathrm{Hom}(H^i_{\mathrm{ur}}(M), H^i_{\mathrm{ur}}(T)).
\end{array}$$

The maps a and c are surjective by definition. Therefore, it suffices to prove the injectivity of e. Take $f \in \mathbf{Chow}^{\mathrm{eff}}_{\Lambda}(T,M)$ such that $e(f) = 0$. By Proposition 2.14 (2) and Lemma 2.15, this implies that, for any prime number $\ell \neq p$, we have

$$f^* = 0 : H^i_{\text{ét}}(M,\mathbb{Z}_\ell(1))_{\mathrm{Tor}} \to H^i_{\text{ét}}(T,\mathbb{Z}_\ell(1))_{\mathrm{Tor}} \quad \text{for } i = 2,3. \qquad (3.3)$$

However, we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{CH}^2(M \otimes T)_{\mathrm{Tor},\mathbb{Z}_\ell} & \hookrightarrow & H^3_{\text{ét}}(M \otimes T, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \\
\| & \searrow^{\mathrm{cyc}} & \downarrow \cong \\
\mathbf{Chow}^{\mathrm{eff}}_{\Lambda}(T,M)_{\mathbb{Z}_\ell} & \longrightarrow & H^4_{\text{ét}}(M \otimes T, \mathbb{Z}_\ell(2))_{\mathrm{Tor}}.
\end{array}$$

Here, cyc is the cycle map. The upper horizontal injective map is the one constructed by Bloch (see [11, Théorème 4.3]). The upper right triangle is commutative by [13, Corollaire 4]. The right vertical map is bijective since we have $H^*_{\text{ét}}(M \otimes T, \mathbb{Q}_\ell(2)) = 0$

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(as M is torsion). We have shown the injectivity of cyc. We consider isomorphisms

$$\begin{aligned} H^4_{\text{ét}}(M\otimes T,\mathbb{Z}_\ell(2))_{\text{Tor}} &\cong \bigoplus_{i=2,3} \text{Tor}(H^{5-i}_{\text{ét}}(M,\mathbb{Z}_\ell(1))_{\text{Tor}}, H^i_{\text{ét}}(T,\mathbb{Z}_\ell(1))_{\text{Tor}}) \\ &\cong \bigoplus_{i=2,3} \text{Hom}(H^i_{\text{ét}}(M,\mathbb{Z}_\ell(1))_{\text{Tor}}, H^i_{\text{ét}}(T,\mathbb{Z}_\ell(1))_{\text{Tor}}) \end{aligned}$$

induced by the Künneth formula, Poincaré duality (together with Proposition 3.4 (3)), and Lemma 8.3 below. Their composition sends α to the correspondence action (that is, $\beta \mapsto \text{pr}_{2*}(\text{pr}_1^*(\beta)\cup\alpha)$, where pr_i are projections on $M\otimes T$). Hence, it fits in the right vertical arrow of a commutative diagram

$$\begin{array}{ccc} \text{CH}^2(M\otimes T)_{\text{Tor},\mathbb{Z}_\ell} & \xhookrightarrow{\text{cyc}} & H^4_{\text{ét}}(M\otimes T,\mathbb{Z}_\ell(2))_{\text{Tor}} \\ \Vert & & \downarrow \cong \\ \mathbf{Chow}^{\text{eff}}_\Lambda(T,M)_{\mathbb{Z}_\ell} & \longrightarrow & \bigoplus_{i=2,3} \text{Hom}(H^i_{\text{ét}}(M,\mathbb{Z}_\ell(1))_{\text{Tor}}, H^i_{\text{ét}}(T,\mathbb{Z}_\ell(1))_{\text{Tor}}), \end{array}$$

where the lower horizontal map is induced by the functors $H^i_{\text{ét}}(-,\mathbb{Z}_\ell(1))_{\text{Tor}}$ for $i=2,3$. Now (3.3) shows that $f=0$ in $\mathbf{Chow}^{\text{eff}}_\Lambda(T,M)_{\mathbb{Z}_\ell}$. We are done.

(2) The relations

$$\exp(\text{NS}(S)_{\text{Tor},\Lambda}) = \exp(\text{Br}(S)_\Lambda) \mid \text{Tor}^{\text{nor}}_\Lambda(S) = \text{Tor}^{\text{nor}}_\Lambda(M) \mid \text{Tor}^{\text{eff}}_\Lambda(M)$$

are seen by Lemma 3.2 (3), Propositions 2.9 and 2.14 (3) applied to $F=\text{Br}(-)_\Lambda$, the equality $S=M$ in $\mathbf{Chow}^{\text{nor}}_\Lambda$, and Remark 2.16 (2), respectively. To conclude, it suffices to apply (1) to $T=S$ and $f=m\cdot\text{id}_S$ with $m\in\mathbb{Z}_{>0}$ to get $\text{Tor}^{\text{eff}}_\Lambda(M) \mid \exp(\text{NS}(S)_{\text{Tor},\Lambda})$. □

We record the following corollary for later use.

Corollary 3.7.

(1) *If $F:\mathbf{SmProj}^{\text{op}}\to\mathbf{Mod}_\Lambda$ is a motivic functor, then $F(M)$ is annihilated by the integer in (3.2). (We used the convention of Remark 2.7.)*

(2) *We have $H^i_{\text{ét}}(M,\mathbb{Z}_\ell)\cong H^i_{\text{ét}}(S,\mathbb{Z}_\ell)_{\text{Tor}}$ for any $i\in\mathbb{Z}$ and any prime $\ell\neq p$.*

Proof. (1) and (2) follows from Propositions 3.6 and 3.4, respectively. □

Problem 3.8. Let $\mathscr{C}$ be the full subcategory of $\mathbf{Chow}^{\text{eff}}_\Lambda$ consisting of torsion direct summands of the motives of surfaces (not necessarily admitting a decomposition of the diagonal). Is the functor $\mathscr{C}\to\mathbf{Chow}^{\text{nor}}_\Lambda$ fully faithful?

We end this section with two remarks concerning the p-adic counterpart of our results.

Remark 3.9. Assume that $p>0$, and let S be as before.

(1) The number δ for S in Lemma 3.3 is *not* necessarily invertible in $\mathbb{Z}$. For example, when S is a unirational (hence supersingular) K3 surface, S admits a decomposition of the diagonal, and we have $\delta=-p^{2\sigma_0}$ for some $1\le\sigma_0\le 10$; cf. [28, Chapter II,

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§7.2]. This example also shows that the decomposition of motives in Proposition 3.4 *does not* hold integrally, in general.

(2) Assume further that δ for S in Lemma 3.3 is invertible in $\mathbb{Z}$; this is the case for an Enriques surface [28, Chapter II, Corollary 7.3.7]. Under this assumption, one can take a torsion motive M of S in $\mathbf{Chow}^{\mathrm{eff}}_{\mathbb{Z}}$, and consider the canonical homomorphism

$$b_p : \mathbf{Chow}^{\mathrm{nor}}_{\mathbb{Z}_p}(T,M) \longrightarrow \bigoplus_{i,j\geq 0} \mathrm{Hom}(H^{i,j}_{\mathrm{ur}}(M)\{p\}, H^{i,j}_{\mathrm{ur}}(T)\{p\}).$$

Here, $H^{i,j}_{\mathrm{ur}}(-)\{p\}$ $(i,j \geq 0)$ is as in §1.4, which is birational and motivic, and normalized for $(i,j) \neq (0,0)$. However, the map b_p is *not* injective in general, even when $T = S$. We explain this claim in what follows. First, note that $H^{i,j}_{\mathrm{ur}}(X)\{p\}$ is zero unless $(i,j) = (0,0),(1,0),(1,1),(2,1),(2,2)$ for any surface $X \in \mathbf{SmProj}$; see [41, Lemma 2.1] for the vanishing of $H^{3,2}_{\mathrm{ur}}(X)\{p\}$. For the torsion motive M, we have $H^{i,j}_{\mathrm{ur}}(M)\{p\} = 0$ unless $(i,j) = (1,0),(1,1),(2,1),(2,2)$. Noting that $H^{i,j}_{\mathrm{ur}}(M)\{p\}$ is killed by $\mathrm{Tor}^{\mathrm{eff}}_{\mathbb{Z}_p}(M)$, we have

$$H^{i,j}_{\mathrm{ur}}(M)\{p\} \cong \varinjlim_{n\geq 1} H^{i-j}_{\text{ét}}(M, W_n\Omega^j_{S,\log}) \cong H^{i-j+1}_{\text{ét}}(S, W\Omega^j_{S,\log})_{\mathrm{Tor}},$$

where the left isomorphism follows from the Gersten resolution and the purity of logarithmic Hodge-Witt sheaves [25], [24]; one also needs the fact that $\mathrm{Pic}(M)$ is killed by $\mathrm{Tor}^{\mathrm{eff}}_{\mathbb{Z}}(M)$ for $(i,j) = (2,1)$. See [28, Chapter I, 5.7.5] for the right isomorphism. Now assume that S is a *supersingular* Enriques surface over k with $\mathrm{ch}(k) = 2$, which satisfies $\mathrm{Pic}^{\tau}_{S/k} \cong \alpha_2$ [28, Chapter II, 7.3.1 (d)]. Then the unramified cohomology groups are computed as follows:

(a) We have $H^2(S, W\mathscr{O}_S) \cong k$, on which the Frobenius operator F is 0 [28, Chapter II, 7.3.2]. Hence, $H^2_{\text{ét}}(S,\mathbb{Z}_2) = H^2(S,W\mathscr{O}_S)^{F=1} = 0$, and $H^{1,0}_{\mathrm{ur}}(M)\{2\} = 0$.

(b) Since $\mathrm{Pic}^{\tau}_{S/k} \cong \alpha_2$, $H^1_{\text{ét}}(S, W\Omega^1_{S,\log})_{2\text{-Tor}}$ is zero (i.e., $H^{1,1}_{\mathrm{ur}}(M)\{2\} = 0$).

(c) Since $H^2(S, W\Omega^1_S) \cong k$ [28, Chapter II, 7.3.6 (b)], we have $H^2_{\text{ét}}(S, W\Omega^1_{S,\log}) \cong \mathbb{Z}/2\mathbb{Z}$ or 0. Since $\mathrm{Pic}^{\tau}_{S/k} \cong \alpha_2$, the perfect group scheme $\underline{H}^0_{\text{ét}}(S, \Omega^1_{S,\log})$ is isomorphic to α_2, and the étale part of $\underline{H}^2_{\text{ét}}(S,\Omega^1_{S,\log})$ is zero by the flat duality of Milne [35, 2.7 (c)] (i.e., $H^2_{\text{ét}}(S,\Omega^1_{S,\log}) = 0$). Therefore, $H^{2,1}_{\mathrm{ur}}(M)\{2\} = 0$.

(d) Since $H^1(S, W\Omega^2_S) = 0$, $H^1_{\text{ét}}(S, W\Omega^2_{S,\log})$ is zero (i.e., $H^{2,2}_{\mathrm{ur}}(M)\{2\} = 0$).

Thus, we have $H^{i,j}_{\mathrm{ur}}(M)\{2\} = 0$ for all i,j. However, we have $H^2(S,\mathscr{O}_S) \cong k$. Since the functor $H^2(-,\mathscr{O}_-)$ is normalized, birational and motivic [10], we have $H^2(M,\mathscr{O}_M) \cong k$ and M is nonzero in $\mathbf{Chow}^{\mathrm{nor}}_{\mathbb{Z}_2}$. These facts imply that b_2 for $T = S$ is not injective.

4. Cohomology of the torsion motive of a surface

We retain the assumptions and notations introduced in Setting 3.1 and 3.5. We prove a few preliminary lemmas in this section. To ease the notation, put

$$N_S := \mathrm{NS}(S)_{\mathrm{Tor},\Lambda} \qquad B_S := \mathrm{Br}(S)_{\Lambda}. \tag{4.1}$$

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For a positive integer m invertible in k, we denote the Bockstein operator for m by

$$Q : H^i_{\text{ét}}(-,\mu_m) \to H^{i+1}_{\text{ét}}(-,\mu_m) \tag{4.2}$$

(i.e., the connecting map associated to the short exact sequence $0 \to \mu_m \to \mu_{m^2} \to \mu_m \to 0$).

Lemma 4.1. *For any $m \in \mathbb{Z}_{>0}$ invertible in k, we have canonical isomorphisms*

$$H^i_{\text{ét}}(M,\mu_m) \cong \begin{cases} 0 & (i \neq 1,2,3), \\ N_S[m] & (i = 1), \\ B_S/mB_S & (i = 3), \end{cases} \tag{4.3}$$

and an exact sequence

$$0 \longrightarrow N_S/mN_S \longrightarrow H^2_{\text{ét}}(M,\mu_m) \longrightarrow B_S[m] \longrightarrow 0. \tag{4.4}$$

If moreover $mN_S = 0$ (so that we have $mB_S = 0$ as well by (3.2)), then we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1_{\text{ét}}(M,\mu_m) & \xrightarrow{Q} & H^2_{\text{ét}}(M,\mu_m) & \xrightarrow{Q} & H^3_{\text{ét}}(M,\mu_m) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\cong} & & \| & & \uparrow{\scriptstyle\cong} & & \\ 0 & \longrightarrow & N_S & \longrightarrow & H^2_{\text{ét}}(M,\mu_m) & \longrightarrow & B_S & \longrightarrow & 0, \end{array}$$

where the vertical isomorphisms are those in (4.3), and the lower sequence is obtained from the exact sequence (4.4) with the identifications $N_S/mN_S = N_S$, $B_S[m] = B_S$.

Proof. The first statement follows from Proposition 2.14, Lemma 3.2 and Corollary 3.7 (2), and the second from the definition of Q. □

Lemma 4.2. *Suppose that $m_0 \in \mathbb{Z}_{>0}$ is invertible in k and $m_0 N_S = 0$. Put $m := m_0^2$ and let Q be the Bockstein operator (4.2) for m. Then there exists a subgroup $\widetilde{B}_S$ of $H^2_{\text{ét}}(M,\mu_m)$ fitting into a commutative diagram with exact row*

$$\begin{array}{ccccccccc} & & N_S & & \widetilde{B}_S & \overset{\cong}{\searrow} & & & \\ & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cap} & & & & \\ 0 & \longrightarrow & H^1_{\text{ét}}(M,\mu_m) & \xrightarrow[Q]{} & H^2_{\text{ét}}(M,\mu_m) & \xrightarrow[Q]{} & H^3_{\text{ét}}(M,\mu_m) & \longrightarrow & 0. \end{array} \tag{4.5}$$

In particular, we have an isomorphism

$$H^2_{\text{ét}}(M,\mu_m) \cong QN_S \oplus \widetilde{B}_S, \tag{4.6}$$

where we identified $N_S = H^1_{\text{ét}}(M,\mu_m)$.

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Proof. Put $H^i_{\text{ét},n}(M) := H^i_{\text{ét}}(M,\mu_n)$. We consider a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & H^2_{\text{ét},m_0}(M) & \longrightarrow & B_S[m_0] & \longrightarrow & 0\\
 & & & & \downarrow & & \downarrow{\cong} & & \\
0 & \longrightarrow & N_S/mN_S & \longrightarrow & H^2_{\text{ét},m}(M) & \longrightarrow & B_S[m] & \longrightarrow & 0\\
 & & \downarrow{\cong} & & \downarrow & & & & \\
0 & \longrightarrow & N_S/m_0N_S & \longrightarrow & H^2_{\text{ét},m_0}(M). & & & &
\end{array}$$

All rows are from (4.4). The left and right vertical bijections come from $m_0N_S = mN_S = 0$ and $B_S[m_0] = B_S[m_0] = B_S$, which follows from our assumption on m_0 and m. We now rewrite it using the latter half of Lemma 4.1:

$$\begin{array}{ccccccccc}
 & & & & H^2_{\text{ét},m_0}(M) & \xrightarrow{Q_0} & H^3_{\text{ét},m_0}(M) & \longrightarrow & 0\\
 & & & & \downarrow{\iota} & & \downarrow{\cong} & & \\
0 & \longrightarrow & H^1_{\text{ét},m}(M) & \xrightarrow{Q} & H^2_{\text{ét},m}(M) & \xrightarrow{Q} & H^3_{\text{ét},m}(M) & \longrightarrow & 0\\
 & & \downarrow{\cong} & & \downarrow{\pi} & & & & \\
0 & \longrightarrow & H^1_{\text{ét},m_0}(M) & \xrightarrow{Q_0} & H^2_{\text{ét},m_0}(M), & & & &
\end{array}$$

where Q_0 denotes the Bockstein operator (4.2) for m_0. We then obtain the assertion from the middle horizontal exact row by putting $\widetilde{B}_S := \operatorname{Im}(\iota) = \ker(\pi)$. □

5. Vishik's method

In [44, §4], Vishik obtained an exact sequence that computes the motivic cohomology with $\mathbb{Z}/5\mathbb{Z}$ coefficients of the classical Godeaux surface over $\mathbb{C}$. In this section, we apply his method to a general surface having a decomposition of the diagonal over an arbitrary algebraically closed field. The main result of this section is Theorem 5.2 below.

We retain the assumptions and notations introduced in Setting 3.1 and 3.5. We also fix the following data:

Setting 5.1. Fix $m_0 \in \mathbb{Z}_{>0}$ that is invertible in k and divisible by (3.2). Put $m := m_0^2$. We also fix an isomorphism $\mathbb{Z}/m\mathbb{Z} \cong \mu_m$ by which we will identify étale and Galois cohomology with different Tate twists. We write

$$H^i_{\text{ét}}(-) := H^i_{\text{ét}}(-,\mathbb{Z}/m\mathbb{Z}), \qquad H^i_{\text{Gal}}(-) := H^i_{\text{Gal}}(-,\mathbb{Z}/m\mathbb{Z}).$$

Using the isomorphism from (2.10), (4.1) and (4.3), we identify

$$H^1_{\text{ur}}(S) \cong H^1_{\text{ét}}(M) \cong N_S, \qquad H^2_{\text{ur}}(S) \cong H^3_{\text{ét}}(M) \cong B_S, \tag{5.1}$$

which are finite abelian groups dual to each other by Lemma 3.2 (3).

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5.1. Motivic cohomology

For $X \in \mathbf{Sm}$, $K \in \mathbf{Fld}$, and $a,b \in \mathbb{Z}$ with $b \geq 0$, we write

$$H^{a,b}_{\mathscr{M}}(X_K,\Lambda) := H^a_{\mathrm{Zar}}(X_K,\Lambda(b)), \qquad H^{a,b}_{\mathscr{M}}(X_K) := H^a_{\mathrm{Zar}}(X_K,\mathbb{Z}/m\mathbb{Z}(b)), \tag{5.2}$$

where $\Lambda(b)$ and $\mathbb{Z}/m\mathbb{Z}(b)$ are Voevodsky's motivic complex [34, Definition 3.1] with coefficients in Λ and $\mathbb{Z}/m\mathbb{Z}$, respectively. We put $H^{a,b}_{\mathscr{M}}(X_K,\Lambda) = H^{a,b}_{\mathscr{M}}(X_K) = 0$ if $b < 0$. We recall the following fundamental facts:

$$H^{a,b}_{\mathscr{M}}(X_K,\Lambda) = H^{a,b}_{\mathscr{M}}(X_K) = 0 \quad \text{if } a > 2b \text{ or } a > b + \dim X. \tag{5.3}$$

$$H^{2b,b}_{\mathscr{M}}(X_K,\Lambda) \cong \mathrm{CH}^b(X_K)_\Lambda, \quad H^{2b,b}_{\mathscr{M}}(X_K) \cong \mathrm{CH}^b(X_K)/m\mathrm{CH}^b(X_K), \tag{5.4}$$

$$H^{a,b}_{\mathscr{M}}(X_K) \cong H^a_{\text{ét}}(X_K) \quad \text{if } a \leq b. \tag{5.5}$$

The case $a > 2b$ of (5.3) and (5.4) are consequences of Voevodsky's comparison theorem on the motivic cohomology with Bloch's higher Chow groups (see [34, Corollary 19.2, Theorem 19.3]). The second case of (5.3) is immediate from the definition (see [34, Theorem 3.6]). The former Beilinson-Lichtenbaum conjecture (5.5) is proved in [48, Theorem 6.17] as a consequence of Rost-Voevodsky's norm residue isomorphism theorem [48, Theorem 6.16], based on the previous works of Suslin-Voevodsky [40] and Geisser-Levine [21].

If we fix a,b and K and let X vary, then $H^{a,b}_{\mathscr{M}}(X_K,\Lambda)$ defines a motivic functor. This follows from [34, Propositions 14.16 and 20.1], as $H^{a,b}_{\mathscr{M}}(X_K,\Lambda)$ is the colimit of $H^{a,b}_{\mathscr{M}}(X \times U,\Lambda)$ where U ranges over all smooth schemes over k with function field K. The same is true of $H^{a,b}_{\mathscr{M}}(X_K)$. Therefore, the notations and results discussed in the previous paragraph are extended to motives; cf. Remark 2.7.

We now state the main result of this section.

Theorem 5.2. *For any $a \in \mathbb{Z}$ and $K \in \mathbf{Fld}$, we have an exact sequence*

$$0 \to H^{a,a-2}_{\mathscr{M}}(M_K) \to \bigoplus_{i=1,2} H^{a-i-1}_{\mathrm{Gal}}(K) \otimes H^i_{\mathrm{ur}}(S) \xrightarrow{\Psi} H^{a-1}_{\mathrm{ur}}(K(S)/K) \to 0.$$

Here, Ψ is given by $\Psi(a \otimes b) = \mathrm{pr}_1^*(a) \cup \mathrm{pr}_2^*(b)$, where pr_i denotes the respective projectors on $\mathrm{Spec}(K) \times S$. (The last term is the unramified cohomology over K and not over k.)

5.2. Étale cohomology

Proposition 5.3. *For any $N \in \mathbf{Chow}_\Lambda$ and $K \in \mathbf{Fld}$, we have an isomorphism*

$$H^*_{\mathrm{Gal}}(K) \otimes H^*_{\text{ét}}(N) \cong H^*_{\text{ét}}(N_K). \tag{5.6}$$

Proof. Vishik proved (5.6) in [44, Proposition 4.2] assuming $k = \mathbb{C}$ and m is a prime, although his proof did not use those assumptions. For the sake of completeness, we include a short proof. We may replace N by $X \in \mathbf{Sm}$. Consider the spectral sequence

$$E_2^{a,b} = H^a_{\mathrm{Gal}}(K, H^b_{\text{ét}}(X_{\overline{K}})) \Rightarrow H^{a+b}_{\text{ét}}(X_K), \tag{5.7}$$

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where $\overline{K}$ is a separable closure of K. By the smooth base change theorem, we have $H^b_{\text{ét}}(X_{\overline{K}}) \cong H^b_{\text{ét}}(X)$ on which the absolute Galois group of K acts trivially, and hence,

$$E_2^{a,b} = H^a_{\text{Gal}}(K, H^b_{\text{ét}}(X_{\overline{K}})) \cong H^a_{\text{Gal}}(K) \otimes H^b_{\text{ét}}(X).$$

Observe that $E_2^{*,*}$ is generated by $H^*_{\text{ét}}(X)$ as a $H^*_{\text{Gal}}(K)$-module, and the differential maps $d_r^{*,*} : E_r^{*,*} \to E_r^{*+r,*-r+1}$ are $H^*_{\text{Gal}}(K)$-linear. It follows from the commutative diagram

$$\begin{array}{ccc} H^j_{\text{ét}}(X_K) & \longrightarrow & E_2^{0,j} = H^0_{\text{Gal}}(K, H^j_{\text{ét}}(X_{\overline{K}})) \\ \uparrow & & \downarrow \\ H^j_{\text{ét}}(X) & \xrightarrow{\ \cong\ } & H^j_{\text{ét}}(X_{\overline{K}}) \end{array}$$

that the edge maps $H^j_{\text{ét}}(X_K) \to E_2^{0,j}$ are surjective for all j, whence $E_2^{0,j} = E_\infty^{0,j}$. We conclude that (5.7) degenerates at E_2-terms and induces the desired isomorphism. □

Remark 5.4. The proof shows that (5.6) remains valid when N is replaced by any $X \in \mathbf{Sm}$.

Corollary 5.5. *For any $K \in \mathbf{Fld}$ and $a \in \mathbb{Z}$, we have an isomorphism*

$$\begin{aligned} H^{a,a}_{\mathscr{M}}(M_K) \cong &(H^{a-1}_{\text{Gal}}(K) \otimes N_S) \oplus (H^{a-2}_{\text{Gal}}(K) \otimes QN_S) \\ &\oplus (H^{a-2}_{\text{Gal}}(K) \otimes \widetilde{B}_S) \oplus (H^{a-3}_{\text{Gal}}(K) \otimes B_S). \end{aligned} \tag{5.8}$$

Proof. Apply Proposition 5.3 to $N = M$ and use (4.6), (5.1) and (5.5). □

5.3. The first coniveau filtration

The isomorphism $\mathbb{Z}/m\mathbb{Z} \cong \mu_m$ fixed in Setting 5.1 yields a homomorphism

$$\tau : H^{a,b}_{\mathscr{M}}(M_K) \to H^{a,b+1}_{\mathscr{M}}(M_K).$$

Proposition 5.6. *For any $K \in \mathbf{Fld}$ and $a \in \mathbb{Z}$, the map*

$$\tau : H^{a,a-1}_{\mathscr{M}}(M_K) \to H^{a,a}_{\mathscr{M}}(M_K) \cong H^a_{\text{ét}}(M_K)$$

is injective, and its image corresponds to the subgroup

$$\begin{aligned} &(H^{a-2}_{\text{Gal}}(K) \otimes QN_S) \oplus (H^{a-3}_{\text{Gal}}(K) \otimes B_S) \\ &\oplus \ker[\alpha_a : (H^{a-1}_{\text{Gal}}(K) \otimes N_S) \oplus (H^{a-2}_{\text{Gal}}(K) \otimes \widetilde{B}_S) \to H^a_{\text{ur}}(M_K)] \end{aligned} \tag{5.9}$$

under the isomorphism (5.8) (see (2.9) for $H^a_{\text{ur}}(M_K)$). Here, α_a is given by the composition

$$(H^{a-1}_{\text{Gal}}(K) \otimes N_S) \oplus (H^{a-2}_{\text{Gal}}(K) \otimes \widetilde{B}_S) \overset{(5.8)}{\hookrightarrow} H^{a,a}_{\mathscr{M}}(M_K) \xrightarrow{\rho} H^a_{\text{ur}}(M_K),$$

where ρ is given by Theorem 5.8 (1) below.

Remark 5.7. We will show that α_a is surjective in Proposition 5.10 below.

For the proof, we recall an important result from [42]:

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Theorem 5.8. *Let $X \in \mathbf{Sm}$, $K \in \mathbf{Fld}$ and $a,b \in \mathbb{Z}$ with $b \geq 0$.*

(1) *There exists a long exact sequence*

$$\cdots \to H^{a,b-1}_{\mathscr{M}}(X_K) \xrightarrow{\tau} H^{a,b}_{\mathscr{M}}(X_K) \xrightarrow{\rho} H^{a-b}_{\mathrm{Zar}}(X_K, \mathscr{H}^b_m) \to H^{a+1,b-1}_{\mathscr{M}}(X_K) \xrightarrow{\tau} \cdots,$$

where $\mathscr{H}^b_m$ is from (2.9).

(2) *Let $E_1^{i,j} = H^{2i+j}_{\mathrm{Zar}}(X_K, \mathscr{H}^{-i}_m) \Rightarrow H^{i+j}_{\text{ét}}(X_K)$ be the τ-Bockstein spectral sequence constructed in [42, p. 4478] (using the long exact sequence in (1)). Let ${}^{\dagger}E_1^{i,j} = \oplus_{x \in (X_K)^{(i)}} H^{j-i}_{\mathrm{Gal}}(K(x)) \Rightarrow H^{i+j}_{\text{ét}}(X_K)$ be the coniveau spectral sequence. Then we have an isomorphism of spectral sequences $E_r^{i,j} \cong {}^{\dagger}E_{r+1}^{2i+j,-i}$.*

(3) *The composition*

$$\mathrm{CH}^a(X_K)/m\mathrm{CH}^a(X_K) \cong H^{2a,a}_{\mathscr{M}}(X_K) \xrightarrow{\tau^a} H^{2a,2a}_{\mathscr{M}}(X_K) \cong H^{2a}_{\text{ét}}(X_K)$$

agrees with the cycle map.

Proof. This is taken from [42, Lemma 2.1, Theorem 2.4]. Here, we only recall that (1) is a consequence of (5.5), (2) is due to Deligne and Paranjape (see [7, p.195, footnote], [36, Corollary 4.4]), and (3) is a consequence of (2). □

We need a simple lemma.

Lemma 5.9.

(1) *The following diagram is commutative:*

$$\begin{array}{ccc} H^{a,b}_{\mathscr{M}}(M_K) & \xrightarrow{\tau} & H^{a,b+1}_{\mathscr{M}}(M_K) \\ {\scriptstyle Q}\uparrow & & \uparrow{\scriptstyle Q} \\ H^{a-1,b}_{\mathscr{M}}(M_K) & \xrightarrow[\tau]{} & H^{a-1,b+1}_{\mathscr{M}}(M_K). \end{array}$$

(2) *We have $Q(H^a_{\mathrm{Gal}}(K) \otimes H^b_{\text{ét}}(M)) = H^a_{\mathrm{Gal}}(K) \otimes Q(H^b_{\text{ét}}(M))$.*

Proof. The m-th power map $H^0_{\mathrm{Gal}}(k, \mu_{m^2}) \to H^0_{\mathrm{Gal}}(k, \mu_m)$ is surjective since k is algebraically closed, and hence, $Q(\zeta) = 0$ for any $\zeta \in \mu_m$. Thus, (1) follows from a formal property of the Bockstein operator $Q(x \cup y) = Q(x) \cup y \pm x \cup Q(y)$ by taking $y = \zeta$ (since $\tau = - \cup \zeta$ by definition). The same formal property reduces (2) to the surjectivity of $H^a_{\mathrm{Gal}}(K, \mu_{m^2}^{\otimes a}) \to H^a_{\mathrm{Gal}}(K, \mu_m^{\otimes a})$, which is a consequence of the norm residue isomorphism theorem (see [48, Theorem 6.16]). □

Proof of Proposition 5.6. The injectivity of τ is a part of the Beilinson-Lichtenbaum conjecture (proved by Voevodsky in [48, Theorem 6.17]). Since $H^{-1}_{\mathrm{Zar}}(S_K, \mathscr{H}^a_m) = 0$ and $H^0_{\mathrm{Zar}}(S_K, \mathscr{H}^a_m) = H^a_{\mathrm{ur},m}(S_K)$ by the definition (2.9), we obtain from Theorem 5.8 (1) with

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$a = b$ an exact sequence sitting in the upper row of a diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^{a,a-1}_{\mathscr{M}}(M_K) & \xrightarrow{\tau} & H^{a,a}_{\mathscr{M}}(M_K) & \xrightarrow{\rho} & H^a_{\mathrm{ur},m}(M_K) \\ & & \Big\uparrow{\scriptstyle Q} & & \Big\uparrow{\scriptstyle Q} & & \\ & & H^{a-1,a-1}_{\mathscr{M}}(M_K) & \underset{\tau}{\xrightarrow{\cong}} & H^{a-1,a}_{\mathscr{M}}(M_K). & & \end{array} \tag{5.10}$$

(This reproves the desired injectivity.) The square in (5.10) is commutative by Lemma 5.9 (1). The lower horizontal arrow in the diagram is an isomorphism by (5.5). By (5.8), we find that $H^{a-1,a-1}_{\mathscr{M}}(M_K)$ and $H^{a,a}_{\mathscr{M}}(M_K)$ are respectively decomposed as

$$(H^{a-2}_{\mathrm{Gal}}(K)\otimes N_S)\oplus(H^{a-3}_{\mathrm{Gal}}(K)\otimes QN_S)\oplus(H^{a-3}_{\mathrm{Gal}}(K)\otimes \widetilde{B}_S)\oplus(H^{a-4}_{\mathrm{Gal}}(K)\otimes B_S),$$
$$(H^{a-1}_{\mathrm{Gal}}(K)\otimes N_S)\oplus(H^{a-2}_{\mathrm{Gal}}(K)\otimes QN_S)\oplus(H^{a-2}_{\mathrm{Gal}}(K)\otimes \widetilde{B}_S)\oplus(H^{a-3}_{\mathrm{Gal}}(K)\otimes B_S).$$

By Lemma 5.9 (2) and (5.10), we get

$$\rho(H^{a-2}_{\mathrm{Gal}}(K)\otimes QN_S) = \rho(Q\tau(H^{a-2}_{\mathrm{Gal}}(K)\otimes N_S)) = \rho(\tau Q(H^{a-2}_{\mathrm{Gal}}(K)\otimes N_S)) = 0.$$

Similarly, we obtain $\rho(H^{a-3}_{\mathrm{Gal}}(K)\otimes B_S) = 0$ since $B_S = Q\widetilde{B}_S$. To conclude (5.9), it suffices now to note that $H^a_{\mathrm{ur},m}(M_K) = H^a_{\mathrm{ur}}(M_K)$ by (2.8) and use Corollary 3.7 (1). □

5.4. The second coniveau filtration

Proposition 5.10. *For any $K \in$ **Fld** and $a \in \mathbb{Z}$, the map*

$$\tau : H^{a,a-2}_{\mathscr{M}}(M_K) \to H^{a,a-1}_{\mathscr{M}}(M_K)$$

is injective, and its image corresponds to the subgroup

$$\ker[\beta_a : (H^{a-2}_{\mathrm{Gal}}(K)\otimes QN_S)\oplus(H^{a-3}_{\mathrm{Gal}}(K)\otimes B_S) \to H^{a-1}_{\mathrm{ur}}(M_K)] \tag{5.11}$$

under the isomorphism (5.9). Here, β_a is defined by the commutativity of

$$\begin{array}{ccc} (H^{a-2}_{\mathrm{Gal}}(K)\otimes QN_S)\oplus(H^{a-3}_{\mathrm{Gal}}(K)\otimes B_S) & & \\ {\scriptstyle Q}\Big\uparrow{\scriptstyle \cong} & \searrow^{\beta_a} & \\ (H^{a-2}_{\mathrm{Gal}}(K)\otimes N_S)\oplus(H^{a-3}_{\mathrm{Gal}}(K)\otimes \widetilde{B}_S) & \xrightarrow[\alpha_{a-1}]{} & H^{a-1}_{\mathrm{ur}}(M_K). \end{array}$$

Moreover, the map α_a in (5.9) is surjective.

Proof. Since $H^{a,b}_{\mathscr{M}}(M_K,\Lambda)$ is annihilated by m for any $a,b \in \mathbb{Z}$, a commutative diagram with an exact row

$$\begin{array}{ccccccc} H^{a-1,b}_{\mathscr{M}}(M_K) & \twoheadrightarrow & H^{a,b}_{\mathscr{M}}(M_K,\Lambda) & \xrightarrow{m=0} & H^{a,b}_{\mathscr{M}}(M_K,\Lambda) & \hookrightarrow & H^{a,b}_{\mathscr{M}}(M_K) \\ & \searrow^{Q} & \Big\uparrow & & & & \\ & & H^{a,b}_{\mathscr{M}}(M_K) & & & & \end{array}$$

shows that the complex $(H^{\bullet,b}_{\mathscr{M}}(M_K), Q)$ is exact. Consider a diagram

$$\begin{array}{ccc} H^{a,a-2}_{\mathscr{M}}(M_K) & \xrightarrow{\tau} & H^{a,a-1}_{\mathscr{M}}(M_K) \\ {\scriptstyle Q}\twoheaduparrow & & \uparrow{\scriptstyle Q} \\ H^{a-1,a-2}_{\mathscr{M}}(M_K) & \xrightarrow[\tau]{} & H^{a-1,a-1}_{\mathscr{M}}(M_K), \end{array}$$

which is commutative by Lemma 5.9 (1). Since $H^{a+1,a-2}_{\mathscr{M}}(M_K) = 0$ by (5.3), the previous remark shows that the left vertical map in the diagram is surjective. The rest of the proof goes along the same lines as Proposition 5.6. We apply (5.9) to obtain direct sum decompositions of $H^{a-1,a-2}_{\mathscr{M}}(M_K)$ and $H^{a,a-1}_{\mathscr{M}}(M_K)$ respectively as

$$(H^{a-3}_{\mathrm{Gal}}(K) \otimes QN_S) \oplus (H^{a-4}_{\mathrm{Gal}}(K) \otimes B_S) \oplus \ker(\alpha_{a-1}),$$
$$(H^{a-2}_{\mathrm{Gal}}(K) \otimes QN_S) \oplus (H^{a-3}_{\mathrm{Gal}}(K) \otimes B_S) \oplus \ker(\alpha_{a}).$$

By Lemma 5.9 (2), the summand $(H^{a-3}_{\mathrm{Gal}}(K) \otimes QN_S) \oplus (H^{a-4}_{\mathrm{Gal}}(K) \otimes B_S)$ of $H^{a-1,a-2}_{\mathscr{M}}(M_K)$ is killed by the left vertical map because $Q^2 = 0$ and $B_S = Q\widetilde{B}_S$. However, $\tau \circ Q$ maps $\ker(\alpha_{a-1})$ injectively into the summand $(H^{a-2}_{\mathrm{Gal}}(K) \otimes QN_S) \oplus (H^{a-3}_{\mathrm{Gal}}(K) \otimes B_S)$ of $H^{a,a-1}_{\mathscr{M}}(M_K)$, showing the first statement.

In particular, we have shown the injectivity of $\tau : H^{a+1,a-1}_{\mathscr{M}}(X_K) \to H^{a+1,a}_{\mathscr{M}}(X_K)$. Thus, the exact sequence from Theorem 5.8 (1) applied with $a = b$ shows that $\rho : H^{a,a}_{\mathscr{M}}(X_K) \to H^{a}_{\mathrm{ur}}(M_K)$ is surjective. The same exact sequence together with Proposition 5.6 shows that $\rho((H^{a-2}_{\mathrm{Gal}}(K) \otimes QN_S) \oplus (H^{a-3}_{\mathrm{Gal}}(K) \otimes B_S)) = 0$. This completes the proof of the last statement. □

Proof of Theorem 5.2. As the unramified cohomology is normalized, birational and motivic (Proposition 2.9), we have $H^i_{\mathrm{ur}}(S) = H^i_{\mathrm{ur}}(M)$ and $H^i_{\mathrm{ur}}(K(S)/K) = H^i_{\mathrm{ur}}(M_K)$. Now Propositions 5.6 and 5.10 complete the proof. □

6. Main exact sequence

We keep the assumptions in Setting 3.1, 3.5 and 5.1.

6.1. Main exact sequence

The following is the main technical result of this paper.

Theorem 6.1. *Suppose that $S \in$ **SmProj** admits a decomposition of the diagonal (see Definition 2.12) and $\dim S = 2$. Then we have an exact sequence for any $K \in$ **Fld***

$$0 \to \mathrm{CH}_0(S_K)_{\mathrm{Tor},\Lambda} \to \bigoplus_{i=1,2} \mathrm{Hom}(H^i_{\mathrm{ur}}(S), H^i_{\mathrm{ur}}(K/k)) \to H^3_{\mathrm{ur}}(K(S)/k) \to 0.$$

(Unlike Theorem 5.2, the last term is the unramified cohomology over k and not over K.)

The proof of Theorem 6.1 will be complete in §6.3 below.

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Remark 6.2. In the situation of Theorem 6.1, we have a canonical isomorphism

$$\mathrm{CH}_0(S_K)_{\mathrm{Tor},\Lambda} \cong \mathrm{Coker}(\mathrm{CH}_0(S)_\Lambda \to \mathrm{CH}_0(S_K)_\Lambda), \tag{6.1}$$

and this group is annihilated by the integer (3.2). To see this, it suffices to note that the degree map $\mathrm{CH}_0(S_K) \to \mathbb{Z}$ is split surjective (as k is algebraically closed), and use Lemma 3.2 (4). As a special case where $K = k(T)$ for $T \in \mathbf{SmProj}$, we also have (see (2.5))

$$\mathrm{CH}_0(S_{k(T)})_{\mathrm{Tor},\Lambda} \cong \mathbf{Chow}^{\mathrm{nor}}_\Lambda(T,S). \tag{6.2}$$

6.2. Auxiliary lemmas

Lemma 6.3. *Let E be a field such that m is invertible in E and $\mu_{m^\infty} \subset E$. Then $H^j_{\mathrm{Gal}}(E)$ is a free $\mathbb{Z}/m\mathbb{Z}$-module for any $j \in \mathbb{Z}$.*

Proof. We may assume $m = \ell^e$ for a prime number $\ell \neq p$ and $e \in \mathbb{Z}_{>0}$. Recall that a module over an Artin local ring is free if and only if it is flat (see, for example, [2, Proposition 2.1.4]). By the norm residue isomorphism theorem (see [48, Theorem 6.16]), $K^M_{j-1}(E) \otimes \mu_{\ell^\infty}$ surjects onto $K^M_j(E)_{\mathrm{Tor}} \otimes \mathbb{Z}_{(\ell)}$, and hence, $K^M_j(E)_{\mathrm{Tor}}$ is divisible by ℓ. It follows that $K^M_j(E)$ is the direct sum of an ℓ-divisible group and a flat $\mathbb{Z}_{(\ell)}$-module. Thus, $K^M_j(E) \otimes \mathbb{Z}/m\mathbb{Z} \cong H^j_{\mathrm{Gal}}(E)$ is a flat $\mathbb{Z}/m\mathbb{Z}$-module. □

By the Poincaré duallty, we have a perfect paring of finite abelian groups for any $i \in \mathbb{Z}$

$$\langle -,- \rangle : H^{4-i}_{\mathrm{\acute{e}t}}(S) \times H^i_{\mathrm{\acute{e}t}}(S) \to \mathbb{Z}/m\mathbb{Z}.$$

For $i = 1,2$, we define the homomorphisms

$$Q'_i : H^{3-i}_{\mathrm{ur}}(S) \to H^{4-i}_{\mathrm{\acute{e}t}}(S), \qquad \pi_i : H^i_{\mathrm{\acute{e}t}}(S) \to H^i_{\mathrm{ur}}(S) \tag{6.3}$$

as follows. For $i = 1$, they are given by (5.1). For $i = 2$, Q'_2 and π_2 are the compositions

$$H^1_{\mathrm{ur}}(S) \cong H^1_{\mathrm{\acute{e}t}}(S) \xrightarrow{Q} H^2_{\mathrm{\acute{e}t}}(S), \qquad H^2_{\mathrm{\acute{e}t}}(S) \xrightarrow{Q} H^3_{\mathrm{\acute{e}t}}(S) \cong H^2_{\mathrm{ur}}(S),$$

where Q are the Bockstein operator (4.2). (Hence, Q'_1 and π_1 are bijective, and we have a split short exact sequence $0 \to H^1_{\mathrm{ur}}(S) \xrightarrow{Q'_2} H^2_{\mathrm{\acute{e}t}}(S) \xrightarrow{\pi_2} H^2_{\mathrm{ur}}(S) \to 0$.)

Lemma 6.4. *We have a perfect paring of finite abelian groups for $i = 1,2$*

$$\langle -,- \rangle : H^{3-i}_{\mathrm{ur}}(S) \times H^i_{\mathrm{ur}}(S) \to \mathbb{Z}/m\mathbb{Z}$$

characterized by the formula

$$\langle Q'_i(a), b \rangle = \langle a, \pi_i(b) \rangle \qquad (a \in H^{3-i}_{\mathrm{ur}}(S), b \in H^i_{\mathrm{\acute{e}t}}(S)). \tag{6.4}$$

Proof. For $i = 1$, (6.4) is nothing other than the paring in Lemma 3.2 (3), whence the result. Assume now $i = 2$. We claim that $Q'_2(H^1_{\mathrm{ur}}(S))$ is the exact annihilator of itself with respect to $\langle -,- \rangle$. For this, we first note that $\langle Q'_2(H^1_{\mathrm{ur}}(S)), Q'_2(H^1_{\mathrm{ur}}(S)) \rangle = 0$ because

$$Q(a) \cup Q(b) = Q(a) \cup Q(b) - a \cup Q^2(b) = Q(a \cup Q(b)) = 0$$

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for $a,b \in H^1_{\text{ét}}(S)$. Here, the first (resp. third) equality holds because $Q^2 = 0$ (resp. $Q : H^3_{\text{ét}}(S) \to H^4_{\text{ét}}(S)$ is the zero map, as $H^4_{\text{ét}}(S,\mathbb{Z}/m\mathbb{Z}) \to H^4_{\text{ét}}(S,\mathbb{Z}/m^2\mathbb{Z})$ is injective). We then use the fact $|Q'_2(H^1_{\text{ur}}(S))| = |H^1_{\text{ur}}(S)| = |H^2_{\text{ur}}(S)| = |H^2_{\text{ét}}(S)/Q'_2(H^1_{\text{ur}}(S))|$ to conclude the claim. It follows that $\langle -,- \rangle$ induces the perfect paring in the statement characterized by (6.4). □

Lemma 6.5. *Let E be a field satisfying the assumption of Lemma 6.3. Then for $i = 1,2$ and for any $j \in \mathbb{Z}$, we have isomorphisms*

$$H^j_{\text{Gal}}(E) \otimes H^{4-i}_{\text{ét}}(S) \cong \text{Hom}(H^i_{\text{ét}}(S), H^j_{\text{Gal}}(E)),$$
$$H^j_{\text{Gal}}(E) \otimes H^{3-i}_{\text{ur}}(S) \cong \text{Hom}(H^i_{\text{ur}}(S), H^j_{\text{Gal}}(E)).$$

Proof. This follows from Lemmas 6.3, 6.4 and 8.1 (2). □

Lemma 6.6. *The canonical map $H^2_{\text{ét}}(\text{Spec}(E \otimes_k k(S))) \to H^2_{\text{Gal}}(E(S))$ is injective for any $E \in \mathbf{Fld}$.*

Proof. We consider a commutative diagram with exact row:

$$\begin{array}{ccccccccc}
 & & & & H^2_{\text{ét}}(\text{Spec}(E \otimes_k k(S))) & \longrightarrow & H^2_{\text{Gal}}(E(S)) & & \\
 & & & & \uparrow & & \uparrow \iota_U & & \\
0 & \longrightarrow & \text{Pic}(U_E)/m\,\text{Pic}(U_E) & \longrightarrow & H^2_{\text{ét}}(U_E) & \xrightarrow[\gamma_U]{} & H^2_{\text{ur},m}(U_E) & \longrightarrow & 0,
\end{array}$$

where U is an open dense subscheme of S. Since the map in question is obtained as the colimit of $\iota_U \circ \gamma_U$ as U ranges over such schemes, it suffices to show the vanishing of the lower left group for sufficiently small $U \subset S$. For this, we take a (possibly reducible) curve $C \subset S$ whose components generate $\text{NS}(S)$. Then we find $\text{Pic}(U_E) = 0$ as soon as $U \subset S \setminus C$ because we have $\text{Pic}(S_E) = \text{NS}(S)$ by Lemma 3.2 (4). We are done. □

Lemma 6.7. *For any $E \in \mathbf{Fld}$, the map*

$$\bigoplus_{i=1,2} H^{i-1}_{\text{Gal}}(E) \otimes H^{3-i}_{\text{ur}}(S) \to H^2_{\text{Gal}}(E(S)), \quad a \otimes b \mapsto \text{pr}_1^*(a) \cup \text{pr}_2^*(b) \tag{6.5}$$

is injective, where pr_i denotes the respective projectors on $\text{Spec}(E) \times S$.

Proof. We decompose (6.5) as follows:

$$\bigoplus_{i=1,2} H^{i-1}_{\text{Gal}}(E) \otimes H^{3-i}_{\text{ur}}(S) \hookrightarrow \bigoplus_{i=1,2} H^{i-1}_{\text{Gal}}(E) \otimes H^{3-i}_{\text{Gal}}(k(S))$$
$$\hookrightarrow H^2_{\text{ét}}(\text{Spec}(E \otimes_k k(S))) \hookrightarrow H^2_{\text{Gal}}(E(S)).$$

The injectivity of the first map follows from Lemma 6.3 since $H^i_{\text{ur}}(S) = H^i_{\text{ur}}(k(S)/k)$ is a subgroup of $H^i_{\text{Gal}}(k(S))$ by definition (see (2.6), (2.11)). The second (resp. third) map is also injective by Remark 5.4 (resp. Lemma 6.6). □

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6.3. End of the proof

We consider a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{CH}_0(S_K)_{\mathrm{Tor},\Lambda} & \longrightarrow & \bigoplus_{i=1,2} H^i_{\mathrm{Gal}}(K)\otimes H^{3-i}_{\mathrm{ur}}(S) & \xrightarrow{\Psi} & H^3_{\mathrm{ur}}(K(S)/K) & \longrightarrow & 0 \\
 & & & & \Big\downarrow \partial_1 & & \Big\downarrow \partial_2 & & \\
 & & & & \bigoplus_{i=1,2}\bigoplus_{v} H^{i-1}_{\mathrm{Gal}}(F_v)\otimes H^{3-i}_{\mathrm{ur}}(S) & \xrightarrow[\psi]{} & \bigoplus_{w} H^2_{\mathrm{Gal}}(F_w). & &
\end{array}$$

The upper row is an exact sequence obtained by setting $a=4$ and replacing i with $3-i$ in Theorem 5.2. In the lower row, v (resp. w) ranges over all discrete valuations of K (resp. $K(S)$) that are trivial on k, and F_v (resp. F_w) denotes the residue field. For each v, let $w(v)$ be an extension of v to $K(S)$. Then the $(v,w(v))$-component of ψ is given by (6.5) for $E=F_v$, and the other components are zero. The two vertical maps are the residue maps recalled in §2.5.

Lemma 6.7 shows that ψ is injective. By Lemma 6.5, we have isomorphisms

$$H^i_{\mathrm{Gal}}(K)\otimes H^{3-i}_{\mathrm{ur}}(S)\cong \mathrm{Hom}(H^i_{\mathrm{ur}}(S),H^i_{\mathrm{Gal}}(K)),$$
$$H^{i-1}_{\mathrm{Gal}}(F_v)\otimes H^{3-i}_{\mathrm{ur}}(S)\cong \mathrm{Hom}(H^i_{\mathrm{ur}}(S),H^{i-1}_{\mathrm{Gal}}(F_v)).$$

By (2.6) and the left exactness of $\mathrm{Hom}(H^i_{\mathrm{ur}}(S),-)$, we obtain

$$\ker(\partial_1)=\bigoplus_{i=1,2}\mathrm{Hom}(H^i_{\mathrm{ur}}(S),H^i_{\mathrm{ur}}(K/k)).$$

However, since $H^3_{\mathrm{ur}}(K(S)/k)\subset H^3_{\mathrm{ur}}(K(S)/K)\subset H^3_{\mathrm{Gal}}(K(S))$, we have

$$\ker(\partial_2)=H^3_{\mathrm{ur}}(K(S)/K)\cap\ker(H^3_{\mathrm{Gal}}(K(S))\to\bigoplus_{w}H^2_{\mathrm{Gal}}(F_w))=H^3_{\mathrm{ur}}(K(S)/k).$$

Now a diagram chase completes the proof of Theorem 6.1.

Remark 6.8. It is not always the case that $H^i_{\mathrm{ur}}(K/k)\otimes H^{3-i}_{\mathrm{ur}}(S)\cong \mathrm{Hom}(H^i_{\mathrm{ur}}(S),$ $H^i_{\mathrm{ur}}(K/k))$.

7. Main results

In this section, we suppose k is algebraically closed and $\Lambda=\mathbb{Z}[1/p]$.

7.1. An exact sequence

Theorem 7.1. *Let* $S,T\in\mathbf{SmProj}$. *Suppose that* S *admits a decomposition of the diagonal and* $\dim S=2$. *Then we have an exact sequence*

$$0\to\mathrm{CH}_0(S_{k(T)})_{\mathrm{Tor},\Lambda}\xrightarrow{\Phi}\bigoplus_{i=1,2}\mathrm{Hom}(H^i_{\mathrm{ur}}(S),H^i_{\mathrm{ur}}(T))\to H^3_{\mathrm{ur}}(S\times T)\to 0. \tag{7.1}$$

Proof. Apply Theorem 6.1 to $K=k(T)$ and use (2.11). Note that the injectivity of Φ follows also from Proposition 3.6 together with (6.2). □

Remark 7.2. Using Lemma 8.3, we may rewrite (7.1) as follows:

$$0 \to \mathrm{CH}_0(S_{k(T)})_{\mathrm{Tor},\Lambda} \to \bigoplus_{i=1,2} \mathrm{Tor}(H^{3-i}_{\mathrm{ur}}(S), H^i_{\mathrm{ur}}(T)) \to H^3_{\mathrm{ur}}(S \times T) \to 0. \tag{7.2}$$

This, together with (5.1) recovers Kahn's exact sequence [29, Corollary 6.4] as a special case $T = S$. It also recovers [29, Corollary 6.5] as the case $\dim T = 1$. The general case should be compared with [29, Theorem 6.3], where the map

$$\mathrm{CH}_0(S_{k(T)})_{\mathrm{Tor},\Lambda} \to \bigoplus_{i=1,2} \prod_{\ell \neq p} \mathrm{Tor}(H^{3-i}_{\text{ét}}(S, \mathbb{Z}_\ell), H^i_{\text{ét}}(T, \mathbb{Z}_\ell))$$

is studied.

7.2. Faithful property of unramified cohomology

Theorem 7.3. *Let $S, T \in \mathbf{SmProj}$. Suppose that S admits a decomposition of the diagonal and $\dim S = 2$. Let $f : T \to S$ be a morphism in $\mathbf{Chow}^{\mathrm{nor}}_\Lambda$. Then the following are equivalent:*

(1) *We have $f = 0$ in $\mathbf{Chow}^{\mathrm{nor}}_\Lambda(T, S)$.*

(2) *The map $F(f) : F(S) \to F(T)$ vanishes for any normalized, birational and motivic functor $F : \mathbf{SmProj}^{\mathrm{op}} \to \mathbf{Mod}_\Lambda$.*

(3) *The map $H^i_{\mathrm{ur}}(f) : H^i_{\mathrm{ur}}(S) \to H^i_{\mathrm{ur}}(T)$ vanishes for $i = 1, 2$.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious, and (3) $\Rightarrow$ (1) follows from Theorem 7.1 and Lemma 7.4 below. □

Lemma 7.4. *Under the identification $\mathrm{CH}_0(S_{k(T)})_{\mathrm{Tor},\Lambda} = \mathbf{Chow}^{\mathrm{nor}}_\Lambda(T, S)$ from (6.2), the map Φ in (7.1) is induced by the functors H^i_{ur} for $i = 1, 2$.*

Proof. Put $K := k(T)$. We use a cartesian diagram

$$\begin{array}{ccc} S_K & \xrightarrow{\mathrm{pr}_2} & \mathrm{Spec}\, K \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle s_2} \\ S & \xrightarrow{s_1} & \mathrm{Spec}\, k, \end{array}$$

where pr_i are the projections and s_i are the structure maps. We first show, by a standard argument, the commutativity of the diagram

$$\begin{array}{ccc} H^4_{\text{ét}}(S_K) & \xrightarrow{\mathrm{cr}} & \bigoplus_i \mathrm{Hom}(H^i_{\text{ét}}(S), H^i_{\mathrm{Gal}}(K)) \\ & {\scriptstyle \mathrm{kü}}\nwarrow{\scriptstyle \cong} & \uparrow{\scriptstyle \cong\ \mathrm{pd}} \\ & & \bigoplus_i H^{4-i}_{\text{ét}}(S) \otimes H^i_{\mathrm{Gal}}(K), \end{array} \tag{7.3}$$

where cr is the correspondence action (that is, $\mathrm{cr}(\xi)(a) = \mathrm{pr}_{2*}(\mathrm{pr}_1^*(a) \cup \xi)$), kü is the Künneth isomorphism, and pd is the isomorphism from Lemma 6.5. We take $a \in H^i_{\text{ét}}(S)$,

$b \in H^{4-i}_{\text{ét}}(S)$ and $x \in H^i_{\text{Gal}}(K)$ and compute

$$\begin{aligned}(\text{cr} \circ \text{kü})(b \otimes x)(a) &= \text{pr}_{2*}(\text{pr}_1^*(a) \cup \text{pr}_1^*(b) \cup \text{pr}_2^*(x)) \\ &= \text{pr}_{2*}(\text{pr}_1^*(a \cup b) \cup \text{pr}_2^*(x)) \overset{(1)}{=} \text{pr}_{2*}(\text{pr}_1^*(a \cup b)) \cup x \\ &\overset{(2)}{=} s_2^* s_{1*}(a \cup b) \cup x = \text{pd}(b \otimes x)(a).\end{aligned}$$

Here, we have used the projection formula for étale cohomology and the base change property in [3, Exposé XVIII, Théorème 2.9] at (1) and (2), respectively. We have shown the commutativity of (7.3).

We now consider the following diagram:

$$\begin{array}{ccccc} \text{CH}_0(S_K)_{\text{Tor},\Lambda} & \xrightarrow{\text{cyc}} & H^4_{\text{ét}}(S_K) & \xrightarrow{(*)} & \bigoplus_i \text{Hom}(H^i_{\text{ét}}(S), H^i_{\text{Gal}}(K)) \\ & \searrow^{(**)} & & & \uparrow \Pi \\ & & & & \bigoplus_i \text{Hom}(H^i_{\text{ur}}(S), H^i_{\text{ur}}(T)). \end{array}$$

Here, cyc is the cycle map, and Π is the direct sum of the compositions

$$\text{Hom}(H^i_{\text{ur}}(S), H^i_{\text{ur}}(T)) \hookrightarrow \text{Hom}(H^i_{\text{ur}}(S), H^i_{\text{Gal}}(K)) \overset{\pi_i^*}{\hookrightarrow} \text{Hom}(H^i_{\text{ét}}(S), H^i_{\text{Gal}}(K)),$$

where π_i^* is induced by π_i in (6.3) (which is split surjective). If we set $\text{pd} \circ \text{kü}^{-1}$ at $(*)$ and Φ at $(**)$, then the diagram commutes by Theorem 5.8 (3) and Lemma 6.4. However, if we set cr at $(*)$ and the induced map by H^*_{ur} at $(**)$, then the diagram commutes by definition. Hence, the assertion follows from the commutativity of (7.3). □

Example 7.5. Let S be an Enriques surface over $\mathbb{C}$ (so that S admits a decomposition of the diagonal by [6] and Remark 2.16 (1)). Let $f : T \to S$ be its universal cover so that $\deg(f) = 2$ and T is a $K3$ surface. In [4, Corollary 5.7], Beauville showed that $H^2_{\text{ur}}(f)$ vanishes if and only if there exists $L \in \text{Pic}(T)$ such that $\sigma(L) = L^{-1}$ and $c_1(L)^2 \equiv 2 \bmod 4$, where $\sigma \in \text{Gal}(f)$ is the nontrivial element. Moreover, it is shown that all the S satisfying those conditions form an infinite countable union of hypersurfaces in the moduli space of Enriques surfaces [4, Corollary 6.5]. Explicit examples of S satisfying those conditions can be found in [20, 27]. As $H^1_{\text{ur}}(f) = 0$ by definition, Theorem 7.3 shows that this condition implies $F(f) = 0$ for any normalized, birational and motivic functor F.

Example 7.6. Let us apply Theorem 7.3 to $T = S$ and $f = m \cdot \text{id}_S$ with $m \in \mathbb{Z}_{>0}$. The minimal m which satisfies the condition (3) is nothing other than the torsion order $\text{Tor}^{\text{nor}}_\Lambda(S)$ in the sense of Definition 2.13. Thus, Theorem 7.3 (together with (5.1)) recovers a main result of [29, Corollary 6.4 (b)], which says $\text{Tor}^{\text{nor}}_\Lambda(S) = \exp(\text{NS}(S)_{\Lambda,\text{Tor}})$.

Theorem 7.3 suggests the following problem.

Problem 7.7. Is the functor H^*_{ur}, viewed as a functor from the full subcategory of torsion objects in $\mathbf{Chow}^{\text{nor}}_\Lambda$ to $\mathbf{Mod}_\Lambda$, faithful? (Compare [29, Question 3.5].)

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7.3. Explicit computation of the Chow group and unramified cohomology

Theorem 7.8. *Suppose the characteristic of k is zero. Let $S \in$ **SmProj** be a surface admitting a decomposition of the diagonal. If $H^1_{\mathrm{ur}}(S)$ is a cyclic group of prime order ℓ, then so are $\mathrm{CH}_0(S_{k(S)})_{\mathrm{Tor},\Lambda}$ and $H^3_{\mathrm{ur}}(S \times S)$.*

Proof. Let $M \in \mathbf{Chow}^{\mathrm{eff}}_\Lambda$ be the Chow motive constructed in Proposition 3.4. Since $\mathrm{CH}_0(S_{k(S)})_{\mathrm{Tor},\Lambda} = \mathbf{Chow}^{\mathrm{nor}}_\Lambda(S,S) = \mathbf{Chow}^{\mathrm{nor}}_\Lambda(M,M)$, Proposition 3.6 (1) and (7.1) yields an exact sequence

$$0 \to \mathbf{Chow}^{\mathrm{eff}}_\Lambda(M,M) \xrightarrow{\Phi} \bigoplus_{i=1,2} \mathrm{Hom}(H^i_{\mathrm{ur}}(S), H^i_{\mathrm{ur}}(S)) \to H^3_{\mathrm{ur}}(S \times S) \to 0. \tag{7.4}$$

We know $\mathrm{id}_M \in \mathbf{Chow}^{\mathrm{eff}}_\Lambda(S,S)$ has order ℓ by Proposition 3.6 (2). Thus, it suffices to show Φ is not surjective. If it were surjective, then by (7.4), there should be a projector $\pi : M \to M$ in $\mathbf{Chow}^{\mathrm{eff}}_\Lambda$ such that $N := \mathrm{Im}(\pi) \subset M$ satisfies $\mathrm{Pic}(N) = 0$ and $\mathrm{Br}(N) \cong \mathbb{Z}/\ell\mathbb{Z}$, but this would contradict the following result of Vishik. □

Theorem 7.9 (Vishik). *Suppose that k is of characteristic zero, and let $N \in \mathbf{Chow}^{\mathrm{eff}}_\Lambda$ be a nontrivial direct summand of a motive of a surface such that $\ell \cdot \mathrm{id}_N = 0$ for some prime number ℓ. Then we have $\mathrm{Pic}(N) \neq 0$.*

Proof. See [44, Corollary 4.22]. □

Remark 7.10. The assumption on the characteristic is used only to invoke Vishik's result. It is likely to hold in any characteristic, as long as ℓ is invertible in k.

Corollary 7.11. *In Theorem 7.8, suppose further that $k = \mathbb{C}$. Then we have*

$$\mathrm{Coker}(\mathrm{CH}^2(S \times S) \to H^4(S \times S(\mathbb{C}), \mathbb{Z}(2)) \cap H^{2,2}(S \times S)) \cong \mathbb{Z}/\ell\mathbb{Z}.$$

In particular, $S \times S$ violates the integral Hodge conjecture in codimension two.

Proof. Set $X := S \times S$. We claim that $\mathrm{CH}_0(X) \cong \mathbb{Z}$. For this, it suffices to show that $\ker(\mathrm{CH}_0(X) \to \mathbb{Z})$ is torsion by Roitman's theorem, but Proposition 3.4 implies that

$$\ker(\mathrm{CH}_0(X) \to \mathbb{Z}) \cong \mathbf{Chow}^{\mathrm{eff}}_\Lambda(\Lambda(0), M \otimes M),$$

which is obviously killed by ℓ. Now the corollary is a consequence of Theorem 7.8 and the following result of Colliot-Thélène and Voisin [14]. □

Theorem 7.12 (Colliot-Thélène, Voisin). *Suppose $k = \mathbb{C}$ and let $X \in$ **SmProj**. Assume that there exist $Y \in$ **SmProj** and a morphism $f : Y \to X$ such that $\dim Y = 2$ and $f_* : \mathrm{CH}_0(Y) \to \mathrm{CH}_0(X)$ is surjective. Then we have an isomorphism of finite abelian groups*

$$H^3_{\mathrm{ur}}(X) \cong \mathrm{Coker}(\mathrm{CH}^2(X) \to H^4(X(\mathbb{C}), \mathbb{Z}(2)) \cap H^{2,2}(X)).$$

Proof. See [14, Théorème 3.9]. □

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Example 7.13.

(1) By applying Theorem 7.8 to an Enriques surface S, we find that $\mathrm{CH}_0(S_{k(S)})_{\mathrm{Tor}}$ is of order two. This answers a question raised by Kahn [29, p. 840, footnote] (in case of characteristic zero).

(2) Similarly, we may apply Theorem 7.8 to a Godeaux surface S over $\mathbb{C}$, as long as Bloch's conjecture holds for S (see Remark 2.16). This is previously known for the classical Godeaux surface by Vishik (see a remark after Proposition 4.6 in [44]). Other Godeaux surfaces for which Bloch's conjecture is verified can be found in [26, 47].

Problem 7.14. Does the equality

$$|\mathrm{CH}_0(S_{k(S)})_{\mathrm{Tor}}| = |H^3_{\mathrm{ur}}(S \times S)|$$

remain valid when $H^1_{\mathrm{ur}}(S) \cong \mathrm{NS}(S)_{\mathrm{Tor},\Lambda}$ is not cyclic of prime order – for example, for a Beauville surface (see [19]) or for a Burniat surface (see [1]) over $\mathbb{C}$? Note that Bloch's conjecture is known for such surfaces, and we have $H^1_{\mathrm{ur}}(S) \cong \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ or $H^1_{\mathrm{ur}}(S) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, respectively.

8. Appendix: elementary homological algebra

In this section, we prove some elementary lemmas that have been used in the body of this paper.

Lemma 8.1.

(1) *Let A, B be abelian groups. Suppose that A is finitely generated and that B is a free $\mathbb{Z}$-module. Then the canonical map*

$$\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}) \otimes B \to \mathrm{Hom}(A, B \otimes \mathbb{Q}/\mathbb{Z}), \qquad \chi \otimes b \mapsto [a \mapsto b \otimes \chi(a)]$$

is an isomorphism.

(2) *Let $m \in \mathbb{Z}_{>0}$ and let A, B be $\mathbb{Z}/m\mathbb{Z}$-modules. Suppose that A is finite and that B is a free $\mathbb{Z}/m\mathbb{Z}$-module. Then the canonical map*

$$\mathrm{Hom}(A, \mathbb{Z}/m\mathbb{Z}) \otimes B \to \mathrm{Hom}(A, B), \qquad \chi \otimes b \mapsto [a \mapsto \chi(a)b]$$

is an isomorphism.

Proof. (1) Write $B = \mathbb{Z}^{\oplus I}$ with some set I. Since tensor product commutes with arbitrary sums, we can identify $- \otimes B = (-)^{\oplus I}$. To conclude, it suffices to note that $\mathrm{Hom}(A, -)$ commutes with arbitrary sums because A is finitely generated. The proof of (2) is identical. □

Lemma 8.2. *Let A, B be abelian groups. Suppose that A is finite and that B is a free $\mathbb{Z}$-module. Then we have canonical isomorphisms*

$$\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}) \otimes B \cong \mathrm{Hom}(A, B \otimes \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Ext}(A, B).$$

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Proof. The first isomorphism is from Lemma 8.1. The second is seen by an exact sequence $0 \to B \to B \otimes \mathbb{Q} \to B \otimes \mathbb{Q}/\mathbb{Z} \to 0$, together with $\mathrm{Hom}(A, B \otimes \mathbb{Q}) = \mathrm{Ext}(A, B \otimes \mathbb{Q}) = 0$ as A is finite and $B \otimes \mathbb{Q}$ is injective. □

Lemma 8.3. *Let A, B be abelian groups with A finite. Then we have canonical isomorphisms*

$$\mathrm{Tor}(\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}), B) \cong \mathrm{Hom}(A, B), \qquad \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}) \otimes B \cong \mathrm{Ext}(A, B).$$

Proof. Set $(-)^\vee := \mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$. We take an exact sequence $0 \to B_1 \to B_0 \to B \to 0$ with free $\mathbb{Z}$-modules B_i. Applying the two functors $A^\vee \otimes -$ and $\mathrm{Hom}(A, -)$, we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Tor}(A^\vee, B) & \longrightarrow & A^\vee \otimes B_1 & \longrightarrow & A^\vee \otimes B_0 & \longrightarrow & A^\vee \otimes B & \longrightarrow & 0 \\
 & & & & \downarrow \cong & & \downarrow \cong & & & & \\
0 & \longrightarrow & \mathrm{Hom}(A, B) & \longrightarrow & \mathrm{Ext}(A, B_1) & \longrightarrow & \mathrm{Ext}(A, B_0) & \longrightarrow & \mathrm{Ext}(A, B) & \longrightarrow & 0,
\end{array}$$

where two middle vertical isomorphisms are from Lemma 8.2. The lemma follows. □

9. Appendix: $\mathbb{P}^1$-invariance and birational motives

The aim of this appendix is to prove Proposition 9.1 below. We freely use the basic notion from [34]. Let F be a Nisnevich sheaf with transfers over our base field k. For $\epsilon = 0, 1$, we denote by $i_\epsilon : \mathrm{Spec}\, k \to \mathbb{A}^1$ the corresponding closed immersions and define

$$h_0(F) := \mathrm{Coker}(i_0^* - i_1^* : \mathrm{Hom}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^1), F) \to F),$$
$$\overline{h}_0(F) := \mathrm{Coker}(i_0^* - i_1^* : \mathrm{Hom}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(\mathbb{P}^1), F) \to F)$$

as presheaf cokernels. For an abelian group A, we write $F \otimes A$ for a presheaf with transfers given by $U \mapsto F(U) \otimes_{\mathbb{Z}} A$. Note that the canonical map

$$(F \otimes A)_{\mathrm{Nis}} \to (F_{\mathrm{Nis}} \otimes A)_{\mathrm{Nis}} \tag{9.1}$$

is an isomorphism (being a map of sheaves that induces isomorphisms on stalks). The following proposition is communicated to us by Bruno Kahn.

Proposition 9.1 (B. Kahn). *Let G be a $\mathbb{P}^1$-invariant Nisnevich sheaf with transfers. For any $X \in$ **Sm** connected and for any $Y \in$ **SmProj**, there is a homomorphism $(*)$ fitting in a commutative diagram*

$$\begin{array}{ccccc}
\mathbf{Cor}(X, Y) & & \longrightarrow & & \mathrm{Hom}_{\mathbf{Ab}}(G(Y), G(X)) \\
\downarrow & & & & \uparrow (*) \\
\mathbf{Cor}(\mathrm{Spec}\, k(X), Y) & = & Z_0(Y_{k(X)}) & \longrightarrow & \mathrm{CH}_0(Y_{k(X)}).
\end{array}$$

In particular, G is birational and motivic in the sense of Definition 2.5 (with $\Lambda = \mathbb{Z}$).

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Proof. We consider the following diagram:

$$\begin{array}{ccc}
\mathbf{Cor}(X,Y)\otimes_{\mathbb{Z}} G(Y) & \xrightarrow{(0)} & G(X) \\
\downarrow & \nearrow^{(1)} & \uparrow{\scriptstyle (2)} \\
(\overline{h}_0(Y)\otimes G(Y))(X) & \searrow & \\
\downarrow & & \\
(\overline{h}_0(Y)_{\mathrm{Nis}}\otimes G(Y))(X) & & (\overline{h}_0(Y)\otimes G(Y))_{\mathrm{Nis}}(X) \\
\downarrow & \swarrow^{\cong}_{(3)} & \\
(\overline{h}_0(Y)_{\mathrm{Nis}}\otimes G(Y))_{\mathrm{Nis}}(X). & &
\end{array}$$

The map (0) factors through (1) since G is $\mathbb{P}^1$-invariant; it also factors through (2) since it is a Nisnevich sheaf. By (9.1), (3) is an isomorphism. However, we have

$$\begin{aligned}(\overline{h}_0(Y)_{\mathrm{Nis}}\otimes G(Y))(X) &\cong (h_0(Y)_{\mathrm{Nis}}\otimes G(Y))(X)\\ &= h_0(Y)_{\mathrm{Nis}}(X)\otimes_{\mathbb{Z}} G(Y)\cong \mathrm{CH}_0(Y_{k(X)})\otimes_{\mathbb{Z}} G(Y),\end{aligned}$$

where the first isomorphism is from [32, Theorem 3.5] and the third from [31, Theorem 3.1.2]. We obtain an induced map $\mathrm{CH}_0(Y_{k(X)})\otimes_{\mathbb{Z}} G(Y)\to G(X)$. The proposition follows by adjunction. □

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