ON TRACE BILINEAR FORMS ON LIE-ALGEBRAS

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To what extent is the structure of a Lie-algebra \( L \) over a field \( F \) determined by the bilinear form

\[ f(a, b) = (a, b)_A \]  

(1)
on \( L \) that is derived from a matrix representation

\[ a \rightarrow \Delta(a) \quad (a \in L) \]

of \( L \) with finite degree \( d(A) \) by forming the trace of the matrix products

\[ f(a, b) = \text{tr}(\Delta a \Delta b) \quad (a, b \in L) \]  

(2)

Such a bilinear form is a function with two arguments in \( L \), values in \( F \) and the properties:

\[ f(a_1 + a_2, b) = f(a_1, b) + f(a_2, b) \]  

(bilinearity)

\[ f(a, b_1 + b_2) = f(a, b_1) + f(a, b_2) \]  

(3)

\[ f(\lambda a, b) = f(a, \lambda b) = \lambda f(a, b) \]  

(symmetry)

\[ f(a, b) = f(b, a) \]  

(symmetry)

\[ f(\lambda a, b) = \lambda f(a, b) \]  

(invariance under \( L \))  

(4)

\( \lambda \in F \); \( a, a_1, b, b_1, c \in L \).

It is clear from the definition that the trace bilinear form (1) depends only on the class of equivalent representations to which \( A \) belongs.

For any subset \( K \) of \( L \), the set \( K^1 \) of all elements \( x \) of \( L \) satisfying \( f(K, x) = 0 \) is a linear subspace of \( L \), because of the bilinearity of \( f \). This linear subspace is called the orthogonal subspace of \( K \). It coincides with the orthogonal subspace of the linear subspace \( \{FK\} \) generated by \( K \). If \( K_1 \subseteq K_2 \) then \( K_1^1 \supseteq K_2^1 \). By the symmetry of \( f \) we have \( K \subseteq (K^1)^1 \). If \( K \) is an ideal of \( L \), then it follows from the invariance of \( f \) that the orthogonal subspace \( K^1 \) is also an ideal. The ideal \( L^1 = L^1(\Delta) \) is called the radical of the representation \( \Delta \). For any ideal \( A \) of \( L \) contained in \( L^1 \), a symmetric invariant bilinear form \( f^A \) is induced on the factor algebra \( L/A \) by setting

\[ f^A(a|A, b|A) = f(a, b) \quad (a, b \in L) \]

(7)

We observe that the kernel of \( \Delta \), i.e. the ideal \( L_\Delta \) of \( L \) formed by the elements \( x \) that are mapped onto 0 by \( \Delta \), lies in the radical of \( \Delta \). By the first isomorphism theorem, \( L/L_\Delta \) is isomorphic to a Lie-subalgebra of the Lie-algebra formed by the matrices of degree \( d(A) \) over \( F \). Hence \( L/L_\Delta \) and \( a \text{ for } L/L^1 \) are finite-dimensional Lie-algebras.

It will be the aim of the investigation to determine the structure of the factor algebra \( L/L^1 \) in terms of simple algebras.

**Theorem 1.** If the characteristic of \( F \) is distinct from 2 and 3, then, for any solvable ideal \( A \) of \( L \), the ideal \( L^1 \) is contained in the radical of any matrix representation \( \Delta \).

\[ \text{For any two subsets } K_1, K_2 \text{ of } L, \text{ denote by } f(K_1, K_2) \text{ the set of all values } f(x_1, x_2), \text{ where } x_i \text{ denotes any element of } K_i (i = 1, 2). \]  

Hence \( f(K, K^1) = f(K^1, K) = 0 \).
Before we enter into the proof of Theorem 1, let us prove

**Lemma 1.** For any irreducible representation $\Delta$ of a Lie-algebra $L$ over the field of reference $F$ all of the irreducible components of the representation $\Delta^T$ obtained by restricting $\Delta$ to the sub-invariant subalgebra $T$ are equivalent, and

**Lemma 2.** If the irreducible representation $\Delta$ of the Lie-algebra $L$ over the field of reference $F$ induces by restriction to the ideal $\Gamma$ of $L$ a nilrepresentation $\Delta^A$ of $A$, then $\Delta^A$ is a null representation of $A$.

**Proof of Lemma 1.** By assumption there is a chain $L = L_0 \supseteq L_1 \supseteq \ldots \supseteq L_m = T$ of Lie-algebras over $F$ from $L$ to $T$ such that $L_i$ is an ideal of $L_{i-1}$ ($i = 1, 2, \ldots, m$). Let $M$ be a representation space of $A$. Since it is of finite dimension over $F$, it must contain an irreducible $A$-subspace $m$. Also there is a maximal $L_i$-$F$-subspace $M_x$ of $M$ such that $m \subseteq M_x$ and all irreducible components of the representation of $L_i$ with representation space $M_x$ are equivalent to the representation $F$ of $L_i$ with representation space $m$. Let $s$ be an element of $L$, $x$ an element of $L_i$, $u$ an element of $M_x$; then

$$x(su) = x(su) - s(xu) + s(xu) = (xs)u + s(xu).$$

Hence $x(su)$ is contained in $sM_x + M_x$ and thus $sM_x + M_x$ is an $L_i$-$F$-module such that the mapping of $u$ onto $su$ is an operator homomorphism of $M_x$ onto $(sM_x + M_x)/M_x$. It follows that the irreducible components of the representation of $L_i$ with representation space $(sM_x + M_x)/M_x$ are equivalent to $F$. By the Jordan-Hölder Theorem, the same applies to the irreducible components of the representation of $L_i$ with representation space $sM_x + M_x$. Because of the maximality of $M_x$ we have $sM_x + M_x = M_x$, $sM_x \subseteq M_x$, $LM_x \subseteq M_x$. Since $M$ is an irreducible $L$-$F$-space, it follows that $M_x = M$ and thus every irreducible component of $\Delta^L_i$ is equivalent to $F$.

The proof of Lemma 1 can now be completed by induction on $m$ and by an application of the Jordan-Hölder Theorem.

**Proof of Lemma 2.** Without restricting the generality we can assume that $\Delta$ is a faithful representation. Hence $\Delta^A$ is faithful. By [4, p. 34, Satz 11], the Lie-algebra $A$ is nilpotent. By [4, p. 29], every irreducible component of $\Delta^A$ is a null representation. Let $M$ be a representation space of $\Delta$. It contains a minimal $A$-$F$-subspace $\neq 0$, say $m$. Hence $Am = 0$. Let $M_1$ be the linear subspace of $M$ consisting of all elements $u$ of $M$ satisfying $Au = 0$. Applying (8) for $s$ of $L$, $x$ of $A$, $u$ of $M_1$, we find that $su$ belongs to $M_1$. Hence $M_1$ is a non-vanishing invariant subspace of the $L$-$F$-space $M$. Since $M_1$ is irreducible, it follows that $M_1 = M$, $AM = 0$ and this proves Lemma 2.

**Proof of Theorem 1.** (1) Let $F$ be algebraically closed, $L^T \neq L$, $\Delta$ be irreducible and faithful and $A(\Delta^A) = 0$. By Lemma 1, the irreducible representation $\Delta$ induces on $A$ a representation $\Delta^A$ all of whose irreducible constituents are equivalent. Since $A$ is nilpotent, it follows from [4, p. 29] that each irreducible representation of $A$ maps each element of $A$ onto a matrix with only one characteristic root (of maximal multiplicity). Hence, for any element $a$ of $A$, the matrix $\Delta(a)$ has only one characteristic root, say $\alpha(a)$, of maximal multiplicity $d(\Delta)$.

If the characteristic of $F$ is 0, then by the trace argument we have

$$\alpha(a + b) = \alpha(a) + \alpha(b).$$

If the characteristic of $F$ does not vanish, then it is by assumption greater than 3 and,
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since $A(AA) = 0$, it follows that (9) again holds by \cite[p. 95, formula (66)]{4}. We observe also that

$$
\Delta(\lambda a) = \lambda \Delta(a) \quad (\lambda \in F, a \in A), \quad \cdots \cdots (10)
$$

so that $\alpha$ is a linear form on $A$.

As a next step we want to show that, for any element $x$ of $L$,

$$
\alpha(xA) = 0. \quad \cdots \cdots (11)
$$

It suffices to show (11) under the additional assumption that

$$(x, x)_A \neq 0. \quad \cdots \cdots (12)$$

Indeed, we know that there are elements $y, z$ of $L$ for which $(y, z)_A \neq 0$, and from the identity

$$(y + z, y + z)_A = (y, y)_A + 2(y, z)_A + (z, z)_A$$

it follows, in view of the assumption that the characteristic of $F$ is not 2, that at least one of the three elements $(y + z, y + z)_A, (y, y)_A, (z, z)_A$ does not vanish. Hence there is an element $x_0$ of $L$ for which $(x_0, x_0)_A \neq 0$. For any element $x$ of $L$ we have the identity

$$(x, x)_A + (x_0, x_0)_A = \frac{1}{4} ((x + x_0, x + x_0)_A + (x - x_0, x - x_0)_A),$$

so that at least one of the three elements $(x, x)_A, (x + x_0, x + x_0)_A, (x - x_0, x - x_0)_A$ does not vanish. Therefore, if we have shown already that $\alpha(x_0A) = 0$ and that at least one of the three conditions $\alpha(xA) = 0, \alpha((x + x_0)A) = 0, \alpha((x - x_0)A) = 0$ is satisfied, it follows from the linearity of $\alpha$ that (11) is true without restrictions on the element $x$ of $L$.

Now let us assume (12).

We want to show that for any subalgebra $U$ of $A$ satisfying $xU \subseteq U$ we have $\alpha(xU) = 0$. We observe that $V = Fx + U$ is a subalgebra of $L$ containing $U$ as an ideal. The representation $\Delta$ induces a representation $\Delta^V$ on $V$. Let $\Gamma$ be an irreducible constituent of $\Delta^V$ with representation space $m$. Since $(x, x)_A$ is the trace of $(Ax)^2$, which can be formed by adding up the traces of $(Fx)^2$ over all irreducible constituents of $\Delta^V$, it follows from (12) that $\Gamma$ may be chosen in such a way that

$$(x, x)_A \neq 0. \quad \cdots \cdots (13)$$

(a) If $V$ is nilpotent then, by \cite[p. 29]{4}, the matrix $\Gamma(x)$ has only one characteristic root $\xi$, so that $(x, x)_A = d(\Gamma)^2 \xi^2$ and thus, by (13), we have $d(\Gamma) \neq 0$ in $F$, $\xi^2 \neq 0$. From \cite[p. 97, Satz 12]{4} it follows that $d(\Gamma) = 1, \Gamma(xU) = 0, \alpha(xU) = 0$.

(b) If $U = Fu$ and

$$xu = \lambda u \quad (\lambda \neq 0), \quad \cdots \cdots (14)$$

then there is a characteristic root $\xi$ of $\Gamma(x)$ and an element $v \neq 0$ of $m$ such that

$$xv = \xi v. \quad \cdots \cdots (15)$$

Set $v_0 = v$ and $v_{i+1} = uv_i$ for $i = 0, 1, 2, \ldots$. It follows by induction that

$$xv_i = (\xi + i\lambda)v_i \quad (i = 0, 1, 2 \ldots). \quad \cdots \cdots (16)$$

Indeed (15) is (16) for $i = 0$. Let (16) be proved for some subscript $i$; then it follows from (14) that

$$xv_{i+1} = x(uv_i) = (ux)v_i + u(xv_i) = uv_i + u(\xi + i\lambda)v_i = \lambda v_{i+1} + (\xi + i\lambda)v_{i+1} = (\xi + (i + 1)\lambda)v_{i+1}.$$

Since $m$ is finite-dimensional, it follows that there is a first element among the elements...
v_0, v_1, ... that is linearly dependent on the preceding elements, say v_g. Hence the linearly independent elements v_0, v_1, ..., v_{g-1} span a linear subspace of m which is invariant under V. Since m is irreducible, it follows that the g elements v_0, ..., v_{g-1} form a basis of m. Hence

\[(x, x)_F = \text{tr} \left( (Tx)^2 \right) = \sum_{i=0}^{g-1} (\xi + i\lambda)^2 \]

\[= g\xi^2 + g(g-1)\xi\lambda + \frac{g(g-1)(2g-1)}{6} \lambda^2 \]

\[= g\left(\xi^2 + (g-1)\xi\lambda + \frac{(g-1)(2g-1)}{6} \lambda^2 \right), \]

since the characteristic of F is different from 2 and 3.

From (13) it follows that \(g \neq 0\) in F. Hence

\[\text{tr}(Y(xu)) = g\alpha(xu) = \text{tr}(Fxu - FuFx) = 0, \quad \alpha(xu) = 0, \quad \alpha(xU) = 0.\]

(c) If \(UU = 0\) and if there is a basis \(u_1, u_2, ..., u_n\) of U over F such that \(xu_i = \lambda u_i + u_{i+1}\) \((\lambda \neq 0, i = 1, 2, ..., \mu; u_{n+1} = 0)\), and if we have shown already that \(\alpha(xu_i) = 0\) for \(i = k, k+1, ..., \mu + 1\), then we find that the linear form \(\alpha\) vanishes on the ideal \(Fu_k + Fu_{k+1} + ... + Fu_{\mu+1}\) of V, so that \(\Gamma\) induces on this ideal a nil representation. By Lemma 2 this nil representation is a null representation. If \(k > 1\), then we can apply (b) to the Lie-algebra \(\Gamma(Fx) + \Gamma(Fu_{k-1})\), substituting \(\Gamma(x)\) for \(x\) and \(\Gamma(u_{k-1})\) for \(u\), and obtain \(\alpha(u_{k-1}) = 0\). Hence, by induction, \(\alpha(u_k) = \alpha(u_{k+1}) = ... = \alpha(u_n) = \alpha(u_{\mu+1}) = 0, \alpha(xU) = 0.\)

(d) If \(UU = 0\), then let us consider a decomposition of U into the direct sum of the characteristic subspaces \(U_i\), that cannot be further decomposed into invariant subspaces. To each of the subalgebras \(Fx + U_i\), either (a) or (c) is applicable and thus we have \(\alpha(xU_i) = 0\); moreover \(\alpha(xU) = 0\) because of the linearity of \(\alpha\).

We may set \(U = AA\) and in this event we find that \(\alpha(x(AA)) = 0.\) As had been shown before, it follows that \(\alpha(L(AA)) = 0.\) Hence the irreducible representation \(L\) induces on the ideal \(L(AA)\) of \(L\) a nil representation and this nil representation is a null representation by Lemma 2. Since it is faithful by assumption, it follows that

\[L(AA) = 0. \quad \text{.................................(17)}\]

(e) Denoting by \(x^*\) the linear transformation \(\begin{pmatrix} a \\ x^a \end{pmatrix}\) of A and by S the set of the characteristic roots of \(x^*\), it follows that there is a decomposition \(A = \sum_{k \in S} A_k\) of \(A^*\) into the direct sum of the characteristic subspaces \(A_k\) of \(x^*\) consisting of all elements \(a\) of A satisfying an equation \((x^* - k)^\mu a = 0\) for some exponent \(\mu\). Moreover, by [4, p. 32], we have \(A_jA_k \subseteq A_{j+k}\), where we set \(A_h = 0\) if \(h\) is not a characteristic root of \(x^*\). From (17) it follows that \(AA\) is contained in \(A_g\). Since the characteristic of F is distinct from 2, it follows that \(A_kA_{k'} \subseteq AA \cap A_{2k} \subseteq A_0 \cap A_{2k} = 0\) if \(k \neq 0\); hence \(A_k\) is an abelian subalgebra of A. In this event \(A_k\) admits a decomposition into the direct sum of abelian subalgebras of A to which (c) is applicable, so that \(\alpha(xA_k) = 0\) if \(k \neq 0\). If \(k = 0\), then (a) is applicable and we find again that \(\alpha(xA_0) = 0.\) Hence \(\alpha(xU_k) = 0\) for all \(k\) of S and hence \(\alpha(xA) = 0\) because of the
It now follows that \( \alpha(\mathcal{A}) = 0 \), as has been shown above. The irreducible representation \( \mathcal{A} \) induces a nil representation on the ideal \( \mathcal{A} \mathcal{L} \). By Lemma 2, this nil representation is a null representation and, since \( \mathcal{A} \) is faithful, it follows that \( \mathcal{A} \mathcal{L} = 0 \).

Let \( \mathcal{B} \) be any solvable ideal of \( \mathcal{L} \) so that \( D^{k-1} \mathcal{B} = 0 \) for some exponent \( k \). There is the chain of ideals

\[
\mathcal{B} \supseteq D \mathcal{B} = \mathcal{B} B \supseteq D^2 \mathcal{B} \supseteq \ldots \supseteq D^k \mathcal{B} = 0.
\]

If \( k > 0 \), then \( D^{k-1} \mathcal{B} \) is an abelian ideal of \( \mathcal{L} \) and then it follows that \( L D^{k-1} \mathcal{B} = 0 \), as was shown above. If \( k > 1 \), then the ideal \( \mathcal{A} = D^{k-2} \mathcal{B} \) satisfies the condition \( \mathcal{A} (\mathcal{A} \mathcal{B}) = 0 \), so that \( L \mathcal{A} = 0 \), as was shown above. Since \( D^{k-1} \mathcal{B} = \mathcal{A} \mathcal{B} \subseteq L \mathcal{A} = 0 \), it follows that \( D^{k-1} \mathcal{B} = 0 \).

Hence \( \mathcal{L} \mathcal{B} = 0 \). \( \mathcal{L} \mathcal{B} \subseteq \mathcal{L}^\perp \).

(2) Let \( \mathcal{F} \) be algebraically closed and \( \mathcal{A} \) be irreducible. If \( \mathcal{L}^\perp = \mathcal{L} \), then it is obvious that \( \mathcal{L} \mathcal{A} \subseteq \mathcal{L}^\perp \). Let \( \mathcal{L}^\perp \neq \mathcal{L} \). The representation \( \mathcal{A} \) induces a faithful irreducible representation of the Lie-algebra \( \mathcal{A} \mathcal{L} \). We denote the Lie-multiplication in \( \mathcal{A} \mathcal{L} \) by \( X \circ Y = XY - YX \).

Since \( \mathcal{A} \) is a solvable ideal of \( \mathcal{L} \), it follows that \( \mathcal{A} \mathcal{A} \) is a solvable ideal of \( \mathcal{A} \mathcal{L} \) and hence it follows, as was shown at the close of (1), that \( \mathcal{A} \mathcal{L} \circ \mathcal{A} \mathcal{A} \subseteq \mathcal{A} (\mathcal{L} \mathcal{A}) \). But \( \mathcal{A} \mathcal{L} \circ \mathcal{A} \mathcal{A} = \mathcal{A} (\mathcal{L} \mathcal{A}) \) and \( (\mathcal{L} \mathcal{A})^\perp = \mathcal{A} (\mathcal{L}^\perp) \); hence \( \mathcal{A} (\mathcal{L} \mathcal{A}) \subseteq \mathcal{A} (\mathcal{L}^\perp) \), \( \mathcal{L} \mathcal{A} \subseteq \mathcal{L} \mathcal{A} + \mathcal{L}^\perp = \mathcal{L}^\perp \).

(3) Let \( \mathcal{F} \) be algebraically closed. Let

\[
\mathcal{A} \sim \begin{pmatrix}
\Delta_1 & * & \cdots & * \\
* & \ddots & \cdots & * \\
\vdots & \ddots & \ddots & \vdots \\
* & \cdots & * & \Delta_r
\end{pmatrix}
\]

be a complete reduction of the representation \( \mathcal{A} \) with irreducible constituents \( \Delta_1, \ldots, \Delta_r \). We have

\[
\text{tr}(\Delta a \Delta b) = \sum_{i=1}^r \text{tr}(\Delta_i a \Delta_i b),
\]

\[
(a, b)_\Delta = \sum_{i=1}^r (a, b)_{\Delta_i};
\]

hence

\[
\mathcal{L}^\perp (\mathcal{A}) \subseteq \bigcap_{i=1}^r \mathcal{L}^\perp (\Delta_i).
\]

Since it was shown in (2) that \( \mathcal{L} \mathcal{A} \subseteq \mathcal{L}^\perp (\mathcal{A}) \), it follows from (20) that \( \mathcal{L} \mathcal{A} \subseteq \mathcal{L}^\perp (\mathcal{A}) \).

(4) Let \( \mathcal{E} \) be an algebraically closed extension of the field of reference. The product algebra \( \mathcal{L}_E = \mathcal{L} \times \mathcal{E} \) over \( \mathcal{F} \) is a Lie algebra over \( \mathcal{E} \) such that any \( \mathcal{F} \)-basis \( \mathcal{B} \) of \( \mathcal{L} \) is an \( \mathcal{E} \)-basis of \( \mathcal{L}_E \). The representation \( \mathcal{A} \) can be uniquely extended to a representation \( \mathcal{A}^\mathcal{E} \) of \( \mathcal{L}_E \) by setting \( \Delta^\mathcal{E} (\sum_{b \in \mathcal{B}} \lambda(b) b) = \sum_{b \in \mathcal{B}} \lambda(b) b \) with coefficients \( \lambda(b) \) in \( \mathcal{E} \). The product algebra \( \mathcal{A}_E = \mathcal{A} \times \mathcal{E} \) over \( \mathcal{F} \) is a solvable ideal of \( \mathcal{L}_E \); hence it follows from (3) that \( \mathcal{L}_E \mathcal{A}_E \subseteq \mathcal{L}_E^\perp \) and thus \( \mathcal{L} \mathcal{A} \subseteq \mathcal{L}_E^\perp \cap \mathcal{L} = \mathcal{L}^\perp \).

From the proof of Theorem 1 and another application of Lemma 2 we derive the

**Corollary of Theorem 1.** Under the same assumptions, for an irreducible representation \( \mathcal{A} \) of \( \mathcal{L} \) either the radical of \( \mathcal{A} \) coincides with \( \mathcal{L} \) or the radical of \( \mathcal{A} \) does not coincide with \( \mathcal{L} \) and \( \mathcal{L} \mathcal{A} \) lies in the kernel of \( \mathcal{A} \).
The example of the solvable linear Lie-algebras formed by all 2 x 2-matrices over any field of characteristic 2 shows that Theorem 1 does not hold for fields of characteristic 2. The example of the solvable linear Lie-algebras formed by the linear combinations of the matrices
\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}, \begin{pmatrix}
0 & -1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
over any field of reference of characteristic 3 shows that the corollary of Theorem 1 does not hold any longer.

The following theorem states that, as far as the structure of \( L \) and the non-degenerate symmetric invariant bilinear form induced on \( L/L^1 \) is concerned, it suffices to assume that \( A \) is fully reducible and faithful, that \( L^1 \) lies in the centre of \( L \) and that every solvable ideal of \( L \) lies in the centre.

**Theorem 2.** If the characteristic of the field of reference is distinct from 2 and 3, then for any Lie-algebra \( L \) with a matrix representation \( A \) there is a subalgebra \( U \) with a fully reducible representation \( \Psi \) and kernel \( U_\Psi \) such that

\[
U + L^1 = L, \quad \text{........................................}(21)
\]
\[
(a, b)_\Psi = (a, b)_A \quad \text{for } a, b \in U, \quad \text{........................................}(22)
\]
\[
UU^1(\Psi) \subseteq U \Psi \subseteq U^1(\Psi), \quad \text{........................................}(23)
\]
\[
UA \subseteq U_\Psi \quad \text{for any ideal } A \text{ of } U \text{ for which } \Psi A \text{ is solvable. } \text{.............}(24)
\]

For the proof of Theorem 2 we need the following

**Lemma 3.** For any ideal \( A \) of a finite-dimensional Lie-algebra \( L \) over the field of reference \( F \), there is a subalgebra \( U \) of \( L \) such that \( U + A = L \) and \( U \cap A \) is nilpotent. If \( L/A \) is nilpotent, then \( U \) can be chosen as a nilpotent subalgebra (cf. [3, Theorem 4]).

**Proof of Lemma 3.** If \( L = 0 \), then Lemma 3 is clear. Let \( L \neq 0 \) and the theorem be proved already for Lie-algebras of dimension less than \( \dim_F L \). For any element \( a \) of \( A \) we form the adjoint linear transformation \( \text{ad}(a) = \begin{pmatrix} x \\ ax \end{pmatrix} \) of \( L \). The set of all elements \( x \) of \( L \) that are annihilated by some power of \( \text{ad}(a) \) forms a subalgebra \( L_0 \), by [4, p. 31]; moreover, \( L \) is the direct sum of \( L_0 \) and another linear subspace \( \hat{L}_0 \) such that \( \text{ad}(a)(\hat{L}_0) = \hat{L}_0 \). Now let \( a \) be an element of \( L \) for which \( \text{ad}(a) \) induces a nilpotent linear transformation of \( L/A \) (e.g. an element of \( A \)). Then it follows that \( \hat{L}_0 = [\text{ad}(a)]^{r-1}\hat{L}_0 \subseteq [\text{ad}(a)]^rL = A \), if \( r \) is large enough; hence \( L_0 + A = L \). If \( \dim_F L_0 \neq \dim_F L \), then, by the induction assumption, it follows that there is a subalgebra \( U \) of \( L_0 \) such that \( U + L_0 \cap A = L_0 \) and \( U \cap (L_0 \cap A) = U \cap A \) is nilpotent. But \( U + A = U + (L_0 \cap A) + A = L_0 + A = L \). Moreover, if \( L/A \) is nilpotent, then, since by the second isomorphism theorem \( L/A \) is isomorphic to \( L/A \), it follows that \( L_0/\gamma(L_0 \cap A) \) is nilpotent, so that it can be assumed that \( U \) is nilpotent.

If the subalgebra \( L_0 \) always coincides with \( L \), then the adjoint representation of \( L \) induces a nil representation of \( A \). The adjoint representation of \( A \) is a constituent of a nil representation; hence it is itself a nil representation and hence \( A \) is nilpotent, by Engel’s Theorem. In this case we may set \( U = L \), if \( L/A \) is not nilpotent. If \( L/A \) is nilpotent, then for every
element $a$ of $L$ the adjoint linear transformation induces a nilpotent linear transformation of $L/A$. Thus by assumption the adjoint representation of $L$ is a nil representation and by Engel’s Theorem it follows that $L$ is nilpotent. In this case we set $U = L$.

**Proof of Theorem 2.** By Lemma 3 there is a subalgebra $U$ of $L$ satisfying (21) such that $U \cap L^2$ is nilpotent. The representation $\Delta_U$ induced by $\Delta$ by restriction to $U$ has a complete reduction

$$\Delta_U \sim \begin{pmatrix} \Delta_1 & \ast & \ldots & \ast \\ \Delta_2 & \ast & \ldots & \ast \\ & \ast & \ldots & \ast \\ & & \ast & \ast \end{pmatrix}$$

with irreducible constituents $\Delta_1$, $\Delta_2$, ..., $\Delta_r$. For the fully reducible representation $\Psi$ that is obtained by adding only those irreducible constituents $\Delta_i$ for which the $\Delta_i$-radical does not coincide with $L$, we clearly obtain (22). Since $U^1(\Psi) = U \cap L^2$ is a nilpotent ideal and therefore $U^{1+} = U^{1+}(\Psi)$ is a solvable ideal of $U$, (23) follows by an application of the corollary of Theorem 1; (24) is proved similarly.

After these preparations we have the following

**Structure Theorem (Theorem 3).** (a) For any Lie-algebra $L$ over a field $F$ of characteristic distinct from 2 and 3 and for any matrix representation $\Delta$ of $L$, the factor algebra $\bar{L}$ of $L$ over the $\Delta$-radical of $L$ permits a decomposition

$$\bar{L} = \sum_{i=1}^r \bar{L}_i$$

into the direct sum of mutually orthogonal and indecomposable ideals $\bar{L}_1$, $\bar{L}_2$, ..., $\bar{L}_r$ distinct from 0.

(b) The ideals $\bar{L}_1, \bar{L}_i$ are perfect ideals and uniquely determined up to the order. The centre $z(\bar{L}_i)$ of $\bar{L}_i$ is of the same dimension over the field of reference as the factor algebra $\bar{L}_i/\bar{L}_i^2$ of $\bar{L}_i$ over $\bar{L}_i^2$.

(c) If the ideal $\bar{L}_i$ is abelian, then it is one-dimensional.

(d) If the centre of $\bar{L}_i$ vanishes, then $\bar{L}_i = \bar{L}_i^{1+}$ is simple non-abelian.

(e) Only if the characteristic of $F$ does not vanish can there be non-abelian components $\bar{L}_i$ with non-vanishing centre $z(\bar{L}_i)$. In this event the ideal $\bar{L}_i^2$ is the sum of the minimal non-vanishing perfect ideals $\bar{L}_{i1}$, ..., $\bar{L}_{imi}$ of $\bar{L}$ contained in $\bar{L}_i$. The algebra $\bar{L}_i^2$ is directly indecomposable but there is the decomposition

$$\bar{L}_i^2/z(\bar{L}_i) = \sum_{j=1}^{m_i} (\bar{L}_{i1} + z(\bar{L}_i))/z(\bar{L}_i)$$

of the factor algebra $\bar{L}_i^2/z(\bar{L}_i)$ into the direct sum of its minimal non-vanishing ideals, each of which is simple non-abelian.

(f) Every minimal non-vanishing perfect ideal of $\bar{L}$ coincides with one of the ideals $\bar{L}_{i1}$. If and only if its centre vanishes, we have $\bar{L}_{i1} = \bar{L}_i$. The minimal non-vanishing perfect ideals are mutually orthogonal.

**Proof of Theorem 3.** From the definition of $\bar{L}$ it follows that the trace bilinear form of $\Delta$ induces on $\bar{L}$ a symmetric invariant bilinear form such that the orthogonal space of $\bar{L}$ vanishes, i.e., a non-degenerate bilinear form. Hence, for every linear subspace $\bar{X}$ of $\bar{L}$, the dimension of $\bar{X}^\perp$ plus the dimension of the orthogonal subspace $\bar{X}^\perp$ is equal to the dimension of $\bar{L}$. Hence
If $\bar{X}$ is non-degenerate, i.e. if $\bar{X} \cdot \bar{X}^\dagger = 0$, then we have in any event the direct decomposition $\bar{L} = \bar{X} \oplus \bar{X}^\dagger$. Thus there is a decomposition (25) of the finite-dimensional Lie-algebra $\bar{L}$ into the direct sum of $r$ mutually orthogonal non-vanishing ideals $\bar{L}_1, \bar{L}_2, \ldots, \bar{L}_r$, such that there is no further decomposition of $\bar{L}_i$ into the direct sum of mutually orthogonal non-vanishing ideals ($i = 1, 2, \ldots, r$). Note that every ideal of $\bar{L}_i$ is also an ideal of $\bar{L}$ and that the trace bilinear form of $\Delta$ induces on $\bar{L}_i$ a non-degenerate symmetric invariant bilinear form.

If $\bar{L}_i$ is abelian, then, since the characteristic of $F$ is distinct from 2, it follows that there is an element $\bar{x}$ of $\bar{L}_i$ for which $(\bar{x}, \bar{x})_A \neq 0$, so that $\bar{L}_i$ is orthogonally decomposable into the direct sum of the ideal $F\bar{x}$ and the orthogonal complement $(F\bar{x})^\perp \subset \bar{L}_i$, and this implies that $\bar{L}_i = F\bar{x}$. Note that $\bar{L}_i^2 = 0$ implies that $\bar{L}_i$ is a perfect ideal.

Let $\bar{L}_i^2 \neq 0$. For the Lie-algebra $M = \bar{L}_i$ with non-degenerate bilinear form $f$ satisfying (2)–(5), we find that

$$f(M^2, z(M)) = f(M, Mz(M)) = f(M, 0) = 0.$$  

Conversely, if $f(M^2, x) = 0$ for the element $x$ of $M$, then $f(M^2, x) = f(M, Mx) = 0$, $Mx = 0$, $x$ lies in $z(M)$; hence $z(M) = (M^2)^\perp$, $z(M)^\perp = M^2$. If for an element $\bar{x}$ of the centre of $\bar{L}_i$ we have $(\bar{x}, \bar{x})_A \neq 0$, then there is the orthogonal decomposition of $\bar{L}_i$ into the ideal $F\bar{x}$ and its orthogonal complement. Since this is impossible and since the characteristic of the field of reference is distinct from 2, it follows that $z(\bar{L}_i)$ is contained in $(z(\bar{L}_i))^\perp = \bar{L}_i^\dagger$. The dimensions of $z(\bar{L}_i)$ and of $\bar{L}_i^\perp$ add up to the dimension of $\bar{L}_i$, so that $z(\bar{L}_i)$ is isomorphic to the factor algebra of $\bar{L}_i$ over $\bar{L}_i^{\perp}$.

By Theorem 1 every solvable ideal of $\bar{L}$ lies in $z(\bar{L})$. For every solvable ideal $A$ of $\bar{L}_i$, it follows from Theorem 1 that $\bar{L}_i^2 A \subseteq (\bar{L}_i^2)^\perp \subset \bar{L}_i = z(\bar{L}_i)$; hence $A$ lies in the second centre of $\bar{L}_i^2$, a solvable ideal of $\bar{L}$, and hence $A$ lies in $z(\bar{L}_i)$. It follows that the factor algebra $\bar{L}_i^2/z(\bar{L}_i)$ contains no abelian ideal $\neq 0$. Moreover $\bar{L}_i^2/z(\bar{L}_i) \neq 0$. The trace bilinear form of $\Delta$ induces a non-degenerate symmetric invariant bilinear form $f^\ast$ on $L_i^\ast = \bar{L}_i/z(\bar{L}_i)$.

There is a decomposition

$$L_i^* = \sum_{j=1}^{m_i} L_{ij}^*$$  

of $L_i^*$ into the direct sum of mutually orthogonal ideals $L_{ij}^*$ which permit no further proper orthogonal decomposition. For an ideal $A^*$ of $L_{ij}^*$, set $B^* = A^{\perp} \cap L_{ij}^*$, so that

$$f^\ast((A^* \cap B^*)^2, L_{ij}^*) = f^\ast(A^* \cap B^*, (A^* \cap B^*)L_{ij}^*) \subseteq f^\ast(A^*, B^*) = 0, (A^* \cap B^*)^2 = 0.$$  

Thus $A^* \cap B^*$ is an abelian ideal of $L_{ij}^*$ and therefore of $L_i^*$. Hence $A^* \cap B^* = 0$, $L_{ij}^* = A^* + B^*$, so that, by assumption, $A^* = L_{ij}^*$, and therefore $L_{ij}^*$ is simple non-abelian. If $X^*$ is any minimal non-vanishing ideal of $L_i^*$ then, as shown above, $X^* \neq 0$; hence $X^*L_{ij}^* \neq 0$, $X^*L_{ij}^* \neq 0$ for some index $j$, $X^* \cap L_{ij}^* \subseteq X^* \cap L_{ij}^*$, $X^* \cap L_{ij}^* \neq 0$. $X^* \cap L_{ij}^* = * = L_{ij}^*$. It follows that the components $L_{ij}^*$ are simple non-abelian ideals characterized as the minimal non-vanishing ideals of $L_i^*$.†

The ideal $\bar{L}_{ij}^* \subseteq \bar{L}_i^2$ formed by the cosets in $L_{ij}^*$ contains a minimal perfect ideal $\bar{L}_{ij} \neq 0$ of $\bar{L}_i^2$. It is clear that $L_{ij}^* \supseteq (L_{ij}^* + z(\bar{L}_i))/z(\bar{L}_i)$ and hence

$$(L_{ij}^* + z(\bar{L}_i))/z(\bar{L}_i) = L_{ij}^*, \quad L_{ij}^* = L_{ij}^* + z(\bar{L}_i), \quad (L_{ij}^*)^2 = (\bar{L}_{ij})^2 = \bar{L}_{ij}.$$  

† Compare [1], [2].
Thus $\mathcal{L}_i$ is uniquely determined by $L_i^*$ as the derived algebra of the algebra $\mathcal{L}_i^*$ formed by the cosets modulo $z(\mathcal{L}_i)$ belonging to $L_i^*$.

Conversely, if $A$ is a minimal perfect ideal $\not= 0$ of $\mathcal{L}$ then, because $\overline{A} = A$, we find that the $i$-th component ideal $A_i = (A + \sum_{j \neq i} A_j) \mathcal{L}_i$ lies in $\mathcal{L}_i^2$ and is isomorphic to $A$. Hence, if $A_i \not= 0$, then $A_i$ is a minimal perfect ideal $\not= 0$ of $\mathcal{L}_i$. Thus $A_i = \mathcal{L}_i^2$ for some $j$, $A_j A_i = A_i \subseteq A_i A \subseteq A$, $A_i A = A_i$, $A_i \subseteq A$. Since $A$ is itself a minimal perfect ideal $\not= 0$ of $\mathcal{L}$, it follows that $A_i = \mathcal{L}_i^2$.

Since the trace bilinear form of $A$ induces on $\mathcal{L}_i^2/z(\mathcal{L}_i)$ a non-degenerate bilinear form, it follows by an argument similar to an earlier one that

$$0 = (D^2L_i, D\mathcal{L}_i \cap (D^2L_i)\mathcal{L}_i) = (D\mathcal{L}_i, D\mathcal{L}_i \cap (D^2\mathcal{L}_i)\mathcal{L}_i),$$

$$D\mathcal{L}_i \cap (D^2\mathcal{L}_i)\mathcal{L}_i = z(\mathcal{L}_i),$$

$$\mathcal{L}_i \cap (D^2\mathcal{L}_i)\mathcal{L}_i$$

is solvable, $\mathcal{L}_i \cap (D^2\mathcal{L}_i)\mathcal{L}_i \subseteq z(\mathcal{L}_i)$,

$$\mathcal{L}_i \cap (D^2\mathcal{L}_i)\mathcal{L}_i = z(\mathcal{L}_i) = \mathcal{L}_i \cap (D\mathcal{L}_i)\mathcal{L}_i,$$

$$D^2\mathcal{L}_i = D\mathcal{L}_i, D^2\mathcal{L}_i = D\mathcal{L}_i.$$ For the perfect ideal $D\mathcal{L}_i$ we find that

$$D\mathcal{L}_i = z(\mathcal{L}_i) + \sum_{j=1}^{r_i} \mathcal{L}_{ij} = D^2\mathcal{L}_i = \sum_{j=1}^{r_i} \mathcal{L}_{ij}.$$
Let $\Gamma$ be an absolutely irreducible constituent of $\Delta$. Then for any element $z$ of $z(H) \cap H^2$ we have, by Schur's Lemma, $\Gamma_z = \xi I$ for some element $\xi$ of an extension of $F$. By [4, p. 29], for any element $h$ of $H$ the matrix $\Gamma(h)$ has only one characteristic root, say $\lambda(h)$, of maximal multiplicity $d(\Gamma)$, so that

\[(z, h) = \operatorname{tr}(\Gamma z \Gamma h) = \xi \operatorname{tr}(\Gamma(h)) = d(\Gamma) \xi \lambda(h).\]

Here either the degree of $\Gamma$ is divisible by the characteristic of $F$ or $d(\Gamma) = 1$, $\Gamma(H^2) = 0$, $\Gamma(z) = 0$, $\xi = 0$. At any rate $(z, h) = 0$. Hence $(z, h) = 0$, $z \subseteq H^1(\Delta^H)$, $z \subseteq L^1(\Delta) \subseteq z(L)$. By assumption, for each irreducible constituent $\Delta_i$ of $\Delta$ we have $L^1(\Delta_i) \subseteq L$; hence $H^1(\Delta^H) \subseteq H$. Since the characteristic of $F$ is not 2, it follows that there is an element $h$ of $H$ such that $(h, h)_{\Delta_i} = 0$. There is an absolutely irreducible constituent $\Gamma$ of $\Delta^H$ for which $(h, h)_{\Gamma} \neq 0$. On the other hand we know that the matrix $\Gamma(h)$ has only one characteristic root $\lambda(h)$ of multiplicity $d(\Gamma)$, so that $0 \neq (h, h) = \operatorname{tr}(\Gamma h)^2 = d(\Gamma) \lambda(h)^2$, $d(\Gamma)$ is not divisible by the characteristic of $F$, $d(\Gamma) = 1$, by [4, p. 97, Satz 12]. Hence $\Gamma(z) = 0$, $\Delta_i(z)$ is a nilpotent matrix. Hence, by Schur's Lemma, $\Delta_i(z)$ is a singular matrix. Hence, by the characteristic of the ideal $Fz$ of $L$, $\Delta_i z = 0$, by Lemma 2. Since $L$ is fully reducible, it follows that $\Delta z = 0$, $z = 0$, $H^2 \cap z(H) = 0$, $H^2 = 0$, q.e.d.

**Proof of the remainder of Theorem 3.** By Theorem 2 and its proof we can assure that $L$ satisfies the assumption of Lemma 4. Moreover we can assume that $0 \subseteq z(L) \subseteq L^2 \subseteq \bar{L} = \bar{L}_i$.

If there is a Cartan subalgebra $H$ of $L$ then, by Lemma 4, it is abelian. Since $H$ is nilpotent and its own normalizer, it follows from [4, pp. 28–29] that there is a decomposition $L = H + \hat{H}$ of $L$ into the direct sum of $H$ and another linear subspace $\hat{H}$ such that $H \hat{H} = \hat{H}$. Hence $H + L^2 = L$. Let $\bar{H} = H/L^2$, so that $H + \bar{L}^2 = \bar{L}$ and $\bar{H}$ is abelian. If there is a decomposition $\bar{L}^2 = \bar{A} + \bar{B}$ of $\bar{L}^2$ into the direct sum of the two ideals $\bar{A}$, $\bar{B}$ of $\bar{L}^2$, then it follows from $D\bar{L}^2 = \bar{L}^2$ that $D\bar{A} = \bar{A}$, $D\bar{B} = \bar{B}$, hence $\bar{A}$, $\bar{B}$ are ideals of $\bar{L}$. Moreover it follows from the relations $A \cap B = 0$, $A + B = \bar{L}^2$ that $A^2 + B^2 = \bar{L}^2$, $A^2 \cap B^2 = \bar{L}^2$, $A^2 \cup B^2 = \bar{L}^2$, $A^2 + B^2 = \bar{L}^2$, $A^2 \cap B^2 = \bar{L}^2$, $A^2 \cup B^2 = \bar{L}^2$, $A^2 + B^2 = \bar{L}^2$, $A^2 \cap B^2 = \bar{L}^2$, $A^2 \cup B^2 = \bar{L}^2$. Since $\bar{L}$ is orthogonally indecomposable, it follows that either $\bar{A}$ or $\bar{B}$ vanishes. Hence $\bar{L}^2$ is indecomposable.

If there is no Cartan subalgebra of $L$ then, by [4, pp. 32–33], it follows that the field of reference is finite. Let $\mathcal{O}(\bar{L}^2)$ be the associative algebra over $F$ that is generated by the adjoint linear transformations of $\bar{L}^2$. Let $\mathcal{O}(\bar{L}^2)$ be the linear associative algebra consisting of all linear transformations of $\bar{L}^2$ that are elementwise permutable with $\mathcal{O}(\bar{L}^2)$. Since $\bar{L}^2$ is perfect, it follows that there is, up to the order of the components, only one decomposition $\bar{L}^2 = \sum_i \bar{A}_i$ of $\bar{L}^2$ into the direct sum of indecomposable ideals $A_i$. Hence the factor algebra $\mathcal{O}(\bar{L}^2)$ over its radical is isomorphic to a ring sum of finitely many division algebras $E_1$, $E_2$, ..., $E_s$ of finite dimension over $F$. By a theorem of Macclagan-Wedderburn, all the $E_i$'s

\[f(H, \hat{H}) = f(H, H \hat{H}) = f(H^3, \hat{H}) = f(H^4, H \hat{H}) = f(H^5, \hat{H}) = \ldots = f(H^{s+1}, \hat{H}) = 0\]

and hence (26) is satisfied.
are finite extensions of $F$. Since the numbers prime to the product $P$ of the degrees of the extensions $E_i$ over $F$ are unbounded, it follows from [4, pp. 32-34] that there is an extension $E$ of $F$ of degree prime to $P$, such that the extended Lie-algebra $L_E$ over $E$ contains a Cartan subalgebra. By the method of the construction of $E$, there is, up to the order of the components, only one decomposition of $L_E^2$ into the direct sum of indecomposable ideals $\neq 0$, viz., the decomposition $(L_E^2)_E = \sum_{i=1}^{t} (A_i)_E$. As we have seen before, there is a decomposition $L_E = \sum_{i=1}^{t} B_i$ of $L_E$ into the direct sum of the mutually orthogonal ideals $B_i$ such that $(A_i)_E$ is contained in $B_i$, for $i = 1, 2, \ldots, s$. We have $(\sum_{i=2}^{t} (A_i)_E)^4 = B_1 + z(L_E) = (\sum_{i=2}^{t} A_i)^4_E$ and there is a linear subspace $X$ of $(\sum_{i=2}^{t} A_i)^4$ such that $B_1 + z(L_E) = (A_1)_E + z(L_E)$, $A_i)_E + X_E$ is an ideal of $L_E$ and $(A_1)_E + X_E)$ such that $B_1 \cap B = 0$ and therefore there is the orthogonal decomposition $L = B \perp B$ of $L$. It follows that $t = 1, \bar{L}$ is indecomposable, q.e.d.

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