ABSTRACT THEORY OF PACKINGS AND COVERINGS. II
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1. Preliminaries and summary. The present paper is closely related to a paper with the same title by A. M. Macbeath [3]. We use many notions which are defined there for a measure-space; nevertheless we define them once more because we consider the slightly different case of a measure-ring.

Let \((\mathcal{S}, \mu)\) be a measure-ring with unity such that the measure \(\mu\) is \(\sigma\)-finite (for definitions cf. [2]). We assume that there exists a countable group \(G\) of transformations which map \(\mathcal{S}\) onto itself and which preserve the measure \(\mu\) and the operations \(\cup, -, \cap\) in \(\mathcal{S}\). We denote by \(gA\) the image of \(A \in \mathcal{S}\) by \(g \in G\). For any subset \(\Gamma\) of \(G\) we write \(\Gamma A\) instead of \(\bigcup_{g \in \Gamma} gA\).

We say that an element \(P\) belonging to \(\mathcal{S}\) is a packing (more precisely, a \(G\)-packing) of an element \(A \in \mathcal{S}\) if \(gP \subset A\) for every \(g \in G\) and the elements \(gP\) are disjoint for different \(g\). We call an element \(C \in \mathcal{S}\) a covering of \(A\) if \(A \subset GC\). If an element \(F \in \mathcal{S}\) is simultaneously a packing and a covering of \(A\), then \(F\) is called a fundamental domain for \(A\). If, in particular, \(A\) is the ring unity \(1\), then we call \(P\) (or \(C\)) a packing (or covering) of \(\mathcal{S}\), and \(F\) a fundamental domain for \(\mathcal{S}\). In Theorem 1 we give a condition on \((\mathcal{S}, \mu)\) and \(G\) which is equivalent to the existence of a fundamental domain \(F\) for \(\mathcal{S}\).

If \(P\) and \(C\) are a packing and a covering of an element \(A \in \mathcal{S}\), then \(\mu P \leq \mu C\). This result is stated in [3] (Theorem I) for the ring \(\mathcal{S}\) of all measurable subsets of a measure-space \((X, \mathcal{S}, \mu)\) and for \(A = X\). However, the proof which is given there is more general and it can be applied to a measure-ring \((\mathcal{S}, \mu)\) and to an arbitrary \(A \in \mathcal{S}\). We shall use this result in several parts of our proof, referring to it as to the theorem about packings and coverings.

Let \(p\) be the upper bound of all measures \(\mu P\), where \(P\) are packings of \(\mathcal{S}\), and let \(c\) be the lower bound of measures \(\mu C\), where \(C\) are coverings of \(\mathcal{S}\). These numbers exist since the zero element \(0 \in \mathcal{S}\) is a packing and the ring unity \(1\) is a covering of \(\mathcal{S}\). By the theorem about packings and coverings we have \(p \leq c\). In Theorem 2 we give a condition on \((\mathcal{S}, \mu)\) and \(G\) which is equivalent to \(p = c\).

The corollaries contain results which are analogous to Theorems 1 and 2 but concern the ring of measurable sets of a measure-space. We construct also examples which show that these theorems fail to be true if the measure is not \(\sigma\)-finite.

2. Results. Let \(e \in G\) be the identity transformation. We denote by \((\pi), (\rho)\) and \((\delta)\) the following properties:

\((\pi)\) If \(A \in \mathcal{S}\), \(g \in G\), \(A \neq 0\) and \(g \neq e\), then there exists a \(B \subset A\) such that \(B \neq 0\) and \(B \cap gB = 0\).

\((\rho)\) If \(A \in \mathcal{S}\) has arbitrarily small coverings, then \(A = 0\).

\((\delta)\) If for some \(A \in \mathcal{S}\) and a certain \(g \in G\), \(g \neq e\) we have \(B \cap gB \neq 0\) for every \(B \subset A\), \(B \neq 0\), then \(A\) has arbitrarily small coverings.

**Theorem 1.** There exists a fundamental domain \(F\) for \(\mathcal{S}\) if and only if both \((\pi)\) and \((\rho)\) hold.

**Theorem 2.** \(p = c\) is equivalent to \((\delta)\).
We shall verify in § 6 that the $\sigma$-finiteness of $\mu$ is in both theorems an indispensable assumption. Let us consider now a measure-space $(X, S, \mu)$, where the measure $\mu$ is $\sigma$-finite and complete (see [2]). Let $G$ be a countable group of transformations of $X$ onto itself which preserve measurability and measure. We denote by $(\pi_0), (\pi_0'), (p_0), (\delta_0)$ the properties:

$(\pi_0)$ If $A \in S$, $g \in G$, $\mu A > 0$ and $g \neq e$, then there exists a $B \subset A$ such that $\mu B > 0$ and $B \cap gB = \emptyset$ ($\phi$ is the empty set).

$(\pi_0')$ No $g \neq e$ has fixed points (i.e. $gx \neq x$ for $x \in X$).

$(p_0)$ If $A \in S$ has arbitrarily small coverings, then $\mu A = 0$.

$(\delta_0)$ If for some $A \in S$ and a certain $g \in G$, $g \neq e$ we have $\mu(B \cap gB) > 0$ for every $B \subset A$ with $\mu B > 0$, then $A$ has arbitrarily small coverings.

Applying Theorems 1 and 2 to the measure-ring defined by $(X, S, \mu)$, we obtain the corollaries:

**COROLLARY 1.** There exists a fundamental domain for $(X, S, \mu)$ if and only if $(\pi_0), (\pi_0'), (p_0)$ hold simultaneously.

**COROLLARY 2.** $\mu = c$ is equivalent to $(\delta_0)$.

Let us consider a locally compact and $\sigma$-compact topological group $H$. We denote by $\mu$ the Haar measure on $H$ and by $S$ the ring of all $\mu$-measurable sets in $H$. Let $G$ be a countable subgroup of $H$. The left translations by elements of $G$ form a group of measure preserving transformations of the measure-space $(H, S, \mu)$. Evidently $(\pi_0)$ holds. Hence $(\delta_0)$ is true and so $\mu = c$.

It follows from Corollary 1 that if $G$ is discrete (in the topology induced by $H$), then a fundamental domain exists. This is however a known result [1]. It follows also from Corollary 1 that if $G$ is not discrete, then no fundamental domain exists. But this can be proved also directly. In fact, a fundamental domain $F$ is of positive measure and we have $F \cap gF = \emptyset$ for $g \in G - \{e\}$. Thus $G - \{e\}$ cannot intersect every neighbourhood of $e$ (see [4]).

### 3. Two lemmas.

Let us call coverings of $S$ simply coverings and let us adopt the same convention for packings and fundamental domains.

**LEMMA 1.** We assume that $(\pi)$ holds. Then every covering $C$ which is not a packing contains a covering $C_0 \neq C$.

**Proof.** We have $C \cap g^{-1}C \neq 0$ for some $g \neq e$. Let $A = C \cap g^{-1}C$ and let $B \subset A$ satisfy $(\pi)$. Thus $gB \subset gA \subset C$ and hence both $B$, $gB$ are contained in $C$. Since they are disjoint it follows that $C_0 = C - B$ is also a covering.

**LEMMA 2.** If $(\pi)$ and $(\rho)$ hold and there exists a covering $C$ with $0 < \mu C < \infty$, then a fundamental domain exists.

**Proof.** Let $C$ be the family of all coverings of finite measure. We observe that a partial order is defined in $C$ by the relation of inclusion. From Zorn’s Lemma it follows that $C$ contains a maximal decreasing chain $M$, i.e. an ordered subfamily $M$ of coverings such that no covering $C_0 \in C - M$ is contained in all $C \in M$. Let $a = \inf_{C \in M} \mu C$. There exist coverings $C_1, C_2, …, C_n, … \in M$ such that $a = \lim_{n \to \infty} \mu C_n$. Put $F = \bigcap_{n=1}^{\infty} C_n$. Thus $\mu F = a$. Let $B = GF$. Since $1 = GO_n$ for each $n$ it follows that $1 - B \subset G(C_n - F)$. We obtain from $\lim_{n \to \infty} \mu(C_n - F) = 0$
that $1 - B$ has arbitrarily small coverings. Hence, by $(\rho)$, we have $B = 1$; i.e. $F$ is a covering.

Let us verify that $F \subseteq C$ holds for every $C \in M$. Indeed, from $F \cap C \neq F$ for some $C \in M$ follows $\mu(F \cap C) < \mu F = \alpha$, and thus $\mu \bigcap_{n=0}^{m} C_n < \alpha$ for $C_0 = C$ and sufficiently large $m$. This is a contradiction since $\bigcap_{n=0}^{m} C_n \in M$. From $F \subseteq C$ for every $C \in M$ we have that no covering $C_0 \neq F$ is contained in $F$. Since $F$ is a covering, we obtain, by Lemma 1, that $F$ is also a packing and thus $F$ is a fundamental domain.

4. Proof of Theorem 1. Suppose first that a fundamental domain $F$ exists. We shall prove that both $(\pi)$ and $(\rho)$ hold. Assume that $(\pi)$ is not true, i.e. that there exists an $A \in S$ and $g \in G$, such that $A \neq 0$, $g \neq e$ and $B \cap gB \neq 0$ whenever $B \subset A$ and $B \neq 0$. From $A \in GF$ we have that, for some $g_0 \in G$, the set $B = A \cap g_0 F$ is not empty. From $B \subset A$ and $B \neq 0$ it follows that $B \cap gB \neq 0$. This is a contradiction, since $B \subset g_0 F$, $gB \subset g_0 F$ and $g \neq e$.

Now suppose that $(\rho)$ is false. We assume that $A \neq 0$ has arbitrarily small coverings. It follows that the same is true for $GA$. Thus, by the theorem about packings and coverings, there exists no packing of $GA$ except 0. Evidently $A \cap g_0 F$ is a packing of $GA$ which is different from 0 and this is a contradiction.

Now let us suppose that $(\pi)$ and $(\rho)$ hold. We take a maximal set $\Phi$ of non-zero elements $A$ of finite measure such that all elements $GA$ ($A \in \Phi$) are disjoint. This set is countable since $\mu$ is $\sigma$-finite. Thus $\Phi = \{A_1, A_2, \ldots, A_n, \ldots\}$. Suppose that $\bigcup_{n=1}^{\infty} GA_n \neq 1$. By the $\sigma$-finiteness of $\mu$ the element $B = 1 - \bigcup_{n=1}^{\infty} GA_n$ contains an element $D \neq 0$ of finite measure. We have $GD \cap GA_n = 0$ for every $n$ and this is a contradiction since $\Phi$ is maximal. Hence $1 = \bigcup_{n=1}^{\infty} GA_n$. Since $A_n$ is a covering of $GA_n$, it follows from Lemma 2 that there exists a fundamental domain $F_n$ for each $GA_n$. Thus $F = \bigcup_{n=1}^{\infty} F_n$ is a fundamental domain for $S$.

5. Proof of Theorem 2. We assume first that $(\delta)$ does not hold and we shall prove that then $p \neq c$. Let $A \in S$ and $g \in G$, $g \neq e$ be such that for $B \subset A$ and $B \neq 0$ we have $B \cap gB \neq 0$, but $A$ does not have arbitrarily small coverings. It follows that the lower bound $m$ of measures of coverings of $GA$ is positive. Let us prove that every packing $P$ of $S$ is disjoint from $GA$. Assume the contrary. Then $g_1 A \cap g_2 P \neq 0$ for some $g_1$, $g_2 \in G$ and thus $B = A \cap g_1^{-1} g_2 P \neq 0$. We have $B \subset A$, $B \neq 0$ and thus it follows from $B \cap gB \neq 0$ that $g_1^{-1} g_2 P \cap g_1^{-1} g_2 P \neq 0$. Therefore $P$ cannot be a packing. We now define $Q = 1 - GA$. If $C$ is an arbitrary covering of $S$, then evidently $M = C \cap GA$ is a covering of $GA$ and $N = C \cap Q$ is a covering of $Q$. Consequently $\mu M \geq m$. We have $M \cup N = C$, $M \cap N = 0$ and this implies $\mu N \leq \mu C - m$. Let $P$ be a packing of $S$. Since $P$ is disjoint from $GA$, $P$ is a packing of $Q$. Thus $\mu P \leq \mu N$, by the theorem about packings and coverings, and we obtain $\mu P \leq \mu C - m$. Therefore $p \neq c$ follows.

We assume now that $(\delta)$ holds and we shall prove that $p = c$. If $(\rho)$ holds, then $(\pi)$ follows by $(\delta)$, and then $p = c$ by Theorem 1. Suppose that $(\rho)$ does not hold and take a maximal set $\Omega$ of non-zero elements such that each $A \in \Omega$ has arbitrarily small coverings and the elements $GA$, where $A \in \Omega$, are disjoint. $\Omega$ is countable by the $\sigma$-finiteness of $\mu$ and
thus also \( Q = \bigcup_{A \in \mathcal{G}} GA \) has arbitrarily small coverings. We shall now prove that there exists a fundamental domain for \( 1 - Q \). This follows from Theorem 1. Indeed, \((p)\) holds for each \( A \subset 1 - Q \) by the construction of \( Q \) and it remains to verify that \((\pi)\) holds also. But if \((\pi)\) is false for some \( A \neq 0 \) and \( g \neq e \), then, by \((\delta)\), \( A \) has arbitrarily small coverings, contradicting \((p)\). Let \( F \) be a fundamental domain for \( 1 - Q \). For each \( \varepsilon > 0 \) there exists a covering \( D \) of \( Q \) with \( \mu D < \varepsilon \). It follows that \( F \cup D \) is a covering and \( F \) a packing of \( S \). Thus \( c \leq p \). By the theorem about packings and coverings, we have \( p \leq c \) and therefore \( p = c \).

6. Rings with a non-\( \sigma \)-finite measure. We give first an example of a measure-ring \((S, \mu)\) and a group \( G \) such that \((\pi)\) and \((p)\) hold but no fundamental domain exists. Let \( S \) be the ring of all Lebesgue-measurable sets of real numbers and let \( N \subset L \) be the ideal of all sets of measure 0. We denote by \( L^* \) the quotient ring \( L/N \). Let \( T \) be an infinite non-countable set and let to each \( \tau \in T \) correspond a replica \( L_{\tau}^* \) of \( L^* \). We consider the product \( S = \prod_{\tau \in T} L_{\tau}^* \). For \( A \in S \) we denote by \( A_{\tau} \) the \( \tau \)-coordinate of \( A \) \( (A_{\tau} \in L_{\tau}^*) \). Let \( m \) denote the Lebesgue measure in \( L^* \). We define \( \mu \) on \( S \) by

\[
\mu A = \sum_{\tau \in T} m A_{\tau},
\]

where the sum of a non-countable collection of positive numbers is defined to be infinite. For \( A, B \in S \) let \( C = A \cup B \) if \( C_{\tau} = A_{\tau} \cup B_{\tau} \) for every \( \tau \). Similarly we define in \( S \) the operations \(-\) and \( \cap \). Let \( G \) be the group of translations of elements of \( L \) by rational numbers. Thus for every \( A \in S \) and \( g \in G \) we can define \( g(A_{\tau}) \) for each \( \tau \). Let us define \( gA \) by \( (gA)_{\tau} = g(A_{\tau}) \).

Consequently \((\pi)\) is true and \((\delta)\) follows. Let us observe that if \( P \) is a packing of \( S \), then each \( P_{\tau} \) is a packing of \( L_{\tau}^* \) and thus \( P_{\tau} = 0 \) by the theorem about packings and coverings. Hence \( 0 \) is the only packing of \( S \) and we have \( p = 0 \). We easily observe that if \( C \) is a covering of \( S \), then \( \mu C = \infty \). Therefore \( p \neq c \).

7. Proofs of the corollaries. Let \( N \) be the ideal of all subsets of \( X \) which are of measure 0; these sets form an ideal since the measure \( \mu \) is complete. We consider the measure-ring \((S^*, \mu)\), where \( S^* \) is the quotient ring \( S/N \). Let us denote by \( A^* \in S^* \) the image of \( A \in S \) by the natural mapping of \( S \) onto \( S^* \).

We first prove Corollary 1. Suppose that \((\pi_0), (\pi'_0)\) and \((\rho_0)\) hold. Then \((\pi)\) and \((\rho)\) hold for \( S^* \). Thus, by Theorem 1, there exists an \( F \in S \) such that \( F^* \) is a fundamental domain for \( S^* \). It follows that

\[
P = F - (G - (\varepsilon))F
\]

is a packing of \( S \) such that \( Q = X - GP \in N \). Evidently \( Q \) is a union of sets \( Gx \) where \( x \in Q \). Let \( D \subset Q \) be any set which contains exactly one element from each of these sets \( Gx \). We have \( D \in N \), and thus \( D \) is measurable. It follows from \((\pi'_0)\) that \( D \) is a fundamental domain for \( Q \). Consequently \( P \cup D \) is a fundamental domain for \( S \).
Conversely, if $F$ is a fundamental domain for $S$, then evidently $F^*$ is a fundamental domain for $S^*$ and the necessity of $(\pi_0)$ and $(\rho_0)$ follows. The necessity of $(\pi'_0)$ is obvious.

To prove Corollary 2 it suffices to observe that to every packing of $S$ corresponds a packing of $S^*$ of the same measure and conversely, and that the same is true for coverings.

REFERENCES

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