AN EXTENSION OF A THEOREM OF GORDON

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In what follows all small Latin letters denote non-negative integers or functions whose values are non-negative integers. Let \( N = (n_1, \ldots, n_j) \) be a \( j \)-dimensional vector and let \( q = q(k; N) = q(k; n_1, \ldots, n_j) \) be the number of partitions of \( N \) into just \( k \) parts, each part being a vector whose components are non-negative integers. We write

\[
Q_j(k) = Q_j(k; X_1, \ldots, X_j) = \sum_{n_1, \ldots, n_j = 0}^{\infty} q(k; n_1, \ldots, n_j) X_1^{n_1} \cdots X_j^{n_j}
\]

for the generating function of \( q \). We have

\[
F_j = \prod_{h_1, \ldots, h_j = 0}^{\infty} (1 - X_1^{h_1} \cdots X_j^{h_j})^{-1} = 1 + \sum_{k=1}^{\infty} Q_j(k) Y^k.
\]

It is well known [3] that

\[
F_1 = \prod_{k=0}^{\infty} (1 - X_1^k Y)^{-1} = 1 + \sum_{k=1}^{\infty} Y^k \prod_{s=1}^{k} (1 - X_1^s)^{-1},
\]

so that

\[
Q_1(k) = \prod_{s=1}^{k} (1 - X_1^s)^{-1} = U(X_1)
\]

(say), but until 1956 the form of \( Q_j(k) \) for \( j > 1 \) was not known. Carlitz [1] and I [4] showed independently that

\[
Q_j(k) = P_j(k; X_1, \ldots, X_j) \prod_{i=1}^{j} U(X_i). \quad (1)
\]

(Carlitz dealt only with \( j = 2 \) but this case presents the essential difficulties.) Here \( P = P_j = P_j(k) \) is a polynomial in the \( X_i \), in which no term consists of a power of a single \( X_i \) only. Thus \( P_1 = 1 \) but, when \( j > 1 \), \( P_j \) is of degree \( g = \frac{1}{2} k(k - 1) \) in each \( X_i \), so that

\[
P_j = \sum_{h_1, \ldots, h_j = 0}^{\infty} \lambda(h_1, \ldots, h_j) X_1^{h_1} \cdots X_j^{h_j}.
\]

Hence, by (1),

\[
q(k; n_1, \ldots, n_j) = \sum_{h_1, \ldots, h_j = 0}^{\infty} \lambda(h_1, \ldots, h_j) \prod_{i=1}^{j} q(1; n_i - h_i). \quad (2)
\]

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In [4] I conjectured that the $\lambda$ are non-negative. Recently Gordon [2] proved this conjecture, essentially by finding the combinatorial interpretation of (2). I have nothing to add to his elegant proof of this result. But he goes on (by a quite different argument) to prove that

$$P_f(k; \xi, \eta, X_3, \ldots, X_j) = 0,$$

(3)

where $\xi, \eta$ are primitive $u$th and $t$th roots of unity respectively and $1 \leq u < t \leq k$. For this purpose he uses a recurrence relation for the $P_f(k)$, which both Carlitz [1] and I [4] found.

There is another expression for $P_f(k)$, which I found in [4] and which appears at first sight to be rather unpromising. In fact, however, it has proved [5, 7] unexpectedly useful to calculate explicit formulae for $q_f(k)$ for general $j$ and not too large $k$ and also asymptotic formulae for large $n_i$ and all $k$. Recently [6] I found the combinatorial explanation of this expression. Here I use the expression to give an alternative proof of Gordon's result (3) and to take this particular approach to the problem of the form of $P_f(k)$ somewhat further.

We write

$$\beta(m) = \prod_{i=1}^{j} (1 - X_i^m), \quad \gamma(m) = \prod_{i=1}^{j} \prod_{\rho} (1 - \rho X_i),$$

where $\rho$ runs through all primitive $m$th roots of unity. Thus

$$\beta(m) = \prod_{d|m} \gamma(d).$$

Again $\pi = \pi(k)$ denotes the partition of $k$ into $h(1)$ parts 1, $h(2)$ parts 2, and so on, and $\sum_{\pi(k)}$ denotes summation over all partitions $\pi$ of $k$. Then (6) and (9) of [4] give us

$$P_f(k) = \sum_{\pi(k)} \Omega(\pi),$$

where

$$\Omega(\pi) = \left\{ \prod_{h=1}^{k} \beta(h) \right\} / \prod_{m} \{ h(m)! \}^k \{ m^k \},$$

a polynomial in the $X$.

Let $1 \leq u \leq k$ and write $v = [k/u]$ and $k = uv + w$, so that $0 \leq w < u$. We consider separately those partitions $\pi_1$ of $k$ which have $v$ parts $u$ and the remaining partitions $\pi_2$ in which there are at most $v-1$ parts $u$. We have

$$P_f(k) = \sum_{\pi_1} \Omega(\pi_1) + \sum_{\pi_2} \Omega(\pi_2) = S_1 + S_2$$

(say). In the numerator of $\Omega(\pi_2)$, the factor $\gamma(u)$ occurs just $v$ times (once in $\beta(h)$ for $h = u, 2u, 3u, \ldots, vu$), while it occurs at most $v-1$ times in the denominator. Hence $\Omega(\pi_2)$ has the factor $\gamma(u)$. Thus

$$S_2 = \sum_{\pi_2} \Omega(\pi_2) = \gamma(u)T_2,$$
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where $T_2$ is a polynomial in the $X$. Again

$$S_1 = \sum_{\pi_1} \Omega(\pi_1)$$

$$= (v!)^{-1} \{u\beta(u)\}^{-v} \prod_{h=1}^{k} \beta(h) \sum_{\pi(w)} \prod_{m} \{h(m)\}^{-1} \{m\beta(m)\}^{-h(m)}$$

$$= (v!)^{-1} \{u\beta(u)\}^{-v} P_f(w) \prod_{h=w+1}^{k} \beta(h) = T_1 P_f(w),$$

where $T_1$ is a polynomial in the $X$. If $u < t \leq k$, then $\gamma(t)$ is a factor of $\prod_{h=w+1}^{k} \beta(h)$, but not of $\beta(u)$. Hence $\gamma(t)$ is a factor of $T_1$. Thus, if $\xi$ is a root of $\gamma(u)$ and $\eta$ a root of $\gamma(t)$, we have

$$S_2(\xi, X_2, \ldots) = 0, \quad S_1(X_1, \eta, X_3, \ldots) = 0, \quad P_f(k; \xi, \eta, X_3, \ldots) = 0,$$

which is Gordon's result.

By a fairly obvious extension of our argument, we find more generally that, if

$$1 \leq u_1 < u_2 < \ldots < u_a \leq k, \quad v_b = [k/u_b], \quad w_b = k - u_b v_b,$$

then

$$P_f(k) = \sum_{b=1}^{a} \frac{P_f(w_b)}{v_b! \{u_b \beta(u_b)\}^{v_b}} + T \prod_{b=1}^{a} \gamma(u_b),$$

where $T$ is a polynomial in the $X$.

REFERENCES


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