

RESEARCH ARTICLE

Unramified logarithmic Hodge–Witt cohomology and \mathbb{P}^1 -invariance

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Abstract

Let X be a smooth proper variety over a field k and suppose that the degree map $\mathrm{CH}_0(X \otimes_k K) \rightarrow \mathbb{Z}$ is isomorphic for any field extension K/k . We show that $G(\mathrm{Spec} k) \rightarrow G(X)$ is an isomorphism for any \mathbb{P}^1 -invariant Nisnevich sheaf with transfers G . This generalises a result of Binda, Rülling and Saito that proves the same conclusion for reciprocity sheaves. We also give a direct proof of the fact that the unramified logarithmic Hodge–Witt cohomology is a \mathbb{P}^1 -invariant Nisnevich sheaf with transfers.

Contents

1	Introduction	1
2	Reminders on homotopy-invariant sheaves with transfers	3
3	\mathbb{P}^1-invariance and the main result	5
4	Projective Suslin complex	7
5	Moving lemma	9
	5.1 The bad locus	9
	5.2 Affine-space case	10
	5.3 The general case	12
6	The unramified cohomology $H_{\mathrm{ur}}^1(-, W_n \Omega_{\log}^j)$	15
A	Proof of Remark 2.5	18

1. Introduction

A proper smooth variety X over a field k is said to be *universally CH_0 -trivial* if for any field extension K/k , the degree map of the Chow group of 0-cycles induces an isomorphism $\mathrm{deg}: \mathrm{CH}_0(X \otimes_k K) \xrightarrow{\cong} \mathbb{Z}$. Basic examples of universally CH_0 -trivial varieties include rational (and more generally stably rational) varieties, and this property may be considered a near-rationality condition. The condition plays a crucial role in the degeneration method established by Voisin [32] and Colliot-Thélène and Pirutka [9], where counterexamples to the Lüroth problem are produced.

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Now it is natural to ask how to disprove the universal CH_0 -triviality for a given variety X . In this direction, Merkurjev [23, Theorem 2.11] proved that X is universally CH_0 -trivial if and only if the function field $k(X)$ has trivial unramified cohomology – that is, $M_*(k) \simeq M_*(k(X))_{\text{ur}}$ for all Rost’s cycle modules M_* over k . As a consequence, if ℓ is a prime number different from the characteristic p of k , then ℓ -primary torsion elements of the Brauer group $\text{Br}(X) := H_{\text{ét}}^2(X, \mathbb{G}_m)$ not coming from $\text{Br}(k)$ obstruct the universal CH_0 -triviality. This is because the ℓ -primary torsion subgroups $\text{Br}(K)[\ell^\infty] \simeq H_{\text{ét}}^2(K, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))$ for all field extensions K/k give rise to a cycle module $M_*: K \mapsto \bigoplus_{i=0}^\infty H_{\text{ét}}^{i+1}(K, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$, to which the theorem of Merkurjev can be applied.

There are, however, more obstructions other than cycle modules. In [28], Totaro adopted as an obstruction the sheaves of differential forms $\Omega_{X/k}^i$ to disprove the universal CH_0 -triviality for a wide class of hypersurfaces. In [3], Auel et al. used $\text{Br}(-)[2^\infty]$ in characteristic $p = 2$ to obtain a similar result for conic bundles over \mathbb{P}^2 . Neither Ω^i nor $\text{Br}(-)[p^\infty]$ in characteristic $p > 0$ constitutes a cycle module. In fact, it is not straightforward to extend Merkurjev’s result to $\text{Br}(-)[p^\infty]$. This gap was filled in by the previous work of Auel et al. [2]:

Theorem 1.1 ([1, Theorem 1.1]). *Let X be a smooth proper variety over a field k which is universally CH_0 -trivial. Then the structure morphism $X \rightarrow \text{Spec } k$ induces an isomorphism $\text{Br}(k) \xrightarrow{\cong} \text{Br}(X)$.*

Our main result extends this theorem to more general invariants:

Theorem 1.2 (see Corollary 3.3). *Let X be a smooth proper variety over a field k which is universally CH_0 -trivial. Then the structure morphism $X \rightarrow \text{Spec } k$ induces an isomorphism $G(k) \xrightarrow{\cong} G(X)$ for any \mathbb{P}^1 -invariant Nisnevich sheaf with transfers G in the sense of Definition 3.1.*

This theorem generalises Theorem 1.1, since the Brauer group has a structure of a \mathbb{P}^1 -invariant Nisnevich sheaf with transfers (see Remark 1.5(1)). The conclusion of Theorem 1.2 for *homotopy-invariant* sheaves with transfers follows from Merkurjev’s result already cited. More recently, Binda et al. proved the same conclusion for another class of Nisnevich sheaves with transfers, called *reciprocity sheaves* [5, Theorem 10.13, Remark 10.14]. Our main theorem also covers their result, since we have by [18, Theorems 3, 8]

$$\text{homotopy invariant} \Rightarrow \text{reciprocity} \Rightarrow \mathbb{P}^1\text{-invariant} \tag{1.1}$$

Both implications are strict (see Remark 3.4). Note also that Ω^i is a reciprocity sheaf by [18, Theorem A.6.2], hence Totaro’s method can be explained either by our results or those of Binda et al. The main technical issue in the proof of Theorem 1.2 is the comparison of $G(X)$ and $h^0(G)(X)$, where $h^0(G)$ is the *maximal homotopy-invariant subsheaf with transfers* of G . We rephrase the problem in terms of algebraic cycles, and settle it by establishing a new moving lemma (Theorem 3.5).

The unramified logarithmic Hodge–Witt cohomology $H_{\text{ur}}^1(-, W_n\Omega_{\text{log}}^i)$ (see Section 6 for the definition) satisfies the hypothesis of Theorem 1.2. Although this fact can also be deduced from known results on reciprocity sheaves (see Remark 6.2), we will give a direct proof which depends on classical results [12, 15] but not on reciprocity sheaves.

Proposition 1.3 (see Proposition 6.1). *The unramified logarithmic Hodge–Witt cohomology $H_{\text{ur}}^1(-, W_n\Omega_{\text{log}}^i)$ is a \mathbb{P}^1 -invariant Nisnevich sheaf with transfers (over a field of positive characteristic) for any integers $n \geq 1$ and $i \geq 0$.*

As a corollary, we obtain a new proof of the following (known) result:

Theorem 1.4. *Let X be a smooth proper variety over a field k of characteristic $p > 0$. Assume that X is universally CH_0 -trivial. Then the canonical map*

$$H_{\text{ur}}^1(\text{Spec } k, W_n\Omega_{\text{log}}^i) \rightarrow H_{\text{ur}}^1(X, W_n\Omega_{\text{log}}^i)$$

is an isomorphism for any integers $n \geq 1$ and $i \geq 0$.

Remark 1.5.

- (1) Since $H_{\text{ur}}^1(X, W_n\Omega_{\log}^1) \simeq \text{Br}(X)[p^n]$, Theorem 1.4 for $i = 1$ follows from Theorem 1.1. For general i , Theorem 1.4 was posed as a problem by Auel et al. in [2, Problem 1.2] and previously proved in [5] and [24], as explained in the following.
- (2) Theorem 1.4 was shown by Binda et al. in [5, Theorem 10.13, Remark 10.14] as a consequence of their general result on reciprocity sheaves already mentioned, along with the fact that $H_{\text{ur}}^1(-, W_n\Omega_{\log}^i)$ has a structure of reciprocity sheaf.
- (3) Independent of [5], almost at the same time, Otabe also obtained Theorem 1.4 for $n = 1$ [24, Theorem 1.2]. His proof is somewhat close to ours, but it is more cycle-module-theoretical. A tame subgroup of the unramified cohomology is used in place of $h^0(G)$ in the present paper. The relation between these two subgroups is left for future research.
- (4) A similar statement to Theorem 1.4 holds when H_{ur}^1 is replaced by H_{ur}^j for any $j \in \mathbb{Z}$. Indeed, the cohomology groups in question are trivial unless $j = 0, 1$, because the natural map $H_{\text{ur}}^j(X, W_n\Omega_{\log}^i) \rightarrow H_{\text{ur}}^j(k(X), W_n\Omega_{\log}^i)$ is injective (if X is connected), and the p -cohomology dimension of any field of characteristic $p > 0$ is at most 1. The case $j = 0$ follows from the results of Bloch, Gabber and Kato [6, Theorem 2.1] and Merkurjev [23, Theorem 2.11].

The organisation of this paper is as follows. In Section 2, we revisit the proof of Theorem 1.2 for homotopy-invariant Nisnevich sheaves with transfers due to Merkurjev [23] and Kahn [16] (see Corollary 2.4). In Section 3, we state the main result in a slightly more general form (see Theorem 3.2) and prove it while admitting the key moving lemma (Theorem 3.5). The proof of Theorem 3.5 occupies the next two sections.

In Section 4, we rephrase the problem in terms of algebraic cycles. To do so, we consider the Suslin complex $C_{\bullet}(X)$ and its variant $\overline{C}_{\bullet}(X)$, where the latter is defined by replacing \mathbb{A}^n with \mathbb{P}^n in the former. Theorem 3.5 is then reduced to a comparison, up to Zariski sheafification, of their 0th homology presheaves (see Theorem 4.1). Its proof is given in Section 5, which is pivotal in our work.

In Section 6, we give a proof of Proposition 1.3. Finally, Appendix A provides a proof of basic properties of universally H_0^S -trivial correspondences (see Definition 2.1).

We close this introduction with a brief discussion of related works. Shimizu [27] and Koizumi [21] have obtained some results resembling our moving lemma (Theorem 3.5) in \mathbb{A}^1 -homotopy theory. Ayoub [4] has considered the notion of \mathbb{P}^1 -localisation, which is much more sophisticated than our \mathbb{P}^n -Suslin complex introduced in Section 4. The relation of their works to ours is to be explored. Kahn has communicated to us that Theorem 3.5 has implications in the theory of birational sheaves [19], which should be an interesting topic for future research (see a brief comment in Remark 3.6).

2. Reminders on homotopy-invariant sheaves with transfers

We fix a field k . Let **Sch** be the category of separated k -schemes of finite type, and **Sm** its full subcategory of smooth k -schemes. We write **Fld** $_k$ for the category of fields over k , and **Fld** $_k^{\text{gm}}$ for its full subcategory consisting of the k -fields which are isomorphic to the function field of some (irreducible) $U \in \mathbf{Sm}$. For $K \in \mathbf{Fld}_k$ and $X \in \mathbf{Sch}$, we write $X_K := X \otimes_k K$.

Let **Cor** be Voevodsky’s category of finite correspondences. By definition it has the same objects as **Sm**, and for $U, V \in \mathbf{Sm}$ the space of morphisms $\mathbf{Cor}(U, X)$ from U to V is the free abelian group on the set of integral closed subschemes of $U \times X$ which is finite and surjective over an irreducible component of U . An additive functor $F : \mathbf{Cor}^{\text{op}} \rightarrow \mathbf{Ab}$ is called a presheaf with transfers. Denote by **PST** the category of presheaves with transfers. If S is a k -scheme that is written as a filtered limit $S = \varprojlim_i S_i$ where $S_i \in \mathbf{Sm}$ and all transition maps are open immersions, then we define

$$F(S) = \varinjlim_i F(S_i) \quad (F \in \mathbf{PST}). \tag{2.1}$$

We abbreviate $F(R) = F(\text{Spec } R)$ for a k -algebra R (when $F(\text{Spec } R)$ is defined). In particular, we may speak of $F(K)$ for $K \in \mathbf{Fld}_k^{\text{gm}}$ and $F(\mathcal{O}_{X,x})$ for $x \in X \in \mathbf{Sm}$. We set $\mathbb{Z}_{\text{tr}}(X) := \mathbf{Cor}(-, X) \in \mathbf{PST}$ for $X \in \mathbf{Sm}$.

We say $F \in \mathbf{PST}$ is homotopy-invariant if the projection $\text{pr} : X \times \mathbb{A}^1 \rightarrow X$ induces an isomorphism $\text{pr}^* : F(X) \cong F(X \times \mathbb{A}^1)$ for any $X \in \mathbf{Sm}$. We write \mathbf{HI} for the full subcategory of \mathbf{PST} of homotopy-invariant presheaves with transfers. We say $F \in \mathbf{PST}$ is a Nisnevich sheaf with transfers if F composed with the inclusion (graph) functor $\mathbf{Sm} \rightarrow \mathbf{Cor}$ is a Nisnevich sheaf on \mathbf{Sm} . We write \mathbf{NST} for the full subcategory of \mathbf{PST} of Nisnevich sheaves with transfers. Set $\mathbf{HI}_{\text{Nis}} := \mathbf{HI} \cap \mathbf{NST}$.

We will use the following facts:

- (V1) The inclusion functor $\mathbf{NST} \hookrightarrow \mathbf{PST}$ admits a left adjoint $a_{\text{Nis}} : \mathbf{PST} \rightarrow \mathbf{NST}$ and $a_{\text{Nis}}(\mathbf{HI}) \subset \mathbf{HI}_{\text{Nis}}$ holds [22, Corollary 11.2, Theorem 22.3]. We write $F_{\text{Nis}} := a_{\text{Nis}}(F)$. We have $F_{\text{Nis}}(K) = F(K)$ for any $K \in \mathbf{Fld}_k^{\text{gm}}$ (because fields are Henselian local).
- (V2) The inclusion functor $\mathbf{HI} \hookrightarrow \mathbf{PST}$ has a left adjoint h_0 given by the formula

$$h_0(F)(U) = \text{Coker}\left(i_0^* - i_1^* : F(U \times \mathbb{A}^1) \rightarrow F(U)\right) \quad (F \in \mathbf{PST}, U \in \mathbf{Sm}),$$

where we write i_ε^* for the pullback along $U \rightarrow U \times \mathbb{A}^1, x \mapsto (x, \varepsilon)$, for $\varepsilon = 0, 1$. This is the maximal homotopy-invariant quotient of F [22, Example 2.20]. For $X \in \mathbf{Sm}$, we write $h_0(X) := h_0(\mathbb{Z}_{\text{tr}}(X))$. We call

$$H_0^S(X_K) := h_0(X)(K) = h_0(X)_{\text{Nis}}(K)$$

the 0th Suslin homology of X_K for $K \in \mathbf{Fld}_k^{\text{gm}}$. There is a canonical surjective map $H_0^S(X_K) \rightarrow \text{CH}_0(X_K)$, which is isomorphic if X is proper over k [22, Exercise 2.21].

- (V3) The inclusion functor $\mathbf{HI} \hookrightarrow \mathbf{PST}$ has a right adjoint h^0 , given by the formula

$$h^0(F)(U) = \mathbf{PST}(h_0(U), F)$$

for $F \in \mathbf{PST}, U \in \mathbf{Sm}$. This is the maximal homotopy-invariant subobject of F [25, Section 4.34].

- (V4) Let $f : F \rightarrow G$ be a morphism in \mathbf{HI}_{Nis} . If f induces an isomorphism $f^* : F(K) \cong G(K)$ for any $K \in \mathbf{Fld}_k^{\text{gm}}$, then f is an isomorphism in \mathbf{HI}_{Nis} [22, Corollary 11.2].
- (V5) Given $F \in \mathbf{PST}$, we denote by F_{Zar} the Zariski sheaf associated to the presheaf on \mathbf{Sm} obtained by restricting F along the graph functor $\mathbf{Sm} \rightarrow \mathbf{Cor}$. In general, it does not admit a structure of presheaf with transfers, but if $F \in \mathbf{HI}$, then we have $F_{\text{Zar}} = F_{\text{Nis}}$ by [22, Theorem 22.2], and hence F_{Zar} acquires transfers by (V1). We say $F \in \mathbf{PST}$ is a Zariski sheaf with transfers if $F = F_{\text{Zar}}$.

Another important fact can be stated as $H_{\text{Zar}}^i(-, F_{\text{Zar}}) = H_{\text{Nis}}^i(-, F_{\text{Nis}}) \in \mathbf{HI}$ for $F \in \mathbf{HI}$, assuming k is perfect [30, Theorems 5.6, 5.7]. We will not use this in the sequel.

Definition 2.1. Let $X, Y \in \mathbf{Sm}$. We say $f \in \mathbf{Cor}(X, Y)$ is *universally H_0^S -trivial* if the induced map $f_{K*} : H_0^S(X_K) \rightarrow H_0^S(Y_K)$ is an isomorphism for each $K \in \mathbf{Fld}_k^{\text{gm}}$.

Remark 2.2. This is an analogue of [9, Définition 1.1], where a proper morphism $f : X \rightarrow Y$ is said to be *universally CH_0 -trivial* if the induced map $f_{K*} : \text{CH}_0(X_K) \rightarrow \text{CH}_0(Y_K)$ is an isomorphism for each $K \in \mathbf{Fld}_k$. When X and Y are proper over k , a universally CH_0 -trivial morphism is also universally H_0^S -trivial by (V2). (We tacitly identify a morphism with its graph.) Note also that a smooth proper variety X is universally CH_0 -trivial (in the sense of Theorem 1.4) if and only if the structure map $X \rightarrow \text{Spec } k$ is universally CH_0 -trivial.

The following result is due to Merkurjev [23, Theorem 2.11] and Kahn [16, Corollary 4.7]. We include a short proof here to keep self-containedness.

Proposition 2.3. *Set $X, Y \in \mathbf{Sm}$. The following conditions are equivalent for $f \in \mathbf{Cor}(X, Y)$:*

- (1) *The finite correspondence f is universally H_0^S -trivial.*
- (2) *The induced map $f_* : h_0(X)_{\mathbf{Nis}} \rightarrow h_0(Y)_{\mathbf{Nis}}$ is an isomorphism in $\mathbf{HI}_{\mathbf{Nis}}$.*
- (3) *The induced map $f^* : F(Y) \rightarrow F(X)$ is an isomorphism for each $F \in \mathbf{HI}_{\mathbf{Nis}}$.*

(Thanks to (V5), we may replace Nis by Zar in this statement.)

Proof. The equivalence of (1) and (2) is a consequence of (V4). We have

$$F(X) = \mathbf{PST}(\mathbb{Z}_{\text{tr}}(X), F) = \mathbf{HI}(h_0(X), F) = \mathbf{HI}_{\mathbf{Nis}}(h_0(X)_{\mathbf{Nis}}, F)$$

for any $F \in \mathbf{HI}_{\mathbf{Nis}}$. Here we used, in order, Yoneda’s lemma, (V2) and (V1). Now another use of Yoneda’s lemma shows the equivalence of (2) and (3). □

Corollary 2.4. *Let X be a smooth and proper scheme over a field k . If X is universally CH_0 -trivial, then we have $F(k) \cong F(X)$ for any $F \in \mathbf{HI}_{\mathbf{Nis}}$.*

Proof. In view of Remark 2.2, this is a special case of Proposition 2.3. □

We will generalise this result in Corollary 3.3.

Remark 2.5. We collect basic properties of universally H_0^S -trivial correspondences. Since they are not used in the sequel, the proof will be given in Appendix A.

- (1) *If f, g are composable finite correspondences and if two out of $f, g, f \circ g$ are universally H_0^S -trivial, then so is the third.*
- (2) *If $f : X \rightarrow Y$ and $f' : X' \rightarrow Y'$ are universally H_0^S -trivial finite correspondences, then so is $f \times f' : X \times X' \rightarrow Y \times Y'$.*
- (3) *Suppose k is perfect. Let $j : U \hookrightarrow X$ be an open dense immersion in \mathbf{Sm} . If $X \setminus j(U)$ is of codimension ≥ 2 , then j is universally H_0^S -trivial.*
- (4) *Suppose k is perfect. A proper birational morphism in \mathbf{Sm} is universally H_0^S -trivial.*

3. \mathbb{P}^1 -invariance and the main result

Recall from [31, Section 3.2] that \mathbf{PST} is equipped with a symmetric monoidal structure \otimes which is uniquely characterised by the facts that it is right exact and the Yoneda functor is monoidal (that is, $\mathbb{Z}_{\text{tr}}(X) \otimes \mathbb{Z}_{\text{tr}}(Y) = \mathbb{Z}_{\text{tr}}(X \times Y)$ for $X, Y \in \mathbf{Sm}$). It admits a right adjoint $\underline{\text{Hom}}$ given by the formula

$$\underline{\text{Hom}}(F, G)(X) = \mathbf{PST}(F \otimes \mathbb{Z}_{\text{tr}}(X), G) \quad (F, G \in \mathbf{PST}, X \in \mathbf{Sm}). \tag{3.1}$$

Definition 3.1. We say $G \in \mathbf{PST}$ is \mathbb{P}^1 -invariant if the structure map $\sigma : \mathbb{P}^1 \rightarrow \text{Spec } k$ induces an isomorphism $G \xrightarrow{\cong} \underline{\text{Hom}}(\mathbb{Z}_{\text{tr}}(\mathbb{P}^1), G)$ – that is, σ induces isomorphisms $G(U) \xrightarrow{\cong} G(U \times \mathbb{P}^1)$ for all $U \in \mathbf{Sm}$. Denote by $\mathbf{PI}_{\mathbf{Nis}}$ the full subcategory of \mathbf{PST} consisting of all Nisnevich sheaves with transfers which are \mathbb{P}^1 -invariant.

Theorem 3.2. *Suppose that X, Y are smooth and proper schemes over a field k and $f \in \mathbf{Cor}(X, Y)$. Then the conditions in Proposition 2.3 are equivalent to the following:*

- (4) *The induced map $f^* : G(Y) \rightarrow G(X)$ is an isomorphism for each $G \in \mathbf{PI}_{\mathbf{Nis}}$.*

As with Corollary 2.4, Theorem 3.2 has an immediate consequence:

Corollary 3.3. *Let X be a smooth proper scheme over a field k . If X is universally CH_0 -trivial, then we have $G(k) \cong G(X)$ for any $G \in \mathbf{PI}_{\mathbf{Nis}}$.*

Remark 3.4. Let T be a smooth quasi-affine scheme over k . It follows from [17, Theorem 6.4.1] that $\mathbb{Z}_{\text{tr}}(T) \in \mathbf{PI}_{\text{Nis}}$. On the other hand, $\mathbb{Z}_{\text{tr}}(T)$ does not have reciprocity in general. (This follows, for example, from [18, Proposition 9.4.4].) We conclude that \mathbb{P}^1 -invariance does not imply reciprocity (i.e., the converse of the second arrow in formula (1.1) does not hold). On the other hand, the conclusion of Corollary 3.3 is obvious for $G = \mathbb{Z}_{\text{tr}}(T)$. Indeed, it is not difficult to show $\mathbf{Cor}(\text{Spec } k, T) \cong \mathbf{Cor}(X, T)$ for any $X \in \mathbf{Sm}$ which is connected and proper over k (but not necessary universally CH_0 -trivial).

For $F \in \mathbf{PST}$ we define

$$\bar{h}_0(F) := \text{Coker}\left(i_0^* - i_1^* : \underline{\text{Hom}}\left(\mathbb{Z}_{\text{tr}}(\mathbb{P}^1), F\right) \rightarrow F\right). \tag{3.2}$$

We write $\bar{h}_0(X) := \bar{h}_0(\mathbb{Z}_{\text{tr}}(X))$ for $X \in \mathbf{Sm}$. The main part in the proof of Theorem 3.2 is the following:

Theorem 3.5 (A moving lemma). *The natural map $\bar{h}_0(X) \rightarrow h_0(X)$ yields an isomorphism $\bar{h}_0(X)_{\text{Zar}} \cong h_0(X)_{\text{Zar}}$ for any smooth proper scheme X over a field k . (Hence we have $\bar{h}_0(X)_{\text{Nis}} \cong h_0(X)_{\text{Nis}}$ as well.)*

The proof of Theorem 3.5 occupies the next two sections. In the rest of this section, we deduce Theorem 3.2 assuming Theorem 3.5.

Remark 3.6. For X as in Theorem 3.5, we have an explicit formula:

$$\bar{h}_0(X)_{\text{Zar}}(U) \cong \text{CH}_0(X \otimes_k k(U)) \quad \text{for any connected } U \in \mathbf{Sm},$$

because we know $h_0(X)_{\text{Nis}}(U) \cong \text{CH}_0(X \otimes_k k(U))$ by [19, Theorem 3.1.2] (and we have $h_0(X)_{\text{Zar}} \cong h_0(X)_{\text{Nis}}$ by (V2) and (V5)). In particular, $\bar{h}_0(X)_{\text{Zar}}$ is *birational* in the sense of [19, Definition 2.3.1] – that is, any open dense immersion $V \hookrightarrow W$ induces an isomorphism $\bar{h}_0(X)_{\text{Zar}}(W) \cong \bar{h}_0(X)_{\text{Zar}}(V)$.

Denote by $i_\varepsilon : \text{Spec } k \rightarrow \mathbb{P}^1$ the closed immersion defined by a rational point $\varepsilon \in \mathbb{P}^1(k)$.

Definition 3.7. Set $F \in \mathbf{PST}$. We say F is \mathbb{P}^1 -rigid if the two induced maps

$$i_0^*, i_1^* : F\left(U \times \mathbb{P}^1\right) \rightarrow F(U)$$

are equal for any $U \in \mathbf{Sm}$. Denote by **PRig** the full subcategory of **PST** consisting of all \mathbb{P}^1 -rigid presheaves with transfers.

Lemma 3.8. *If $F \in \mathbf{PST}$ is \mathbb{P}^1 -invariant, then it is \mathbb{P}^1 -rigid. The converse holds if F is separated for Zariski topology.*

Proof. See [18, Proposition 6.1.4]. □

Lemma 3.9.

(1) For $F \in \mathbf{PST}$, the following conditions are equivalent:

- (a) F is \mathbb{P}^1 -rigid.
- (b) The two induced maps $i_0^*, i_1^* : \underline{\text{Hom}}(\mathbb{Z}_{\text{tr}}(\mathbb{P}^1), F) \rightarrow F$ are equal.
- (c) The canonical surjection $F \rightarrow \bar{h}_0(F)$ is an isomorphism.
- (d) The canonical injection $\mathbf{PST}(\bar{h}_0(U), F) \rightarrow \mathbf{PST}(\mathbb{Z}_{\text{tr}}(U), F)$ is an isomorphism for each $U \in \mathbf{Sm}$.

(2) Rormula (3.2) defines a left adjoint $\bar{h}_0 : \mathbf{PST} \rightarrow \mathbf{PRig}$ of the inclusion functor $\mathbf{PRig} \hookrightarrow \mathbf{PST}$.

(3) We have $\mathbf{HI}_{\text{Nis}} \subset \mathbf{PI}_{\text{Nis}}$.

Proof. (1) The equivalence of (a), (b) and (c) follows from formulas (3.1) and (3.2). If (c) holds, then any morphism $\mathbb{Z}_{\text{tr}}(U) \rightarrow F$ factors as $\mathbb{Z}_{\text{tr}}(U) \rightarrow \bar{h}_0(U) \rightarrow \bar{h}_0(F) \cong F$, whence (d). If (d) holds, then any morphism $\mathbb{Z}_{\text{tr}}(U) \rightarrow F$ factors as $\mathbb{Z}_{\text{tr}}(U) \rightarrow \bar{h}_0(U) \rightarrow F$, whence (b).

(2) We need to show $\mathbf{PST}(\overline{h_0}(G), F) \cong \mathbf{PST}(G, F)$ for any $F \in \mathbf{PRig}$ and $G \in \mathbf{PST}$. This is (d) if $G = \mathbb{Z}_{\text{tr}}(U)$ for $U \in \mathbf{Sm}$, to which the general case is reduced by taking a resolution of the form

$$\bigoplus_{\beta} \mathbb{Z}_{\text{tr}}(V_{\beta}) \rightarrow \bigoplus_{\alpha} \mathbb{Z}_{\text{tr}}(U_{\alpha}) \rightarrow G \rightarrow 0,$$

where $U_{\alpha}, V_{\beta} \in \mathbf{Sm}$ [31, Section 3.2].

(3) Given $F \in \mathbf{PST}$, we have a chain of canonical surjections $F \twoheadrightarrow \overline{h_0}(F) \twoheadrightarrow h_0(F)$, and F is homotopy-invariant if and only if the composition is an isomorphism. It follows from (1) that $\mathbf{HI} \subset \mathbf{PRig}$. Now the assertion follows from Lemma 3.8. \square

Lemma 3.10. *Suppose $G \in \mathbf{PST}$ is a Zariski sheaf. Then $h^0(G)$ is a Nisnevich sheaf.*

Proof. This is essentially shown in [25, Section 4.34], but we include a short proof for the sake of completeness. We consider a commutative diagram

$$\begin{array}{ccc} h^0(G) & \hookrightarrow & G \\ j \downarrow & & \parallel \\ h^0(G)_{\text{Zar}} & \hookrightarrow & G_{\text{Zar}} \end{array}$$

The bottom arrow is injective, since the sheafification is exact. This shows the injectivity of j . Fact (V5) shows that $h^0(G)_{\text{Zar}} = h^0(G)_{\text{Nis}}$, and (V1) shows that it is homotopy-invariant. Hence j must be isomorphic, since $h^0(G) \subset G$ is the maximal subobject in \mathbf{HI} . \square

Proof of Theorem 3.2, admitting Theorem 3.5. That (4) implies (3) follows from Lemma 3.9(3). To show the converse, we assume (3) and take $G \in \mathbf{PI}_{\text{Nis}}$. Set $F := h^0(G)$. By Lemma 3.10 we find $F \in \mathbf{HI}_{\text{Nis}}$, and hence we have $f^* : F(Y) \cong F(X)$ by assumption (3). It remains to show $G(X) = F(X)$ for any proper $X \in \mathbf{Sm}$. By Theorem 3.5 we have $\overline{h_0}(X)_{\text{Nis}} = h_0(X)_{\text{Nis}} \in \mathbf{HI}_{\text{Nis}} \subset \mathbf{PST}$. We now proceed as follows:

$$\begin{aligned} G(X) &= \mathbf{PST}(\mathbb{Z}_{\text{tr}}(X), G) && \text{(Yoneda)} \\ &= \mathbf{PST}(\overline{h_0}(X), G) && \text{(Lemma 3.9(d))} \\ &= \mathbf{PST}(\overline{h_0}(X)_{\text{Nis}}, G) && (G \text{ is a Nisnevich sheaf)} \\ &= \mathbf{PST}(h_0(X)_{\text{Nis}}, G) && (\overline{h_0}(X)_{\text{Nis}} = h_0(X)_{\text{Nis}}, \text{ by Theorem 3.5)} \\ &= \mathbf{PST}(h_0(X), G) && (G \text{ is a Nisnevich sheaf)} \\ &= \mathbf{HI}(h_0(X), F) && (h^0 \text{ is a right adjoint to the inclusion)} \\ &= \mathbf{PST}(\mathbb{Z}_{\text{tr}}(X), F) && (h_0 \text{ is a left adjoint to the inclusion)} \\ &= F(X) && \text{(Yoneda).} \end{aligned}$$

This completes the proof. \square

4. Projective Suslin complex

Fix $X \in \mathbf{Sm}$ in this section. After a brief review of the definition of the Suslin complex of X , we define its variant using the projective spaces \mathbb{P}^n . This will be used in the proof of the moving lemma in the next section. (We will use them only up to degree 2.)

For each nonnegative integer n , we write

$$\Delta^n := \text{Spec } k[t_0, \dots, t_n]/(t_0 + \dots + t_n - 1) \cong \mathbb{A}^n.$$

For $j = 0, \dots, n$, we define

$$i_{n,j} : \Delta^{n-1} \rightarrow \Delta^n; \quad (t_0, \dots, t_{n-1}) \mapsto (t_0, \dots, t_{j-1}, 0, t_j, \dots, t_{n-1}). \tag{4.1}$$

The *Suslin complex* $C_\bullet(X)$ of X is a complex in **PST** defined by

$$C_n(X) := \underline{\text{Hom}}(\mathbb{Z}_{\text{tr}}(\Delta^n), \mathbb{Z}_{\text{tr}}(X)),$$

$$\partial_n := \sum_{j=0}^n (-1)^j i_{n,j}^* : C_n(X) \rightarrow C_{n-1}(X).$$

Its homology is denoted by $h_n(X) \in \mathbf{PST}$. For $n = 0$, it recovers $h_0(X)$ from (V2).

For each nonnegative integer n , we set

$$\overline{\Delta}^n := \text{Proj } k[t_0, \dots, t_{n+1}]/(t_0 + \dots + t_n - t_{n+1}) \cong \mathbb{P}^n.$$

We have a canonical open immersion $\iota_n : \Delta^n \hookrightarrow \overline{\Delta}^n$, which is isomorphic to $\mathbb{A}^n \hookrightarrow \mathbb{P}^n$. It induces a map in **PST**

$$\overline{C}_n(X) := \underline{\text{Hom}}\left(\mathbb{Z}_{\text{tr}}(\overline{\Delta}^n), \mathbb{Z}_{\text{tr}}(X)\right) \hookrightarrow C_n(X), \tag{4.2}$$

which is injective for any n and isomorphic for $n = 0$. Indeed, its section over $U \in \mathbf{Sm}$ is given by

$$(\text{Id}_U \times \iota_n)^* : \mathbf{Cor}(U \times \overline{\Delta}^n, X) \rightarrow \mathbf{Cor}(U \times \Delta^n, X),$$

which is injective in general and isomorphic for $n = 0$. We regard $\overline{C}_n(X)$ as a subobject in **PST** of $C_n(X)$. Since the morphisms $i_{n,j}$ from formula (4.1) extend (uniquely) to morphisms $\overline{\Delta}^{n-1} \rightarrow \overline{\Delta}^n$, we obtain a subcomplex $\overline{C}_\bullet(X)$ of $C_\bullet(X)$. We write its homology by $\overline{h}_n(X) \in \mathbf{PST}$. For $n = 0$, it recovers $\overline{h}_0(X)$ from formula (3.2).

We write

$$Q_\bullet(X) := C_\bullet(X)/\overline{C}_\bullet(X) \tag{4.3}$$

for the quotient complex, and its homology presheaf is denoted by $H_n(Q_\bullet(X)) \in \mathbf{PST}$. We have $H_0(Q_\bullet(X)) = 0$, for formula (4.2) is isomorphic for $n = 0$. Since the sheafification is exact, we obtain an exact sequence

$$H_1(Q_\bullet(X))_{\text{Zar}} \rightarrow \overline{h}_0(X)_{\text{Zar}} \rightarrow h_0(X)_{\text{Zar}} \rightarrow H_0(Q_\bullet(X))_{\text{Zar}} = 0.$$

Theorem 3.5 is now reduced to the following:

Theorem 4.1. *If $X \in \mathbf{Sm}$ is proper, then we have $H_1(Q_\bullet(X))_{\text{Zar}} = 0$.*

Remark 4.2.

- (1) In general, $\overline{h}_n(X)_{\text{Zar}} \rightarrow h_n(X)_{\text{Zar}}$ is not isomorphic for $n > 0$. Indeed, one easily checks $H_2(Q_\bullet(\mathbb{P}^1))_{\text{Zar}}(k) = H_2(Q_\bullet(\mathbb{P}^1))(k) \neq 0$.
- (2) The properness assumption on X is essential. Indeed, it follows from [17, Theorem 6.4.1] that if X is quasiaffine, then all the boundary maps of $\overline{C}_\bullet(X)$ are the zero maps.

5. Moving lemma

We shall prove Theorem 4.1 in the following (equivalent) form:

Theorem 5.1. *Let $X \in \mathbf{Sm}$ be proper. For every irreducible affine $V \in \mathbf{Sm}$ and local scheme U at a closed point of V , the restriction map*

$$H_1(Q_\bullet(X))(V) \rightarrow H_1(Q_\bullet(X))(U)$$

is the zero map. (The target $H_1(Q_\bullet(X))(U)$ is defined as a colimit – see equation (2.1).)

Note that in proving Theorem 5.1, we may assume k is infinite; for if k is finite, we can use the usual norm argument.

5.1. The bad locus

In the notation of Theorem 5.1, let $\Gamma \subset V \times X \times \Delta^n$ be an irreducible closed subset which is finite and surjective over $V \times \Delta^n$. Let $\bar{\Gamma}$ be its closure in $V \times X \times \bar{\Delta}^n$. We call

$$B(\Gamma) := \left\{ p \in V \times \bar{\Delta}^n \mid \bar{\Gamma} \rightarrow V \times \bar{\Delta}^n \text{ is not finite over } p \right\} \tag{5.1}$$

the *bad locus* of Γ , which witnesses how far Γ is from being a member of $\bar{C}_n(X)(V)$.

Lemma 5.2.

- (1) *We have $\Gamma \in C_n(X)(V)$ if and only if $B(\Gamma) = \emptyset$.*
- (2) *The bad locus $B(\Gamma)$ is a closed proper subset of $V \times (\bar{\Delta}^n \setminus \Delta^n)$.*
- (3) *If $n \leq 1$, the image of the projection $B(\Gamma) \rightarrow V$ is a closed proper subset.*

Proof. Assertion (1) is clear from definitions, and put for later reference.

To prove (2), consider the set upstairs:

$$\tilde{B}(\bar{\Gamma}) := \left\{ x \in \bar{\Gamma} \mid \text{locally around } x, \text{ the fibre of } \bar{\Gamma} \rightarrow V \times \bar{\Delta}^n \text{ has dimension } \geq 1 \right\}.$$

It is a closed subset of $V \times X \times \bar{\Delta}^n$ by Chevalley’s theorem [11, IV₃ 13.1.3]. It is contained in $\bar{\Gamma} \setminus \Gamma$:

$$\tilde{B}(\bar{\Gamma}) \subset \bar{\Gamma} \setminus \Gamma, \tag{5.2}$$

because Γ is assumed to be finite over $V \times \Delta^n$. Since $B(\Gamma)$ is by definition the image of the map $\tilde{B}(\bar{\Gamma}) \rightarrow V \times \bar{\Delta}^n$ which is proper because X is, it follows that $B(\Gamma)$ is a closed subset of $V \times (\bar{\Delta}^n \setminus \Delta^n)$.

To show that $B(\Gamma)$ is a proper subset of $V \times (\bar{\Delta}^n \setminus \Delta^n)$, let $\xi \in \tilde{B}(\bar{\Gamma})$ be an arbitrary point and $\eta \in B(\Gamma)$ its image. By formula (5.2) we have

$$\begin{aligned} \dim(\bar{\Gamma} \setminus \Gamma) &\geq \mathbf{trdeg}(k(\xi)/k) \\ &= \mathbf{trdeg}(k(\xi)/k(\eta)) + \mathbf{trdeg}(k(\eta)/k) \\ &\geq 1 + \mathbf{trdeg}(k(\eta)/k), \end{aligned}$$

and by $\dim(V) + n - 1 \geq \dim(\bar{\Gamma} \setminus \Gamma)$ we obtain

$$\dim(V) + (n - 2) \geq \mathbf{trdeg}(k(\eta)/k). \tag{5.3}$$

Since $\dim(B(\Gamma)) = \sup_{\eta \in B(\Gamma)} \mathbf{trdeg}(k(\eta)/k)$, we conclude $\dim(V \times (\bar{\Delta}^n \setminus \Delta^n)) > \dim(B(\Gamma))$. We are done.

Assertion (3) is a direct consequence of (2) (or of formula (5.3)). □

5.2. Affine-space case

In this subsection we consider the case $V = \mathbb{A}^N$ of Theorem 5.1, with $N \geq 0$ an integer. Recall that we may assume k is infinite, which we do here.

Let $\Gamma \in C_1(X)(\mathbb{A}^N)$ be an arbitrary irreducible cycle. By a diagram chase in the following diagram (see formula (4.3) for the definition of $Q_\bullet(X)$), it suffices to find $\tilde{\Gamma} \in C_2(X)(\mathbb{A}^N)$ such that $(\Gamma - \partial_2 \tilde{\Gamma})|_U \in \overline{C}_1(X)(U)$:

$$\begin{array}{ccccc}
 \tilde{\Gamma} & ? & & & \Gamma \\
 \cap & & & & \cap \\
 Q_2(X)(\mathbb{A}^N) & \xrightarrow{\partial_2} & Q_1(X)(\mathbb{A}^N) & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \\
 Q_2(X)(U) & \xrightarrow{\partial_2} & Q_1(X)(U) & \longrightarrow & 0.
 \end{array}$$

For a vector $v \in \mathbb{A}^N(k)$, consider the translation $+v : \mathbb{A}^N \rightarrow \mathbb{A}^N$ by v . The next assertion suggests how it can be useful:

Lemma 5.3. *There is a closed proper subset $B \subset \mathbb{A}^N$ such that for every vector $v \in \mathbb{A}^N(k) \setminus B$, if we denote by $\tau_v : \mathbb{A}^N \times X \times \mathbb{A}^1 \rightarrow \mathbb{A}^N \times X \times \mathbb{A}^1$ the base change of the translation $\mathbb{A}^N \xrightarrow{+v} \mathbb{A}^N$, then we have*

$$(\tau_v^* \Gamma)|_U \in \overline{C}_1(X)(U).$$

Proof. Let $s \in U$ be the unique closed point of V contained in U . Let $B(\Gamma) \subset V \times \overline{\Delta}^{-1}$ be as in formula (5.1). By Lemma 5.2 we know that its projection $\text{pr}_V(B(\Gamma)) \subset V = \mathbb{A}^N$ is a closed proper subset. Consider the closed subset

$$B_0 := \text{pr}_V(B(\Gamma)_{k(s)}) - s = \left\{ v \in \mathbb{A}^N_{k(s)} \mid v + s \in \text{pr}_V(B(\Gamma)_{k(s)}) \right\} \subseteq \mathbb{A}^N_{k(s)}$$

and let B be its image by the finite projection $\pi : \mathbb{A}^N_{k(s)} \rightarrow \mathbb{A}^N$:

$$B := \pi(B_0) \subseteq \mathbb{A}^N. \tag{5.4}$$

Take an arbitrary $v \in \mathbb{A}^N(k) \setminus B$. By definitions, we know $s + v \in \mathbb{A}^N \setminus \text{pr}_V(B(\Gamma))$. Since the right-hand side is an open subset of \mathbb{A}^N , this relation remains true if we replace s by any point specialising to s . In particular,

$$U + v \subset \mathbb{A}^N \setminus \text{pr}_V(B(\Gamma)). \tag{5.5}$$

Let us denote by $\bar{\tau}_v$ the endomorphism of $\mathbb{A}^N \times X \times \overline{\Delta}^{-1}$ obtained as the base change of the translation $+v$. By formula (5.5), the maps τ_v and $\bar{\tau}_v$ restrict themselves as in the following commutative diagram:

$$\begin{array}{ccc}
 U \times X \times \Delta^1 & \xrightarrow{\tau_v} & (\mathbb{A}^N \setminus \text{pr}_V(B(\Gamma))) \times X \times \Delta^1 \supset \Gamma|_{\mathbb{A}^N \setminus \text{pr}_V(B(\Gamma))} \\
 \cap & & \cap \\
 U \times X \times \overline{\Delta}^{-1} & \xrightarrow{\bar{\tau}_v} & (\mathbb{A}^N \setminus \text{pr}_V(B(\Gamma))) \times X \times \overline{\Delta}^{-1} \supset \bar{\Gamma}|_{\mathbb{A}^N \setminus \text{pr}_V(B(\Gamma))} \\
 \downarrow \text{pr} & & \downarrow \text{pr} \\
 U \times \overline{\Delta}^{-1} & \xrightarrow{(+v) \times \text{Id}} & (\mathbb{A}^N \setminus \text{pr}_V(B(\Gamma))) \times \overline{\Delta}^{-1} \xleftarrow{\text{finite}}
 \end{array}$$

The slanted arrow is finite because $\overline{\Gamma} \rightarrow \mathbb{A}^N \times \overline{\Delta}^1$ is finite outside $B(\Gamma)$ precisely by definition (5.1), and we have the inclusion $(\mathbb{A}^N \setminus \text{pr}_V B(\Gamma)) \times \overline{\Delta}^1 \subset (\mathbb{A}^N \times \overline{\Delta}^1) \setminus B(\Gamma)$. It follows that $\overline{\tau_v^* \Gamma}$ is finite over $U \times \overline{\Delta}^1$. As a general fact about closure and continuity, we have the inclusion $\overline{(\tau_v^* \Gamma)} \subset \overline{\tau_v^* \Gamma}$. We conclude that $(\tau_v^* \Gamma)|_U$ belongs to $\overline{C}_1(X)(U)$, and this completes the proof. \square

Let B be as in Lemma 5.2 and fix a vector $v \in \mathbb{A}^N(k) \setminus B$. Let $\varphi_v : \mathbb{A}^N \times \mathbb{A}^1 \rightarrow \mathbb{A}^N$ be the map $(a, t) \mapsto a + tv$ and let

$$\Phi_{v,n} : \mathbb{A}^N \times X \times \mathbb{A}^1 \times \Delta^n \rightarrow \mathbb{A}^N \times X \times \Delta^n$$

be its base change by $X \times \Delta^n \rightarrow \text{Spec } k$. Since finite morphisms are stable under base change, we know that

$$\text{the inverse image } \Phi_{v,1}^{-1} \Gamma \subset \mathbb{A}^N \times X \times \mathbb{A}^1 \times \Delta^1 \text{ is finite over } \mathbb{A}^N \times \mathbb{A}^1 \times \Delta^1. \tag{5.6}$$

We shall use the following triangulation maps, as in Figure 1:

$$\sigma_1, \sigma_2 : \Delta^2 \xrightarrow{\cong} \mathbb{A}^1 \times \Delta^1. \tag{5.7}$$

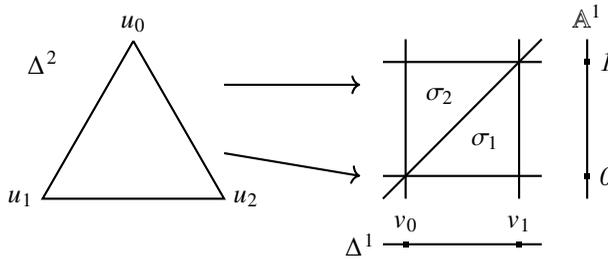


Figure 1. Triangulation.

Explicitly, the maps σ_1, σ_2 are the unique affine-linear ones satisfying

$$\begin{aligned} \sigma_1(u_0) &= (v_0, 0), & \sigma_1(u_1) &= (v_1, 0), & \sigma_1(u_2) &= (v_1, 1), \\ \sigma_2(u_0) &= (v_0, 0), & \sigma_2(u_1) &= (v_0, 1), & \sigma_2(u_2) &= (v_1, 1) \end{aligned}$$

in $(\Delta^1 \times \mathbb{A}^1)(k)$, where $v_j = i_{1,j}(\Delta^0)$ for $j = 0, 1$ and $u_0 = i_{2,2}(v_1), u_1 = i_{2,0}(v_1), u_2 = i_{2,1}(v_0)$.

Let $\Phi_{v,1}^{(1)}, \Phi_{v,1}^{(2)} : \mathbb{A}^N \times X \times \Delta^2 \rightarrow \mathbb{A}^N \times X \times \Delta^1$ be the composite maps $\Phi_{v,1} \circ (\text{Id}_{\mathbb{A}^N \times X} \times \sigma_i)$ ($i = 1, 2$). By fact (5.6), we conclude $\Phi_{v,1}^{(1)*} \Gamma, \Phi_{v,1}^{(2)*} \Gamma \in C_2(X)(\mathbb{A}^N)$. Now set

$$\tilde{\Gamma} := \Phi_{v,1}^{(1)*} \Gamma - \Phi_{v,1}^{(2)*} \Gamma.$$

We want to show $(\Gamma - \partial_2 \tilde{\Gamma})|_U \in \overline{C}_1(X)(U)$.

By a routine calculation of $\partial_2 \tilde{\Gamma}$, we have

$$\Gamma - \partial_2 \tilde{\Gamma} = \tau_v^* \Gamma + \Phi_{v,0}^* (i_{1,1}^* \Gamma) - \Phi_{v,0}^* (i_{1,0}^* \Gamma). \tag{5.8}$$

By Lemma 5.3, for $v \in \mathbb{A}^N(k) \setminus B$ we know that the first term of the right-hand side maps into $\overline{C}_1(X)(U)$. So it suffices to show the following, which we apply to $\gamma := i_{1,j}^* \Gamma$:

Lemma 5.4. *For every $\gamma \in C_0(X)(\mathbb{A}^N)$, there is a closed proper subset $C \subseteq \mathbb{A}^N$ such that for all $v \in \mathbb{A}^N(k) \setminus C$, the cycle $\Phi_{v,0}^* \gamma$ on $\mathbb{A}^N \times X \times \Delta^1$ belongs to $\overline{C}_1(X)(\mathbb{A}^N)$.*

Proof. Let us observe that as long as $v \neq 0$, the morphism φ_v extends (uniquely) to a morphism

$$\begin{aligned} \overline{\varphi}_v : \mathbb{A}^N \times \mathbb{P}^1 &\rightarrow \mathbb{P}^N \\ (a, [t_0 : t_1]) &\mapsto [t_0 : t_0 a + t_1 v]. \end{aligned} \tag{5.9}$$

For this proof, we may assume γ is irreducible. Let $\overline{\gamma} \subset \mathbb{P}^N \times X$ be the closure of γ . Consider the set

$$C_\infty := \{a \in \mathbb{P}^N \setminus \mathbb{A}^N \mid \overline{\gamma} \rightarrow \mathbb{P}^N \text{ is not finite over } a\}. \tag{5.10}$$

By the $V = \text{Spec}(k)$ case of Lemma 5.2(2) (via $X \times \overline{\Delta}^n \cong \mathbb{P}^N \times X$), we find that C_∞ is a closed proper subset of $\mathbb{P}^N \setminus \mathbb{A}^N$. Let $C \subset \mathbb{A}^N$ be the cone associated to C_∞ , namely,

$$C := \begin{cases} \{0\} \cup q^{-1}(C_\infty) \subset \mathbb{A}^N & (N > 0), \\ \emptyset & (N = 0), \end{cases}$$

where $q : \mathbb{A}^N \setminus \{0\} \rightarrow \mathbb{P}^N \setminus \mathbb{A}^N$ is the projection with centre 0. This is a closed proper subset of \mathbb{A}^N .

Now suppose $v \in \mathbb{A}^N(k) \setminus C$. Then by formula (5.9) we see that $\overline{\varphi}_v$ maps $\mathbb{A}^N \times \mathbb{P}^1$ into $\mathbb{P}^N \setminus C_\infty$. We obtain the following commutative diagram:

$$\begin{array}{ccc} \mathbb{A}^N \times X \times \Delta^1 & \xrightarrow{\Phi_{v,0}} & \mathbb{A}^N \times X & \supset & \gamma \\ \cap & & \cap & & \cap \\ \mathbb{A}^N \times X \times \mathbb{P}^1 & \xrightarrow{\overline{\varphi}_v \times \text{Id}_X} & (\mathbb{P}^N \setminus C_\infty) \times X & \supset & \overline{\gamma}|_{\mathbb{P}^N \setminus C_\infty} \\ \downarrow \text{pr} & & \downarrow \text{pr} & \swarrow & \\ \mathbb{A}^N \times \mathbb{P}^1 & \xrightarrow{\overline{\varphi}_v} & \mathbb{P}^N \setminus C_\infty & & \end{array}$$

Since the slanted arrow is finite by the definition (5.10) of C_∞ , the inverse image $(\overline{\varphi}_v \times \text{Id}_X)^* \overline{\gamma}$ is finite over $\mathbb{A}^N \times \mathbb{P}^1$. By the inclusion $(\Phi_{v,0}^* \gamma) \subset (\overline{\varphi}_v \times \text{Id}_X)^* \overline{\gamma}$ of subsets of $\mathbb{A}^N \times X \times \mathbb{P}^1$, we conclude that $\Phi_{v,0}^* \gamma$ belongs to $\overline{C}_1(X)(\mathbb{A}^N)$, completing the proof. □

Lemmas 5.3 and 5.4 applied to equation (5.8) prove Theorem 5.1 in the case $V = \mathbb{A}^N$.

5.3. The general case

Let $V \in \mathbf{Sm}$ be affine and irreducible. Let $N = \dim(V)$ be its dimension. Fix a closed embedding $V \hookrightarrow \mathbb{A}^{N'}$ into an affine space. For a technical reason (see Proposition 5.6), we assume it is obtained as the composition of a preliminary one $V \hookrightarrow \mathbb{A}^{N'_0}$ and the 2-fold Veronese embedding $\mathbb{A}^{N'_0} \hookrightarrow \mathbb{A}^{N'}$, $N' := \binom{N'_0 + 2}{2} - 1$ defined by

$$(x_i)_{i=1, \dots, N'_0} \mapsto (x_1, \dots, x_{N'_0}; (x_i x_j)_{i \leq j}). \tag{5.11}$$

Let $M_{NN'} \cong \mathbb{A}^{NN'}$ be the k -scheme parametrising $N \times N'$ matrices. The choice of a k -rational point $f \in M_{NN'}(k)$ determines a morphism, which we denote by the same symbol $f : \mathbb{A}^{N'} \rightarrow \mathbb{A}^N$.

Proposition 5.5 (Noether’s normalisation lemma). *There is a closed proper subset $D_1 \subset M_{NN'}$ such that the composite map*

$$\pi_f : V \hookrightarrow \mathbb{A}^{N'} \xrightarrow{f} \mathbb{A}^N$$

is finite and flat whenever $f \in M_{NN'}(k) \setminus D_1$.

Proof. For the existence of D_1 which guarantees finiteness, see, for example, [1, p. 69] or [20, Section 3.1]. Flatness is then automatic by the smoothness of V and \mathbb{A}^N ; see [13, Exercise III-10.9, p. 276]. \square

By Proposition 5.5, for $f \in M_{NN'}(k) \setminus D_1$ we have push-forward maps $\pi_{f*} : H_n(X)(V) \rightarrow H_n(X)(\mathbb{A}^N)$. Let $s \in U$ be the unique closed point of V contained in U . Let $U_0 \subset \mathbb{A}^N$ be the local scheme at $\pi_f(s)$. Since π_f carries U into U_0 , we have the following commutative diagram:

$$\begin{CD} C_1(X)(V) @>>> H_1(Q_\bullet(X))(V) \\ @. @VV\pi_{f*}V \\ @. H_1(Q_\bullet(X))(\mathbb{A}^N) @>(-)|_{U_0=0}>> H_1(Q_\bullet(X))(U_0) \\ @. @V\pi_f^*VV @VV(\pi_f|_U)^*V \\ @. H_1(Q_\bullet(X))(V) @>(-)|_U>> H_1(Q_\bullet(X))(U). \end{CD} \tag{5.12}$$

Here the restriction map $(-)|_{U_0}$ is the zero map by the conclusion of Section 5.2.

Now toward the proof of Theorem 5.1, let

$$\Gamma \in C_1(X)(V)$$

be an irreducible cycle. The commutative diagram shows that $(\pi_f^* \pi_{f*} \Gamma)|_U = 0$ in $H_1(Q_\bullet(X))(U)$. In other words, we have

$$\Gamma|_U = (\Gamma - \pi_f^* \pi_{f*} \Gamma)|_U \text{ in } H_1(Q_\bullet(X))(U). \tag{5.13}$$

This right-hand side turns out to be easier to handle.

Let $\bar{\Gamma} \subset V \times X \times \bar{\Delta}^{-1}$ be the closure and let $B(\Gamma) \subset V \times \bar{\Delta}^{-1}$ be the bad locus in formula (5.1). By Lemma 5.2, we know it is a closed proper subset of $V \times \{\infty\}$. Let $\bar{B}(\Gamma) \subseteq V$ be its projection (which is isomorphic to $B(\Gamma)$).

Proposition 5.6. *There is a closed proper subset $D_2 \subseteq M_{NN'}$ such that whenever $f \in M_{NN'}(k) \setminus (D_2 \cup D_1)$, we have the equality of zero cycles:*

$$\pi_f^* \pi_{f*} s = s + \sum_{i=1}^m x_i \text{ on } V, \tag{5.14}$$

where s, x_1, \dots, x_m are distinct and $x_i \in V \setminus \bar{B}(\Gamma)$ for all i .

Proof. This can be shown using Chow’s techniques [8, pp. 458–460], [7, p. 3-08, Lemma 2]. We present a proof based on a more recent account [20].

First, since we are using the Veronese embedding (5.11), we can invoke [20, Propositions 3.2 and 3.3], which state that there is a closed proper subset $D' \subseteq M_{NN'}$ such that for all $f \in M_{NN'}(k) \setminus (D' \cup D_1)$, the map $\pi_f : V \rightarrow \mathbb{A}^N$ is étale over $\pi_f(s)$ and the restriction $s \rightarrow \pi_f(s)$ is an isomorphism. This gives an equality of the form (5.14), with s and x_i s distinct.

It remains to show that $x_i \in V \setminus \overline{B}(\Gamma)$. We need the following statement:

Proposition 5.7 (a special case of [20, Proposition 3.5]). *Let $B \subset V$ be a proper closed subset and $s \in V$ a closed point. Then there is a closed proper subset $D'' \subsetneq M_{NN'}(k) \setminus D''$, we have the equality of subsets of V :*

$$\left(\pi_f^{-1} \pi_f(s) \setminus \{s\}\right) \cap B = \emptyset.$$

(To extract Proposition 5.7 from [20, Proposition 3.5], set $X := V, Y := \text{Spec}(k), p = 0, W := B$ and $V := \{s\}$. Also note that the only topological space having dimension ≤ -1 is the empty set.)

Now set $D_2 := D' \cup D''$ and suppose $f \in M_{NN'}(k) \setminus (D_1 \cup D_2)$. Then we know $\pi_f^{-1} \pi_f(s) \setminus \{s\} = \{x_1, \dots, x_m\}$, by the first half of this proof. Applying Proposition 5.7 to $B := \overline{B}(\Gamma)$, we get the desired result. This completes the proof of Proposition 5.6. \square

Corollary 5.8. *Take any $f \in M_{NN'}(k) \setminus (D_1 \cup D_2)$ and form the fibre product $V \times_{\mathbb{A}^N} U$ of $\pi_f : V \rightarrow \mathbb{A}^N$ and $\pi_f|_U : U \rightarrow \mathbb{A}^N$. Then it decomposes as*

$$V \times_{\mathbb{A}^N} U \cong U \sqcup T,$$

where T is finite and étale over U by the second projection and maps into $V \setminus \overline{B}(\Gamma)$ along the first projection.

Proof. Since π_f is finite and étale over U_0 by Proposition 5.6 and U maps into U_0 , the second projection $V \times_{\mathbb{A}^N} U \rightarrow U$ is étale (and finite by default, because $f \in M_{NN'}(k) \setminus D_1$). Since it has the diagonal splitting $U \rightarrow V \times_{\mathbb{A}^N} U$, we have $V \times_{\mathbb{A}^N} U \cong U \sqcup T$. By Proposition 5.6, the set of its closed points can be computed as $V \times_{\mathbb{A}^N} \{s\} \cong \{s, x_1, \dots, x_m\}$, with the right-hand side having the reduced structure, and we know that x_i s map into $V \setminus \overline{B}(\Gamma)$ by the first projection. This completes the proof. \square

Consider the following Cartesian diagram, where Id denotes $\text{Id}_{X \times \Delta^1}$:

$$\begin{array}{ccc} \Gamma & & (U \sqcup T) \times X \times \Delta^1 & (5.15) \\ \cap & & \cong \Big| \text{Corollary 5.8} & \\ V \times X \times \Delta^1 & \xleftarrow{\text{pr}_1 \times \text{Id}} & (V \times_{\mathbb{A}^N} U) \times X \times \Delta^1 & \\ \downarrow \pi_f \times \text{Id} & \square & \downarrow \text{pr}_2 \times \text{Id} & \\ \mathbb{A}^N \times X \times \Delta^1 & \xleftarrow{(\pi_f|_U) \times \text{Id}} & U \times X \times \Delta^1 & \end{array}$$

The element $(\pi_f^* \pi_{f*} \Gamma)|_U$ in equation (5.13) is represented by the cycle $((\pi_f|_U) \times \text{Id})^* (\pi_f \times \text{Id})_* \Gamma$ on $U \times X \times \Delta^1$. By a slight abuse of notation, let us omit Ids from the notation in what follows; for example, the previous expression is shortened as $(\pi_f|_U)^* \pi_{f*} \Gamma$. By the base-change formula for flat pullback and proper push-forward of algebraic cycles, we know this equals $\text{pr}_{2*} \text{pr}_{1*}^* \Gamma$. Via the vertical isomorphism in diagram (5.15), if we write pr_{iU} and pr_{iT} ($i = 1, 2$) for the restrictions of the projections, we get

$$\left(\pi_f^* \pi_{f*} \Gamma\right)|_U = \text{pr}_{2U*} \text{pr}_{1U}^* \Gamma + \text{pr}_{2T*} \text{pr}_{1T}^* \Gamma. \tag{5.16}$$

We know that $\text{pr}_{1U} : U \rightarrow V$ is the inclusion map and $\text{pr}_{2U} : U \rightarrow U$ is the identity map. Thus the first term is $\Gamma|_U$. Therefore we can compute the right-hand side in equation (5.13) as

$$\left(\Gamma - \pi_f^* \pi_{f*} \Gamma\right)|_U = -\text{pr}_{2T*} \text{pr}_{1T}^* \Gamma. \tag{5.17}$$

We have $F^{q,i}(S) = F_{\text{ur}}^{q,i}(S)$ for any local S .

Theorem 6.3. *For any $X \in \mathbf{Sm}$ and any $q = 0, 1$, we have an exact sequence*

$$0 \rightarrow F_{\text{ur}}^{q,i}(X) \rightarrow \bigoplus_{x \in X^{(0)}} F^{q,i}(x) \rightarrow \bigoplus_{x \in X^{(1)}} G_x^{q,i}(X),$$

where we set $G_x^{q,i}(X) := H_x^{q+1}(X, W_n \Omega_{X, \log}^i)$. Moreover, if $q = 0$ there exists a canonical isomorphism

$$\theta_x^i : F^{0,i-1}(k(x)) \xrightarrow{\cong} G_x^{0,i}(X)$$

for any codimension 1 point $x \in X^{(1)}$.

Proof. For the first assertion, see [12, Theorem 1.4] or [26, Theorem 4.1]. For the last assertion, see [26, Theorem 3.2]. □

Proposition 6.4. *The étale sheaf $W_n \Omega_{\log}^i$ on \mathbf{Sm} has a structure of presheaf with transfers. Hence so does the étale cohomology group $H^j(-, W_n \Omega_{\log}^i)$ for any $j \geq 0$.*

Proof. According to [22, 6.21], the second assertion is immediate from the first one. Therefore, it suffices to show that $W_n \Omega_{\log}^i \in \mathbf{PST}$. However, thanks to Theorem 6.3 together with the theorem of Bloch, Gabber and Kato [6], for any $X \in \mathbf{Sm}$ we have a natural exact sequence

$$0 \rightarrow H^0(X, W_n \Omega_{\log}^i) \rightarrow \bigoplus_{x \in X^{(0)}} K_i^M(k(x))/p^n \xrightarrow{(\partial_x^M)} \bigoplus_{x \in X^{(1)}} K_{i-1}^M(k(x))/p^n,$$

where ∂_x^M is the tame symbol at each $x \in X^{(1)}$. This implies that the sheaf $W_n \Omega_{\log}^i$ has a structure of (homotopy-invariant) presheaf with transfers (see [16]). This completes the proof of the proposition. □

Proposition 6.5. *Let $f : Y \rightarrow X$ be an étale morphism in \mathbf{Sm} which induces an isomorphism $k(y) \cong k(x)$ for some $y \in Y^{(1)}$ and $x := f(y)$. Then for $q = 0, 1$, in the commutative diagram*

$$\begin{array}{ccc} F^{q,i}(\text{Frac } \mathcal{O}_{X,x})/F^{q,i}(\mathcal{O}_{X,x}) & \longrightarrow & F^{q,i}(\text{Frac } \mathcal{O}_{Y,y})/F^{q,i}(\mathcal{O}_{Y,y}) \\ \downarrow & & \downarrow \\ G_x^{q,i}(X) & \longrightarrow & G_y^{q,i}(Y) \end{array}$$

all maps are bijective.

Proof. For $q = 0$, the assertion follows from the last claim of Theorem 6.3. So let us assume that $q = 1$. Then the bijectivity of the left vertical map is a consequence of the localisation sequence

$$F^{1,i}(\mathcal{O}_{X,x}) \rightarrow F^{1,i}(\text{Frac } \mathcal{O}_{X,x}) \rightarrow G_x^{1,i}(X) \rightarrow H^2(\mathcal{O}_{X,x}, W_n \Omega_{X, \log}^i) = 0,$$

and the same for the right vertical map. The statement for the upper horizontal map in the case when $n = 1$ is a consequence of [29, Theorem 4.3]. Indeed, given any discrete valuation ring over k , the cited theorem shows that there is an exhaustive filtration $F^{1,i}(R) = U_{-1} \subset U_0 \subset U_1 \subset \dots \subset F^{1,i}(\text{Frac } R)$ whose graded quotients are described solely in terms of the residue field. In our situation, $F^{1,i}(\text{Frac } \mathcal{O}_{X,x}) \rightarrow F^{1,i}(\text{Frac } \mathcal{O}_{Y,y})$ respects this filtration because $\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{Y,y}$ is étale. Hence the desired bijectivity follows from the assumption $k(y) \cong k(x)$. For $n > 1$, thanks to the exact sequence

$$0 \rightarrow W_{n-1} \Omega_{X, \log}^i \rightarrow W_n \Omega_{X, \log}^i \rightarrow \Omega_{X, \log}^i \rightarrow 0$$

for any regular scheme X over \mathbb{F}_p (see [26, Proposition 2.12]), one can inductively see the bijectivity of the map $F^{1,i}(\text{Frac } \mathcal{O}_{X,x})/F^{1,i}(\mathcal{O}_{X,x}) \rightarrow F^{1,i}(\text{Frac } \mathcal{O}_{Y,y})/F^{1,i}(\mathcal{O}_{Y,y})$. This completes the proof. \square

Proof of Proposition 6.1. We first show that $F_{\text{ur}}^{1,i}$ is a sheaf for Nisnevich topology. For this, we take a Cartesian diagram in **Sm**

$$\begin{array}{ccc} V & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ U & \xrightarrow{j} & X, \end{array}$$

where j is an open dense immersion and f is an étale morphism that is an isomorphism over $X \setminus j(U)$. By [22, 12.7], it suffices to prove the exactness of the upper row in the commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F_{\text{ur}}^{1,i}(X) & \longrightarrow & F_{\text{ur}}^{1,i}(U) \oplus F_{\text{ur}}^{1,i}(Y) & \longrightarrow & F_{\text{ur}}^{1,i}(V) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \bigoplus_{x \in X^{(0)}} F^{1,i}(x) & \longrightarrow & \bigoplus_{x \in U^{(0)}} F^{1,i}(x) \oplus \bigoplus_{y \in Y^{(0)}} F^{1,i}(y) & \longrightarrow & \bigoplus_{y \in V^{(0)}} F^{1,i}(y) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \bigoplus_{x \in X^{(1)}} G_x^{1,i}(X) & \xrightarrow{(*)} & \bigoplus_{x \in U^{(1)}} G_x^{1,i}(U) \oplus \bigoplus_{y \in Y^{(1)}} G_y^{1,i}(Y). & & \end{array}$$

The second row is exact for obvious reason, and all columns are exact by Theorem 6.3. The map $(*)$ is injective by Proposition 6.5 (and the assumption on f). Now the claim follows by diagram chasing. As a consequence, we can find that $F_{\text{ur}}^{1,i}$ is the same as the Nisnevich sheaf associated with $F^{1,i} \in \mathbf{PST}$ (see Proposition 6.4). Therefore, we conclude that $F_{\text{ur}}^{1,i} \in \mathbf{PST}$ by [22, 13.1].

Finally, to show that $F_{\text{ur}}^{1,i}$ is \mathbb{P}^1 -invariant, we apply Lemma 6.6. Condition (1) is due to Izhboldin [15] (see also [29, Theorem 4.4]), and (2) is a part of Theorem 6.3. \square

Lemma 6.6. *Set $F \in \mathbf{PST}$. Suppose the following conditions hold:*

- (1) *For any $K \in \mathbf{Fld}_k^{\text{gm}}$, $\sigma_K^* : F(\text{Spec } K) \cong F(\mathbb{P}_K^1)$ is an isomorphism, where σ_K denotes the base change of the structure morphism $\sigma : \mathbb{P}^1 \rightarrow \text{Spec } k$.*
- (2) *Any open immersion $U \hookrightarrow V$ in **Sm** induces an injection $F(U) \rightarrow F(V)$.*

Then F is \mathbb{P}^1 -invariant.

Proof. Define $G \in \mathbf{PST}$ by the formula

$$G(U) := \text{Coker}\left(\sigma^* : F(U) \rightarrow \underline{\text{Hom}}\left(\mathbb{Z}_{\text{tr}}(\mathbb{P}^1), F\right)(U) = F\left(U \times \mathbb{P}^1\right)\right) \quad (U \in \mathbf{Sm}). \tag{6.3}$$

We have a direct sum decomposition $\underline{\text{Hom}}(\mathbb{Z}_{\text{tr}}(\mathbb{P}^1), F) \cong F \oplus G$ (provided by a k -rational point of \mathbb{P}^1), and $G(U) = 0$ holds if and only if the map in formula (6.3) is an isomorphism. By (1), we have $G(\text{Spec } K) = 0$ for any $K \in \mathbf{Fld}_k^{\text{gm}}$. Property (2) for F implies the same property for $\underline{\text{Hom}}(\mathbb{Z}_{\text{tr}}(\mathbb{P}^1), F)$ and hence for G . We conclude that $G(U) \hookrightarrow G(k(U)) = 0$ for any (irreducible) $U \in \mathbf{Sm}$, which means F is \mathbb{P}^1 -invariant. \square

A. Proof of Remark 2.5

We give a proof of Remark 2.5. Property (1) is obvious. Property (2) is a consequence of the formula $h_0(X \times X') = h_0(X) \otimes h_0(X')$. For (3) and (4), we shall freely use Voevodsky’s triangulated category $\mathbf{DM}_{\text{eff}}^-(k) \subset D^-(\mathbf{NST})$ of effective motivic complexes over k [31]. For $V \in \mathbf{Sm}$, we denote the motivic complex of V by $M(V) := C_*(\mathbb{Z}_{\text{tr}}(V)) \in \mathbf{DM}_{\text{eff}}^-(k)$. Recall that its homology sheaves $h_n(V) := H_n(M(V))$ are trivial if $n < 0$, and for $n = 0$ it recovers $h_0(V)$ defined in (V2).

Lemma A.1. *Let $c > 0$ be a positive integer and let $j : U \rightarrow X$ be an open immersion in \mathbf{Sm} . Define $M(X/U)$ to be the cone of $j_* : M(U) \rightarrow M(X)$. If each component of $Z := (X \setminus U)_{\text{red}}$ is of codimension $\geq c$ in X , then $h_n(X/U) := H_n(M(X/U))$ vanishes for any $n < c$.*

Proof. If Z is smooth over k , then by the Gysin triangle [22, Theorem 15.15] we have

$$M(X/U) \cong M(Z)(c)[2c] \cong M(Z) \otimes (\mathbb{G}_m[0])^{\otimes c}[c],$$

from which the statement follows. In general, let $Z' \subset Z$ be the singular locus of Z , and define $U' := X \setminus Z'$. There is a distinguished triangle

$$M(U'/U) \rightarrow M(X/U) \rightarrow M(X/U') \rightarrow M(U'/U)[1].$$

Since each component of Z' has codimension $\geq c + 1$ in X , we may assume $h_n(X/U') = 0$ for any $n \leq c$ by induction. Since $U' \setminus U = Z \setminus Z'$ is smooth over k , we have shown $h_n(U'/U) = 0$ for $n < c$. It follows that $h_n(X/U) = 0$ for $n < c$. □

Now (3) immediately follows from Lemma A.1. To show (4), let $f : X \rightarrow Y$ be a proper birational morphism in \mathbf{Sm} . Let $V \subset Y$ be the open dense subset on which f^{-1} is defined. Then f restricts to an isomorphism $U := f^{-1}(V) \xrightarrow{\cong} V$ and $Y \setminus V$ has codimension ≥ 2 in Y . We have a commutative diagram

$$\begin{array}{ccccccc} h_0(U) & \longrightarrow & h_0(X) & \longrightarrow & h_0(X/U) = 0 \\ \cong \downarrow & & \downarrow & & \downarrow \\ 0 = h_1(Y/V) & \longrightarrow & h_0(V) & \longrightarrow & h_0(Y) & \longrightarrow & h_0(Y/V) = 0, \end{array}$$

in which we used Lemma A.1 for the vanishing. This proves (4).

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