

RESEARCH ARTICLE

Iitaka fibrations for dlt pairs polarized by a nef and log big divisor

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Abstract

We study lc pairs polarized by a nef and log big divisor. After proving the minimal model theory for projective lc pairs polarized by a nef and log big divisor, we prove the effectivity of the Iitaka fibrations and some boundedness results for dlt pairs polarized by a nef and log big divisor.

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1. Introduction

Throughout this paper, we will work over the complex number field \mathbb{C} .

In this paper, we study triples (X, Δ, M) such that (X, Δ) is a projective lc pair and M is a nef \mathbb{R} -divisor on X that is log big with respect to (X, Δ) (see Definition 2.13). By definition, the log bigness coincides with the bigness in the case of klt pairs, and a nef \mathbb{R} -divisor D on a normal projective variety V is log big with respect to a projective lc pair (V, B) if and only if $(D^{\dim V}) > 0$ and $(S \cdot D^n) > 0$ for any n -dimensional lc center S of (V, B) . Especially, all ample divisors are nef and log big with respect to any projective lc pair. The triples (X, Δ, M) can be regarded as generalized pairs defined by Birkar–Zhang [7]. Generalized pairs are the main objects in the study of lc-trivial fibrations. Because

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the log bigness is one of the special cases of the property of being log abundant (see Definition 2.13), the recent progress of the canonical bundle formula for lc-trivial fibrations by Floris–Lazić [11] (see also [34, Theorem 1.2] by Hu) implies that the above triples (X, Δ, M) appear as the structures on the base varieties of special lc-trivial fibrations. The goal of this paper is to develop the theory of the above triples (X, Δ, M) from viewpoints of the minimal model theory.

We start with the minimal model theory for the above triples (X, Δ, M) . If (X, Δ) is a klt pair, then we can run a minimal model program (MMP, for short) on $K_X + \Delta + M$ and we get a birational contraction $\phi: X \dashrightarrow X'$ such that $\phi_*(K_X + \Delta + M)$ is semiample or X' has the structure of a Mori fiber space with respect to $\phi_*(K_X + \Delta + M)$. This fact is a consequence of the celebrated result by Birkar–Cascini–Hacon–McKernan [6]. Even if (X, Δ) is not klt, the same statement is known by the author and Hu [33] under the assumption of the ampleness of M . In the general case, the minimal model theory for $K_X + \Delta + M$ is not known. However, the abundance theorem (more strongly, the effective base point free theorem in the \mathbb{Q} -boundary case) was proved by Fujino [14] if $K_X + \Delta + M$ is nef and Cartier. Moreover, Fujino [16] proved the abundance theorem for $K_X + \Delta + M$ in the case where (X, Δ) is slc, a more general situation than the lc case.

The first main result of this paper is the minimal model theory for $K_X + \Delta + M$.

Theorem 1.1. *Let (X, Δ) be a projective lc pair, and let M be a nef \mathbb{R} -divisor on X that is log big with respect to (X, Δ) . Suppose that M is a finite $\mathbb{R}_{>0}$ -linear combination of nef \mathbb{Q} -Cartier divisors on X . Then there exists a birational contraction $\phi: X \dashrightarrow X'$, which is a sequence of steps of a $(K_X + \Delta + M)$ -MMP such that X' satisfies one of the following conditions.*

- $\phi_*(K_X + \Delta + M)$ is semiample, or
- There is a contraction $X' \rightarrow Z$ to a projective variety Z such that $\dim Z < \dim X'$, $-\phi_*(K_X + \Delta + M)$ is ample over Z and the relative Picard number is one.

Note that $K_{X'} + \phi_*\Delta$ is \mathbb{R} -Cartier and $(X', \phi_*\Delta)$ is an lc pair.

Theorem 1.1 is proved by generalizing [31, Theorem 3.5] to the context of generalized pairs. For the generalization, see Theorem 3.15. To prove Theorem 1.1, we only need to prove the termination of the MMP. It is because the abundance theorem in the situation of Theorem 1.1 was already proved by Fujino [14]. The proof in [14] heavily depends on a vanishing theorem for quasi-log schemes (see [17, Theorem 5.7.3]). On the other hand, for the termination of the MMP of Theorem 1.1, we need to discuss the MMP in the framework of generalized pairs (see, for example, [25] by Han–Li) and we need the techniques developed in [31]. By Theorem 3.15, we also know the existence of a minimal model for generalized lc pairs with a polarization (Theorem 3.17), which was mentioned in [32]. We note that we do not need the existence of flips for \mathbb{Q} -factorial generalized lc pairs by Hacon–Liu [23] in the proof of Theorem 1.1 or Theorem 3.17.

We apply Theorem 1.1 to study the effectivity of the Iitaka fibrations for the above triples (X, Δ, M) such that (X, Δ) is a dlt pair. The effectivity of the Iitaka fibrations for higher dimensional lc pairs was studied by Fujino–Mori [20], Viehweg–Zhang [41], Birkar–Zhang [7] and Hacon–Xu [24]. Those results are based on the canonical bundle formula for the Iitaka fibrations [20]. If we know the effectivity of the Iitaka fibrations for lc pairs, then we know the existence of an integer m , which is independent of the variety, such that every projective lc pair $(Y, 0)$ of fixed dimension with $\kappa(Y, K_Y) \geq 0$ satisfies $H^0(Y, \mathcal{O}_Y(mK_Y)) \neq 0$. Especially, if K_Y is numerically trivial, then mK_Y is Cartier. Hence, the effectivity of the Iitaka fibrations for lc pairs is stronger than the boundedness of the Cartier indices of numerically trivial lc varieties of fixed dimension.

In the case of klt pairs whose boundary divisors are big, the effectivity of the Iitaka fibrations is known by Hacon–Xu [24]. In the non-klt case, the effectivity of the Iitaka fibrations for lc pairs of log general type was proved by Hacon–McKernan–Xu [21], and their result was generalized to the context of generalized lc pairs (see Definition 2.6) by Birkar–Zhang [7]. The theorem by Birkar–Zhang [7] implies the effectivity of the Iitaka fibrations for $K_X + \Delta + M$ of the triples (X, Δ, M) such that $K_X + \Delta + M$ are big and M are finite $\mathbb{R}_{>0}$ -linear combinations of nef \mathbb{Q} -Cartier divisors. In this paper, we study the case where $K_X + \Delta + M$ has an intermediate Iitaka dimension.

Theorem 1.2. *Let d and p be positive integers, and let $\Phi \subset \mathbb{Q}_{\geq 0}$ be a DCC set. Then there exists a positive integer m , depending only on d, p and Φ , satisfying the following. Let (X, Δ) be a projective dlt pair, and let M be a nef \mathbb{Q} -divisor on X such that*

- $\dim X = d$,
- the coefficients of Δ belong to Φ ,
- pM is Cartier and
- M is log big with respect to (X, Δ) and $K_X + \Delta + M$ is pseudo-effective.

Then, the linear system $|[lm(K_X + \Delta + M)]|$ is not empty and it defines a map birational to the Iitaka fibration for every positive integer l .

More precisely, we prove the existence of a generalized lc pair such that the coefficients of the boundary divisor and the b-Cartier index of the nef part are controlled.

Theorem 1.3. *Let d be a positive integer, and let $\Phi \subset \mathbb{Q}_{\geq 0}$ be a DCC set. Then there exist positive integers n and q and a DCC set $\Omega \subset \mathbb{Q}_{\geq 0}$, depending only on d and Φ , satisfying the following. Let (X, Δ) be a projective dlt pair, and let $M = \sum_i \mu_i M_i$ be a nef \mathbb{R} -divisor on X , which is the sum of nef and log big Cartier divisors M_i with respect to (X, Δ) , such that*

- $\dim X = d$,
- the coefficients of Δ and μ_i belong to Φ and
- $K_X + \Delta + M$ is pseudo-effective but not big.

Then the variety $Z := \mathbf{Proj} \bigoplus_{l \in \mathbb{Z}_{>0}} H^0(X, \mathcal{O}_X([l(K_X + \Delta + M)]))$, which is well defined by Theorem 1.1, has the structure of a generalized lc pair $(Z, \Delta_Z, \mathbf{N})$ such that

- $H^0(X, \mathcal{O}_X([ln(K_X + \Delta + M)])) \simeq H^0(Z, \mathcal{O}_Z([ln(K_Z + \Delta_Z + \mathbf{N}_Z)]))$ for every positive integer l ,
- the coefficients of Δ_Z belong to Ω and
- $q\mathbf{N}$ is b-Cartier.

Theorem 1.2 directly follows from Theorem 1.3 and [7, Theorem 1.3].

We further study the structures of the Iitaka fibrations. In [39], Li defined the Iitaka volumes of \mathbb{Q} -Cartier divisors on normal projective varieties (see Definition 5.1). In our context, with notations as in Theorem 1.3, the Iitaka volume of $K_X + \Delta + M$ coincides with the volume of $K_Z + \Delta_Z + \mathbf{N}_Z$. The following theorem is the DCC for the Iitaka volumes of $K_X + \Delta + M$.

Theorem 1.4. *Let d be a positive integer and $\Phi \subset \mathbb{Q}_{\geq 0}$ a DCC set. Then there exists a DCC set $\Omega \subset \mathbb{Q}_{>0}$, depending only on d and Φ , satisfying the following. Let (X, Δ) be a projective dlt pair, and let $M = \sum_i \mu_i M_i$ be a nef \mathbb{R} -divisor on X , which is the sum of nef and log big Cartier divisors M_i with respect to (X, Δ) , such that*

- $\dim X = d$,
- the coefficients of Δ and μ_i belong to Φ and
- $K_X + \Delta + M$ is pseudo-effective.

Then the Iitaka volume $\text{Ivol}(K_X + \Delta + M)$ is an element of Ω .

We give remarks on general fibers of the Iitaka fibrations. As shown in the example below, compared to [24, Theorem 1.3], we cannot prove a kind of the boundedness of general fibers of the Iitaka fibrations.

Example 1.5 (see Example 4.6). We put $d = 2, p = 1$, and $\Phi = \{1\}$. We consider the category \mathcal{C} whose objects are the triples (X, Δ, M) , where (X, Δ) is a projective dlt pair and M is a nef Cartier divisor on X which is log big with respect to (X, Δ) such that $\dim X = 2, \Delta$ is a Weil divisor, and $K_X + \Delta + M \sim_{\mathbb{Q}} 0$. Then, for some set

$$\mathcal{D} \subset \{X \mid (X, \Delta, M) \text{ is a object of } \mathcal{C} \text{ for some } \Delta \text{ and } M\},$$

\mathcal{D} is unbounded. Indeed, for each positive integer n , defining $X_n := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$, then there are divisors Δ_n and M_n on X_n such that (X_n, Δ_n, M_n) is an object of \mathcal{C} . However, we see that the set $\{X_n\}_{n \geq 1}$ is unbounded. For details, see Example 4.6.

Despite the example, it is possible to prove the effective nonvanishing (and the boundedness of complements) for general fibers of the Iitaka fibrations for $K_X + \Delta + M$.

Theorem 1.6. *Let d be a positive integer, and let $\Phi \subset \mathbb{Q}_{\geq 0}$ be a DCC set. Then there exists a positive integer n , depending only on d and Φ , satisfying the following. Let (X, Δ) be a projective dlt pair, and let $M = \sum_i \mu_i M_i$ be a nef \mathbb{R} -divisor on X , which is the sum of nef and log big Cartier divisors M_i with respect to (X, Δ) such that*

- $\dim X = d$,
- the coefficients of Δ and μ_i belong to Φ and
- $K_X + \Delta + M$ is pseudo-effective.

Let $\phi: X \dashrightarrow X'$ be finite steps of a $(K_X + \Delta + M)$ -MMP such that $\phi_(K_X + \Delta + M)$ is semiample (see Theorem 1.1), and let F be a general fiber of the contraction induced by $\phi_*(K_X + \Delta + M)$. Put $\Delta'_F = \phi_*\Delta|_F$ and $M'_F = \phi_*M|_F$. Then*

- $n(K_F + \Delta'_F + M'_F) \sim 0$, and
- there is $B'_F \in |nM'_F|$ such that $(F, \Delta'_F + \frac{1}{n}B'_F)$ is an lc pair.

Theorem 1.6 is a consequence of the boundedness of complements for a special kind of dlt pairs (Theorem 5.4). In Example 5.5, we show that the log bigness of M_i cannot be relaxed to the bigness in Theorem 1.2 and Theorem 1.6. It is not clear that the dlt property can be generalized to log canonicity.

Finally, we discuss the effective finite generation of the generalized log canonical rings for the above (X, Δ, M) . A kind of the effective finite generation for projective klt threefolds (V, B) was proved by Cascini–Zhang [8] when $K_V + B$ is big or B is nef and big. In this paper, we prove the following theorem.

Theorem 1.7. *Let d and p be positive integers, and let v be a positive real number. Then there exists a positive integer m , depending only on d, p and v , satisfying the following. Let (X, Δ) be a projective dlt pair, let M be a nef \mathbb{Q} -divisor on X that is log big with respect to (X, Δ) and let $\phi: X \dashrightarrow X'$ be finite steps of a $(K_X + \Delta + M)$ -MMP such that*

- $\dim X = d$,
- pM is Cartier and $p\phi_*(K_X + \Delta + M)$ is nef and Cartier and
- the Iitaka volume $\text{Ivol}(K_X + \Delta + M)$ is less than or equal to v .

Putting $R_l = H^0(X, \mathcal{O}_X(\lfloor l(K_X + \Delta + M) \rfloor))$ for every $l \in \mathbb{Z}_{\geq 0}$, then

- $\bigoplus_{l \in \mathbb{Z}_{\geq 0}} R_{lm}$ is generated by R_m as a graded \mathbb{C} -algebra, and
- the variety $Z := \mathbf{Proj} \bigoplus_{l \in \mathbb{Z}_{\geq 0}} R_l$ belongs to a bounded family \mathfrak{F} that depends only on d, p and v .

The proof of the theorem heavily depends on the effective base point free theorem for spacial generalized dlt pairs (Theorem 5.8) and the boundedness of generalized lc pairs in a special case (Lemma 5.10), thus we will need some extra work to remove the upper bound of the Cartier index of $\phi_*(K_X + \Delta + M)$ in Theorem 1.7.

The contents of the paper are as follows: In Section 2, we collect notations and definitions. In Section 3, we prove Theorem 1.1. In Section 4, we study the effectivity of the Iitaka fibrations for projective dlt pairs polarized by nef and log big divisors, and we prove Theorem 1.2 and Theorem 1.3. In Section 5, we show some boundedness results, and we prove Theorem 1.4, Theorem 1.6 and Theorem 1.7. In Section 6, which is an appendix, we discuss the definition of generalized dlt pairs.

2. Preliminaries

In this section, we collect notations and definitions. We will freely use the notations in [38] and [6].

2.1. Divisors, morphisms and singularities

We collect notations and definitions on divisors, morphisms and singularities of generalized pairs.

We will use the standard definitions of nef \mathbb{R} -divisor, ample \mathbb{R} -divisor, semiample \mathbb{R} -divisor and pseudo-effective \mathbb{R} -divisor. All big \mathbb{R} -divisors in this paper are \mathbb{R} -Cartier. For a morphism $f: X \rightarrow Y$ and an \mathbb{R} -Cartier divisor D on Y , we sometimes denote f^*D by $D|_X$. For a prime divisor P over X , the image of P on X is denoted by $c_X(P)$. A projective morphism $f: X \rightarrow Y$ of varieties is called a *contraction* if $f_*\mathcal{O}_X \simeq \mathcal{O}_Y$. For a variety X and an \mathbb{R} -divisor D' on it, a *log resolution of (X, D')* denotes a projective birational morphism $f: Y \rightarrow X$ from a smooth variety Y such that the exceptional locus $\text{Ex}(f)$ of f is pure codimension one and $\text{Ex}(f) \cup \text{Supp}f_*^{-1}D'$ is an snc divisor.

Definition 2.1. We say that a subset of \mathbb{R} satisfies *the descending chain condition (DCC, for short)* if the subset does not contain a strictly decreasing infinite sequence. We say that a subset of \mathbb{R} satisfies *the ascending chain condition (ACC, for short)* if the subset does not contain a strictly increasing infinite sequence. A subset of \mathbb{R} is called a *DCC set* (resp. an *ACC set*) if the subset satisfies the DCC (resp. ACC).

Let a be a real number. Then we define $\lfloor a \rfloor$ to be the unique integer satisfying $\lfloor a \rfloor \leq a < \lfloor a \rfloor + 1$, and we define $\{a\}$ by $\{a\} = a - \lfloor a \rfloor$.

Let X be a normal variety, and let D be an \mathbb{R} -divisor on X . Let $D = \sum_i d_i D_i$ be the prime decomposition. Then we define $\lfloor D \rfloor := \sum_i \lfloor d_i \rfloor D_i$, $\{D\} := \sum_i \{d_i\} D_i$ and $\lceil D \rceil := -\lfloor -D \rfloor$. By definition, we have $D = \lfloor D \rfloor + \{D\}$.

Definition 2.2 (Hyperstandard set). Let $\mathfrak{R} \subset \mathbb{R}$ be a subset. Throughout this paper, $\Phi(\mathfrak{R})$ is defined by

$$\Phi(\mathfrak{R}) = \left\{ 1 - \frac{a}{r} \mid a \in \mathfrak{R}, r \in \mathbb{Z}_{>0} \right\},$$

and we call it a *hyperstandard set associated to \mathfrak{R}* .

Definition 2.3 (Asymptotic vanishing order). Let X be a normal projective variety and D a pseudo-effective \mathbb{R} -Cartier divisor on X . Let P be a prime divisor over X . We define *asymptotic vanishing order of P with respect to D* , which we denote $\sigma_P(D)$, as follows. We take a projective birational morphism $f: Y \rightarrow X$ such that P appears as a prime divisor on Y . When D is big, we define $\sigma_P(D)$ by

$$\sigma_P(D) = \inf \{ \text{coeff}_P(D') \mid f^*D \sim_{\mathbb{R}} D' \geq 0 \}.$$

When D is not necessarily big, $\sigma_P(D)$ is defined by

$$\sigma_P(D) = \lim_{\epsilon \rightarrow +0} \sigma_P(D + \epsilon A)$$

for an ample \mathbb{R} -divisor A on X . It is easy to see that $\sigma_P(D)$ is independent of $f: Y \rightarrow X$. We have $\sigma_P(D) \geq 0$, and $\sigma_P(D)$ is also independent of A ([40, III, 1.5 (2) Lemma]). We can easily check $\sigma_P(D) = \sup \{ \sigma_P(D + H) \mid H \text{ is ample} \}$.

Definition 2.4 (Negative part of Nakayama–Zariski decomposition). For X and D as in Definition 2.3, the *negative part of Nakayama–Zariski decomposition of D* , denoted by $N_{\sigma}(D)$, is defined by

$$N_{\sigma}(D) = \sum_{\substack{P: \text{prime divisor} \\ \text{on } X}} \sigma_P(D)P.$$

Note that $N_{\sigma}(D)$ is not necessarily \mathbb{R} -Cartier in this paper.

When X is smooth, the definition of $N_\sigma(D)$ coincides with [40, III, 1.12 Definition].

We refer to [31, Remark 2.3] for basic properties of asymptotic vanishing order and the negative part of Nakayama–Zariski decomposition.

Definition 2.5 (b-divisor). Let X be a normal variety. Then an \mathbb{R} -b-divisor \mathbf{D} on X is a (possibly infinite) \mathbb{R} -linear combination of divisorial valuations v_P

$$\mathbf{D} = \sum_{\substack{P: \text{prime divisor} \\ \text{over } X}} r_P v_P \quad (r_P \in \mathbb{R})$$

such that, for every projective birational morphism $Y \rightarrow X$, the set

$$\{P \mid c_Y(P) \text{ is a divisor on } Y \text{ such that } r_P \neq 0\}$$

is a finite set. When $r_P \in \mathbb{Q}$ for all P , we call \mathbf{D} a \mathbb{Q} -b-divisor. For a projective birational model Y of X , the trace of an \mathbb{R} -b-divisor $\mathbf{D} = \sum_P r_P v_P$ on Y , which we denote \mathbf{D}_Y , is defined by

$$\mathbf{D}_Y = \sum_{\substack{c_Y(P) \text{ is a divisor} \\ \text{on } Y}} r_P c_Y(P).$$

By definition, \mathbf{D}_Y is an \mathbb{R} -divisor on Y .

Let \mathbf{D} be an \mathbb{R} -b-divisor on X . If there exists an \mathbb{R} -Cartier divisor D on a projective birational model Y of X such that

$$\mathbf{D} = \sum_{\substack{P: \text{prime divisor} \\ \text{over } X}} \text{ord}_P(D) \cdot v_P,$$

then we say that \mathbf{D} is \mathbb{R} -b-Cartier, we say that \mathbf{D} descends to Y and we denote $\mathbf{D} = \overline{D}$. When $\mathbf{D} = \overline{D}$ and D is \mathbb{Q} -Cartier, we say that \mathbf{D} is \mathbb{Q} -b-Cartier. When $\mathbf{D} = \overline{\mathbf{D}_{Y'}}$ and $\mathbf{D}_{Y'}$ is Cartier for some projective birational model Y' , we say that \mathbf{D} is b-Cartier.

Suppose that X has a projective morphism $X \rightarrow Z$ to a variety Z , and suppose in addition that $\mathbf{D} = \overline{\mathbf{D}_{Y'}}$ and $\mathbf{D}_{Y'}$ is nef over Z (resp. big over Z) for some projective birational model Y' of X . Then, we say that \mathbf{D} is b-nef/ Z (resp. b-big/ Z).

Let X and X' be normal varieties which are projective over a variety Z , and let $\phi: X \dashrightarrow X'$ be a birational map over Z . Let \mathbf{D} and \mathbf{D}' be \mathbb{R} -b-divisors on X and X' , respectively. Then, we say that $\mathbf{D} = \mathbf{D}'$ by ϕ if $\mathbf{D}_Y = \mathbf{D}'_{Y'}$ for all projective birational morphisms $f: Y \rightarrow X$ and $f': Y \rightarrow X'$ such that $f' = \phi \circ f$.

Definition 2.6 (Singularities of generalized pairs). A generalized pair $(X, \Delta, \mathbf{M})/Z$ consists of

- a projective morphism $X \rightarrow Z$ from a normal variety to a variety,
- an effective \mathbb{R} -divisor Δ on X and
- a b-nef/ Z \mathbb{R} -b-Cartier \mathbb{R} -b-divisor \mathbf{M} on X

such that $K_X + \Delta + \mathbf{M}_X$ is \mathbb{R} -Cartier. When $\mathbf{M} = 0$, the generalized pair $(X, \Delta, \mathbf{M})/Z$ is a usual pair (X, Δ) with $X \rightarrow Z$. When Z is a point, we simply denote (X, Δ, \mathbf{M}) .

Let $(X, \Delta, \mathbf{M})/Z$ be a generalized pair and P a prime divisor over X . Let $f: Y \rightarrow X$ be a projective birational morphism such that \mathbf{M} descends to Y and P appears as a divisor on Y . Then there is an \mathbb{R} -divisor Γ on Y such that

$$K_Y + \Gamma + \mathbf{M}_Y = f^*(K_X + \Delta + \mathbf{M}_X).$$

Then the generalized log discrepancy $a(P, X, \Delta + \mathbf{M}_X)$ of P with respect to $(X, \Delta, \mathbf{M})/Z$ is defined to be $1 - \text{coeff}_P(\Gamma)$. When $\mathbf{M} = 0$, the generalized log discrepancy $a(P, X, \Delta)$ coincides with the log discrepancy of P with respect to the usual pair (X, Δ) .

A generalized pair $(X, \Delta, \mathbf{M})/Z$ is called a *generalized klt* (resp. *generalized lc*) pair if $a(P, X, \Delta + \mathbf{M}_X) > 0$ (resp. $a(P, X, \Delta + \mathbf{M}_X) \geq 0$) for all prime divisors P over X . A *generalized lc center* of $(X, \Delta, \mathbf{M})/Z$ is the image on X of a prime divisor P over X satisfying $a(P, X, \Delta + \mathbf{M}_X) = 0$.

A generalized pair $(X, \Delta, \mathbf{M})/Z$ is a *generalized dlt pair* if it is generalized lc and for any generic point η of any generalized lc center of $(X, \Delta, \mathbf{M})/Z$, (X, Δ) is log smooth near η and \mathbf{M} descends to X on a neighborhood of η .

Lemma 2.7. *Let $(X, \Delta, \mathbf{M})/Z$ be a generalized dlt pair such that Z is quasi-projective, and let S be a component of $[\Delta]$. Then there is a plt pair (X, B) such that $S = [B]$. In particular, S is normal, and (X, B') is a klt pair for some B' .*

Proof. In this proof, we will use the notion of *generalized sub-lc pair*, whose definition is the same as that of generalized lc pairs except that the boundary part is not necessarily effective. This is an analogous definition of sub-lc pairs.

First, we reduce the lemma to the case, where $[\Delta] = S$. We can take an open subset $U \subset X$ such that $(U, \Delta|_U)$ is log smooth, U contains the generic points of all generalized lc centers of $(X, \Delta, \mathbf{M})/Z$ and $\mathbf{M}|_U$ descends to U . We take a log resolution $f: Y \rightarrow X$ of (X, Δ) such that

- f is an isomorphism over U , and
- $-E$ is f -ample for some effective f -exceptional divisor E on Y .

We note that we do not assume \mathbf{M}_Y to be nef over Z . We may write

$$K_Y + \Gamma + \mathbf{M}_Y = f^*(K_X + \Delta + \mathbf{M}_X).$$

Since U contains the generic points of all generalized lc centers of $(X, \Delta, \mathbf{M})/Z$, there is a real number $t > 0$ such that $(Y, \Gamma + tE, \mathbf{M})/Z$ is generalized sub-lc. Take an ample divisor A on X such that $-\frac{t}{2}E + f^*A$ is ample. By perturbing the coefficients of Γ with $-\frac{t}{2}E + f^*A$ and pushing down, we get an \mathbb{R} -divisor $G \geq 0$ on X such that

- $K_X + \Delta + \mathbf{M}_X + A \sim_{\mathbb{R}, Z} K_X + G + \mathbf{M}_X$,
- $(U, G|_U)$ is log smooth,
- $[G] = S$ and
- writing $K_Y + G_Y + \mathbf{M}_Y = f^*(K_X + G + \mathbf{M}_X)$, then $(Y, G_Y, \mathbf{M})/Z$ is a generalized sub-lc pair whose generalized lc centers intersect $f^{-1}(U)$.

In this way, we get a generalized dlt pair $(X, G, \mathbf{M})/Z$ such that $[G] = S$.

In the case where $[\Delta] = S$, we see that $(U, \Delta|_U)$ is plt. Hence, $a(P, X, \Delta + \mathbf{M}_X) > 0$ for all prime divisors P over X except $P = S$. Let $g: Y' \rightarrow X$ be a log resolution of (X, Δ) such that

- \mathbf{M} descends to Y' , and
- $-F'$ is g -ample for some g -exceptional divisor F' .

Let H be an ample divisor on X such that $-F' + g^*H$ is ample. Since $\mathbf{M}_{Y'}$ is nef over Z and $a(F_i, X, \Delta + \mathbf{M}_X) > 0$ for every component F_i of F' , we can apply the argument of perturbation of coefficients. We get a plt pair (X, B) such that $[B] = S$. □

We introduce properties of divisorial adjunction for generalized pairs.

Remark 2.8 (Divisorial adjunction). Let $(X, \Delta, \mathbf{M})/Z$ be a generalized lc pair, let S be a component of $[\Delta]$ with the normalization S^\vee and let $(S^\vee, \Delta_{S^\vee}, \mathbf{N})/Z$ be a generalized lc pair defined with divisorial adjunction for generalized pairs.

- If $p\mathbf{M}$ is b-Cartier, then $p\mathbf{N}$ is b-Cartier.
- We fix a DCC set $\Lambda \subset \mathbb{R}_{\geq 0}$. By [7, Proposition 4.9], there is a DCC set $\Omega \subset \mathbb{R}_{\geq 0}$, depending only on Λ , such that if the coefficients of Δ belong to Λ and \mathbf{M} is the sum of finitely many b-nef/ Z b-Cartier b-divisors with the coefficients in Λ , then the coefficients of Δ_{S^\vee} belong to Ω .

- We fix a finite set of rational numbers $\mathfrak{R} \subset [0, 1]$ and a positive integer p . By [4, Lemma 3.3], there is a finite set of rational numbers $\mathfrak{S} \subset [0, 1]$, depending only on \mathfrak{R} and p , such that if the coefficients of Δ belong to $\Phi(\mathfrak{R})$ and $p\mathbf{M}$ is b -Cartier, then the coefficients of Δ_{S^v} belong to $\Phi(\mathfrak{S})$.

Definition 2.9 (Restriction morphism). Let $(X, \Delta, \mathbf{M})/Z$ be a generalized lc pair, and let S be a component of $[\Delta]$ with the normalization S^v . With divisorial adjunction for generalized pairs we define a generalized lc pair $(S^v, \Delta_{S^v}, \mathbf{N})/Z$. Suppose that $p\mathbf{M}$ is b -Cartier. Let $\tau: S^v \rightarrow S \hookrightarrow X$ be the natural morphism. With the idea of [4, 3.1], we will define a morphism

$$\mathcal{O}_X(\lfloor p(K_X + \Delta + \mathbf{M}_X) \rfloor) \longrightarrow \tau_* \mathcal{O}_{S^v}(\lfloor p(K_{S^v} + \Delta_{S^v} + \mathbf{N}_{S^v}) \rfloor).$$

We take a log resolution $f: Y \rightarrow X$ of (X, Δ) such that \mathbf{M} descends to Y , and we put $T = f_*^{-1}S$. Note that $p\mathbf{M}_Y$ is Cartier since $p\mathbf{M}$ is b -Cartier. Hence, we may replace $p\mathbf{M}_Y$ by a linear equivalence class so that $\text{Supp } \mathbf{M}_Y$ does not contain T . We can write

$$p(K_Y + T + \Gamma + \mathbf{M}_Y) = p f^*(K_X + \Delta + \mathbf{M}_X),$$

and we have

$$\lfloor p(K_Y + T + \Gamma + \mathbf{M}_Y) \rfloor|_T \sim pK_T + \lfloor p\Gamma \rfloor|_T + p\mathbf{M}_Y|_T.$$

Thus, we obtain a natural morphism

$$f_* \mathcal{O}_Y(\lfloor p(K_Y + T + \Gamma + \mathbf{M}_Y) \rfloor) \longrightarrow \tau_* f_{T*} \mathcal{O}_T(pK_T + \lfloor p\Gamma \rfloor|_T + p\mathbf{M}_Y|_T), \tag{*}$$

where $f_T: T \rightarrow S^v$ and $\tau: S^v \rightarrow S \rightarrow X$ are the natural morphisms. With the relation

$$K_Y + T + \Gamma + \mathbf{M}_Y = f^*(K_X + \Delta + \mathbf{M}_X),$$

we have

$$f_* \mathcal{O}_Y(\lfloor p(K_Y + T + \Gamma + \mathbf{M}_Y) \rfloor) \simeq \mathcal{O}_X(\lfloor p(K_X + \Delta + \mathbf{M}_X) \rfloor) \tag{**}$$

By construction of divisorial adjunction for generalized pairs, we have

$$f_T^*(K_{S^v} + \Delta_{S^v} + \mathbf{N}_{S^v}) = K_T + \Gamma|_T + \mathbf{M}_Y|_T$$

as \mathbb{R} -divisors. Since $p\mathbf{M}_Y|_T$ is a Weil divisor, we see that

$$\lfloor f_T^*(p(K_{S^v} + \Delta_{S^v} + \mathbf{N}_{S^v})) \rfloor = pK_T + \lfloor p\Gamma \rfloor|_T + p\mathbf{M}_Y|_T.$$

From this fact, we see that

$$f_{T*} \mathcal{O}_T(pK_T + \lfloor p\Gamma \rfloor|_T + p\mathbf{M}_Y|_T) \simeq \mathcal{O}_{S^v}(\lfloor p(K_{S^v} + \Delta_{S^v} + \mathbf{N}_{S^v}) \rfloor). \tag{***}$$

By equations (*), (**) and (***), we can define the desired morphism.

Definition 2.10 (Models, cf. [25]). Let $(X, \Delta, \mathbf{M})/Z$ be a generalized lc pair, and let $(X', \Delta', \mathbf{M}')/Z$ be a generalized pair. Let $\phi: X \dashrightarrow X'$ be a birational map over Z .

We say that $(X', \Delta', \mathbf{M}')/Z$ is a *generalized log birational model* of $(X, \Delta, \mathbf{M})/Z$ if $\mathbf{M} = \mathbf{M}'$ by ϕ and $\Delta' = \phi_* \Delta + \sum_i E_i$, where E_i runs over ϕ^{-1} -exceptional prime divisors.

We say that a generalized log birational model $(X', \Delta', \mathbf{M}')/Z$ of $(X, \Delta, \mathbf{M})/Z$ is a *weak generalized log canonical model* (*weak generalized lc model*, for short) over Z if

- $K_{X'} + \Delta' + \mathbf{M}'_{X'}$ is nef over Z , and
- $a(D, X, \Delta + \mathbf{M}_X) \leq a(D, X', \Delta' + \mathbf{M}'_{X'})$ for every prime divisor D on X which is exceptional over X' .

We say that a weak generalized lc model $(X', \Delta', \mathbf{M}')/Z$ of $(X, \Delta, \mathbf{M})/Z$ is a *minimal model over Z* if the inequality on generalized log discrepancies is strict.

We say that a minimal model $(X', \Delta', \mathbf{M}')/Z$ of $(X, \Delta, \mathbf{M})/Z$ is a *good minimal model over Z* if $K_{X'} + \Delta' + \mathbf{M}'_{X'}$ is semiample over Z .

2.2. Abundant divisors

We define the invariant Iitaka dimension and the numerical dimension ([40]) for \mathbb{R} -Cartier divisors on normal projective varieties, then we define abundant divisors, log abundant divisors and log big divisors.

Definition 2.11 (Invariant Iitaka dimension). Let X be a normal projective variety, and let D be an \mathbb{R} -Cartier divisor on X . We define the *invariant Iitaka dimension* of D , denoted by $\kappa_l(X, D)$, as follows ([9, Definition 2.2.1]; see also [17, Definition 2.5.5]): If there is an \mathbb{R} -divisor $E \geq 0$ such that $D \sim_{\mathbb{R}} E$, set $\kappa_l(X, D) = \kappa(X, E)$. Here, the right-hand side is the usual Iitaka dimension of E . Otherwise, we set $\kappa_l(X, D) = -\infty$.

Let $X \rightarrow Z$ be a projective morphism from a normal variety to a variety and D an \mathbb{R} -Cartier divisor on X . Then the *relative invariant Iitaka dimension* of D , denoted by $\kappa_l(X/Z, D)$, is similarly defined: If there is an \mathbb{R} -divisor $E \geq 0$ such that $D \sim_{\mathbb{R}, Z} E$, then we set $\kappa_l(X/Z, D) = \kappa_l(F, D|_F)$, where F is a sufficiently general fiber of the Stein factorization of $X \rightarrow Z$, and otherwise we set $\kappa_l(X/Z, D) = -\infty$.

Definition 2.12 (Numerical dimension). Let X be a normal projective variety, and let D be an \mathbb{R} -Cartier divisor on X . We define the *numerical dimension* of D , denoted by $\kappa_{\sigma}(X, D)$, as follows ([40, V, 2.5 Definition]): For any Cartier divisor A on X , we set

$$\sigma(D; A) = \max \left\{ k \in \mathbb{Z}_{\geq 0} \mid \limsup_{m \rightarrow \infty} \frac{\dim H^0(X, \mathcal{O}_X(\lfloor mD \rfloor + A))}{m^k} > 0 \right\}$$

if $\dim H^0(X, \mathcal{O}_X(\lfloor mD \rfloor + A)) > 0$ for infinitely many $m \in \mathbb{Z}_{>0}$, and otherwise we set $\sigma(D; A) := -\infty$. Then, we define

$$\kappa_{\sigma}(X, D) := \max \{ \sigma(D; A) \mid A \text{ is a Cartier divisor on } X \}.$$

Let $X \rightarrow Z$ be a projective morphism from a normal variety to a variety, and let D be an \mathbb{R} -Cartier divisor on X . Then, the *relative numerical dimension* of D over Z is defined by $\kappa_{\sigma}(F, D|_F)$, where F is a sufficiently general fiber of the Stein factorization of $X \rightarrow Z$. Then $\kappa_{\sigma}(F, D|_F)$ does not depend on the choice of F , so the relative numerical dimension is well-defined. In this paper, we denote $\kappa_{\sigma}(F, D|_F)$ by $\kappa_{\sigma}(X/Z, D)$.

We refer to [31, Remark 2.14] (see also [40, V, 2.7 Proposition], [33, Remark 2.8]) for basic properties of the invariant Iitaka dimension and the numerical dimension.

Definition 2.13 (Abundant divisor, log abundant divisor, log big divisor). Let $X \rightarrow Z$ be a projective morphism from a normal variety to a variety, and let D be an \mathbb{R} -Cartier divisor on X . We say that D is *abundant over Z* if $\kappa_l(X/Z, D) = \kappa_{\sigma}(X/Z, D)$.

Let $X \rightarrow Z$ and D be as above, and let $(X, \Delta, \mathbf{M})/Z$ be a generalized lc pair. We say that D is *log abundant over Z with respect to $(X, \Delta, \mathbf{M})/Z$* if D is abundant over Z and for any generalized lc center S of $(X, \Delta, \mathbf{M})/Z$ with the normalization $S^{\nu} \rightarrow S$, the pullback $D|_{S^{\nu}}$ is abundant over Z .

We say that D is *log big over Z with respect to $(X, \Delta, \mathbf{M})/Z$* if D is big over Z and for any generalized lc center S of $(X, \Delta, \mathbf{M})/Z$ with the normalization $S^{\nu} \rightarrow S$, the pullback $D|_{S^{\nu}}$ is big over Z .

When Z is a point in the above definitions, we remove ‘over Z ’ in each terminology.

Lemma 2.14 (cf. [33, Lemma 2.11], [40], [18]). *Let (X, Δ, \mathbf{M}) be a generalized lc pair such that $K_X + \Delta + \mathbf{M}_X$ is abundant and there is an effective \mathbb{R} -divisor D on X such that $D \sim_{\mathbb{R}} K_X + \Delta + \mathbf{M}_X$. Let $X \dashrightarrow V$ be the Iitaka fibration associated to D . We take a log resolution $f: Y \rightarrow X$ of (X, Δ) such that*

\mathbf{M} descends to Y and the induced map $Y \dashrightarrow V$ is a morphism. Let (Y, Γ, \mathbf{M}) be a generalized lc pair such that

$$K_Y + \Gamma + \mathbf{M}_Y = f^*(K_X + \Delta + \mathbf{M}_X) + E$$

for some effective f -exceptional \mathbb{R} -divisor E . Then $\kappa_\sigma(Y/V, K_Y + \Gamma + \mathbf{M}_Y) = 0$.

Proof. The argument in [33, Proof of Lemma 2.11] works with no changes because we may apply the discussion in [18, Section 3]. □

Lemma 2.15. *Let (X, Δ) be a projective lc pair and $M = \sum_i \mu_i M_i$ a finite $\mathbb{R}_{>0}$ -linear combination of nef Cartier divisors M_i on X such that M is log big with respect to (X, Δ) and $\mu_i > 2 \cdot \dim X$ for all i . Then $K_X + \Delta + M$ is log big with respect to (X, Δ) .*

Proof. We need to prove that if S is X or an lc center of (X, Δ) with the normalization S^\vee , then $(K_X + \Delta + M)|_{S^\vee}$ is big. Suppose by contradiction that there is an lc center S of (X, Δ) or $S = X$ such that $(K_X + \Delta + M)|_{S^\vee}$ is not big. By replacing M with $(1 - t)M$ for some $0 < t \ll 1$, we may assume that $(K_X + \Delta + M)|_{S^\vee}$ is not pseudo-effective.

Taking a dlt blowup of (X, Δ) and applying Ambro’s canonical bundle formula as in [19, Corollary 3.2], we get a generalized lc pair $(S^\vee, \Delta_{S^\vee}, \mathbf{N})$ such that

$$K_{S^\vee} + \Delta_{S^\vee} + \mathbf{N}_{S^\vee} \sim_{\mathbb{R}} (K_X + \Delta)|_{S^\vee}.$$

Let (Y, Γ, \mathbf{N}) be a \mathbb{Q} -factorial generalized dlt model of $(S^\vee, \Delta_{S^\vee}, \mathbf{N})$, and let M_Y be the pullback of $M|_{S^\vee}$ to Y . Then M_Y is big and $K_Y + \Gamma + \mathbf{N}_Y + M_Y$ is not pseudo-effective.

By running a $(K_Y + \Gamma + \mathbf{N}_Y + M_Y)$ -MMP with scaling of an ample divisor, we get a birational contraction $\phi: Y \dashrightarrow Y'$ to a normal projective variety Y' which has the structure of a Mori fiber space $Y' \rightarrow V$ with respect to $\phi_*(K_Y + \Gamma + \mathbf{N}_Y + M_Y)$. By the length of extremal rays (cf. [25, Proposition 3.17]), the birational transform of M_Y is numerically trivial with respect to the extremal contraction in each step of the MMP. Thus, $\phi_* M_Y$ is big and $\phi_* M_Y \sim_{\mathbb{R}, V} 0$, a contradiction because $\dim Y' > \dim V$.

In this way, we see that $(K_X + \Delta + M)|_{S^\vee}$ is big, and therefore $K_X + \Delta + M$ is log big with respect to (X, Δ) . □

The following lemma plays a crucial role in Section 4 and Section 5.

Lemma 2.16. *Let $(X, \Delta, \mathbf{M})/Z$ be a generalized lc pair with a morphism $\pi: X \rightarrow Z$ such that*

- Δ is a \mathbb{Q} -divisor and \mathbf{M} is a \mathbb{Q} -b-Cartier \mathbb{Q} -b-divisor,
- $K_X + \Delta + \mathbf{M}_X$ is nef over Z and
- there is a log resolution $f: Y \rightarrow X$ of (X, Δ) such that
 - \mathbf{M} descends to Y , and
 - writing $K_Y + \Gamma + \mathbf{M}_Y = f^*(K_X + \Delta + \mathbf{M}_X) + E$, where $\Gamma \geq 0$ and $E \geq 0$ have no common components, then \mathbf{M}_Y is log big over Z with respect to (Y, Γ) .

Let S be a component of $[\Delta]$, let S^\vee be the normalization of S and let $(S^\vee, \Delta_{S^\vee}, \mathbf{N})/Z$ be a generalized lc pair with a morphism $\pi_{S^\vee}: S^\vee \rightarrow Z$ defined with divisorial adjunction for $(X, \Delta, \mathbf{M})/Z$ and S . Let p be a positive integer such that $p\Delta$ is a Weil divisor and $p\mathbf{M}_Y$ is Cartier. Then, the morphism

$$\pi_* \mathcal{O}_X(l p(K_X + \Delta + \mathbf{M}_X)) \longrightarrow \pi_{S^\vee *} \mathcal{O}_{S^\vee}([\!|lp(K_{S^\vee} + \Delta_{S^\vee} + \mathbf{N}_{S^\vee})|\!|])$$

induced by Definition 2.9 is surjective for every positive integer l .

Proof. The idea is very similar to [12]. We fix l .

We put

$$L = \mathbf{M}_Y + (lp - 1)f^*(K_X + \Delta + \mathbf{M}_X).$$

Since $K_X + \Delta + \mathbf{M}_X$ is nef over Z and \mathbf{M}_Y is nef and log big over Z with respect to (Y, Γ) , it follows that L is nef and log big over Z with respect to (Y, Γ) . We have

$$K_Y + \Gamma + L = lpf^*(K_X + \Delta + \mathbf{M}_X) + E. \tag{1}$$

Since the coefficients of $p\Delta$ are integers and $p\mathbf{M}_Y$ is Cartier, we can write

$$lpf^*(K_X + \Delta + \mathbf{M}_X) = \lfloor lpf^*(K_X + \Delta + \mathbf{M}_X) \rfloor + E' \tag{2}$$

with an effective f -exceptional \mathbb{Q} -divisor E' . Put $T = f_*^{-1}S$. By restricting sides of equation (2) to T and taking the round down, we have

$$\lfloor (lpf^*(K_X + \Delta + \mathbf{M}_X)) \rfloor|_T = \lfloor lpf^*(K_X + \Delta + \mathbf{M}_X) \rfloor|_T. \tag{3}$$

We define $\Gamma' := \Gamma - T$. By the definition, $(Y, T + \Gamma')$ is a log smooth lc pair and L is nef and log big over Z with respect to $(Y, T + \Gamma')$. The relations (1) and (2) show

$$K_Y + T + \Gamma' + L = \lfloor lpf^*(K_X + \Delta + \mathbf{M}_X) \rfloor + E' + E. \tag{4}$$

We define a Weil divisor F on Y so that

$$\text{coeff}_P(F) = \begin{cases} 0 & (\text{coeff}_P(\Gamma' - \{(E + E')\}) \geq 0) \\ 1 & (\text{coeff}_P(\Gamma' - \{(E + E')\}) < 0) \end{cases}$$

for every prime divisor P . We put

$$B = \Gamma' - \{(E + E')\} + F \quad \text{and} \quad G = \lfloor (E + E') \rfloor + F.$$

By the definition, we can check that B is a boundary divisor, $\lfloor B \rfloor \leq \lfloor \Gamma' \rfloor$, and G is an effective f -exceptional Weil divisor. Since $\lfloor B \rfloor \leq \lfloor \Gamma' \rfloor$, the divisor L is nef and log big over Z with respect to $(Y, T + B)$. Furthermore, the relation (4) implies

$$K_Y + T + B + L = \lfloor lpf^*(K_X + \Delta + \mathbf{M}_X) \rfloor + G. \tag{5}$$

We put $D = lpf^*(K_X + \Delta + \mathbf{M}_X)$. Since G is f -exceptional, we have

$$\mathcal{O}_X(lp(K_X + \Delta + \mathbf{M}_X)) \simeq f_*\mathcal{O}_Y(\lfloor D \rfloor + G).$$

By construction, (Y, B) is a log smooth lc pair and L is nef and log big over Z with respect to (Y, B) . Thus, the Kodaira type vanishing theorem [12, Lemma 1.5] implies

$$R^1(\pi \circ f)_*\mathcal{O}_Y(\lfloor D \rfloor + G - T) = R^1(\pi \circ f)_*\mathcal{O}_Y(K_Y + B + L) = 0.$$

From these facts, the morphism

$$(\pi \circ f)_*\mathcal{O}_Y(\lfloor D \rfloor + G) \longrightarrow (\pi_{S^v} \circ f_T)_*\mathcal{O}_T(\lfloor D \rfloor + G)|_T$$

is surjective, where $f_T: T \rightarrow S^v$ is the natural morphism. Since $\lfloor D \rfloor|_T = \lfloor D|_T \rfloor$ by equation (3), a natural morphism

$$\mathcal{O}_{S^v}(\lfloor lp(K_{S^v} + \Delta_{S^v} + \mathbf{N}_{S^v}) \rfloor) \longrightarrow f_{T*}\mathcal{O}_T(\lfloor D|_T \rfloor) \hookrightarrow f_{T*}\mathcal{O}_T(\lfloor D|_T \rfloor + G|_T)$$

is defined. From these facts, we obtain the diagram

$$\begin{array}{ccc}
 \pi_* f_* \mathcal{O}_Y([\!|D|\!] + G) & \xrightarrow{\quad} & \pi_{S^v*} f_{T*} \mathcal{O}_T([\!|D|\!]_T + G|_T) \\
 \uparrow \cong & & \uparrow \text{J} \\
 \pi_* \mathcal{O}_X(lp(K_X + \Delta + \mathbf{M}_X)) & \xrightarrow{\quad} & \pi_{S^v*} \mathcal{O}_{S^v}([\!|lp(K_{S^v} + \Delta_{S^v} + \mathbf{N}_{S^v})|\!])
 \end{array}$$

and therefore the morphism

$$\pi_* \mathcal{O}_X(lp(K_X + \Delta + \mathbf{M}_X)) \longrightarrow \pi_{S^v*} \mathcal{O}_{S^v}([\!|lp(K_{S^v} + \Delta_{S^v} + \mathbf{N}_{S^v})|\!])$$

is surjective. □

3. Minimal model theory for generalized pairs

The goal of this section is to prove Theorem 1.1. All the arguments in this section are very similar to those in [33], [29], [31] and [32].

The following results will be used without any mention.

Lemma 3.1. *Let $(X, \Delta, \mathbf{M})/Z$ be a generalized lc pair and $(X', \Delta', \mathbf{M})/Z$ a generalized lc pair with a birational morphism $X' \rightarrow X$ such that $(X', \Delta', \mathbf{M})/Z$ is a generalized log birational model of $(X, \Delta, \mathbf{M})/Z$ as in Definition 2.10. Then $(X, \Delta, \mathbf{M})/Z$ has a minimal model (resp. a good minimal model) if and only if $(X', \Delta', \mathbf{M})/Z$ has a minimal model (resp. a good minimal model).*

Lemma 3.2. *Let $(X, \Delta, \mathbf{M})/Z$ be a generalized lc pair and $(X, \Delta, \mathbf{M}) \dashrightarrow (X', \Delta', \mathbf{M})$ finite steps of a $(K_X + \Delta + \mathbf{M}_X)$ -MMP over Z . Then $(X', \Delta', \mathbf{M})/Z$ is a generalized lc pair, and $(X, \Delta, \mathbf{M})/Z$ has a minimal model (resp. a good minimal model) if and only if $(X', \Delta', \mathbf{M})/Z$ has a minimal model (resp. a good minimal model).*

Lemma 3.3 (cf. [25, Theorem 4.1]). *Let $(X, \Delta, \mathbf{M})/Z$ be a generalized lc pair such that Z is quasi-projective, $(X, 0)$ is a \mathbb{Q} -factorial klt pair and \mathbf{M} is a finite $\mathbb{R}_{>0}$ -linear combination of b -nef/ Z \mathbb{Q} - b -Cartier \mathbb{Q} - b -divisors. If $(X, \Delta, \mathbf{M})/Z$ has a minimal model, then any sequence of steps of a $(K_X + \Delta + \mathbf{M}_X)$ -MMP over Z with scaling of an ample divisor terminates with a minimal model.*

3.1. Auxiliary results

We collect results used in this section. Note that all results in this subsection are known in the case of usual pairs.

Lemma 3.4 (cf. [33, Lemma 2.16]). *Let $(X, \Delta, \mathbf{M})/Z$ be a generalized lc pair such that Z is quasi-projective and (X, B) is a dlt pair for some B . Let \mathcal{T} be an empty set or a finite set of exceptional prime divisors over X such that $0 < a(P, X, \Delta + \mathbf{M}_X) < 1$ for any $P \in \mathcal{T}$. Then there is a \mathbb{Q} -factorial variety \tilde{X} and a projective birational morphism $f: \tilde{X} \rightarrow X$ such that f -exceptional prime divisors are exactly elements of \mathcal{T} .*

Proof. Replacing B , we may assume that (X, B) is klt. We pick a real number $0 < t \ll 1$ so that the generalized klt pair $(X, (1 - t)\Delta + tB, (1 - t)\mathbf{M})/Z$ satisfies

$$0 < a(P, X, ((1 - t)\Delta + tB) + (1 - t)\mathbf{M}_X) < 1$$

for any $P \in \mathcal{T}$. Replacing $(X, \Delta, \mathbf{M})/Z$ with $(X, (1 - t)\Delta + tB, (1 - t)\mathbf{M})/Z$, we may assume that $(X, \Delta, \mathbf{M})/Z$ is generalized klt. By the perturbation of coefficients, we can find Δ' such that (X, Δ') is klt and $0 < a(P, X, \Delta') < 1$ for any $P \in \mathcal{T}$, where $a(\cdot, X, \Delta')$ is the log discrepancy. Then the lemma follows from [33, Lemma 2.16]. □

Lemma 3.5 (cf. [31, Lemma 2.6]). *Let (X, Δ, \mathbf{M}) and $(X', \Delta', \mathbf{M}')$ be generalized dlt pairs with a birational map $\phi: X \dashrightarrow X'$ by which $\mathbf{M} = \mathbf{M}'$. Let S and S' be generalized lc centers of (X, Δ, \mathbf{M}) and $(X', \Delta', \mathbf{M}')$, respectively, such that ϕ is an isomorphism near the generic point of S and the restriction $\phi|_S$ defines a birational map $\phi|_S: S \dashrightarrow S'$. Suppose that $K_X + \Delta + \mathbf{M}_X$ is pseudo-effective. Suppose in addition that*

- $a(D', X', \Delta' + \mathbf{M}'_{X'}) \leq a(D', X, \Delta + \mathbf{M}_X)$ for every prime divisor D' on X' , and
- $\sigma_P(K_X + \Delta + \mathbf{M}_X) = 0$ for every prime divisor P over X such that $c_X(P) \cap S \neq \emptyset$ and $a(P, X, \Delta + \mathbf{M}_X) < 1$, where $\sigma_P(\cdot)$ is as in Definition 2.3.

Let $(S, \Delta_S, \mathbf{N})$ and $(S', \Delta_{S'}, \mathbf{N}_{S'})$ be generalized dlt pairs which are constructed by applying divisorial adjunctions for generalized pairs to (X, Δ, \mathbf{M}) and $(X', \Delta', \mathbf{M}')$ repeatedly. Then

$$a(Q, S', \Delta_{S'} + \mathbf{N}_{S'}) \leq a(Q, S, \Delta_S + \mathbf{N}_S)$$

for all prime divisors Q on S' .

Proof. We closely follow [31, Proof of Lemma 2.6].

Taking an appropriate common log resolution $f: Y \rightarrow X$ and $f': Y \rightarrow X'$ of the birational map $(X, \Delta) \dashrightarrow (X', \Delta')$, we may find a subvariety T of Y which is birational to S and S' such that the induced morphisms $T \rightarrow S$ and $T \rightarrow S'$ form a common resolution of $\phi|_S$. Note that \mathbf{M} does not necessarily descend to Y . We may write

$$f^*(K_X + \Delta + \mathbf{M}_X) = f'^*(K_{X'} + \Delta' + \mathbf{M}'_{X'}) + G_+ - G_-$$

such that $G_+ \geq 0$ and $G_- \geq 0$ have no common components. Then G_+ is f' -exceptional by the first condition of Lemma 3.5. Since $K_X + \Delta + \mathbf{M}_X$ is pseudo-effective, we see that $K_{X'} + \Delta' + \mathbf{M}'_{X'}$ is pseudo-effective. We have $\text{Supp}G_+ \not\supset T$ and $\text{Supp}G_- \not\supset T$ since S and S' are generalized lc centers of (X, Δ, \mathbf{M}) and $(X', \Delta', \mathbf{M}')$, respectively.

We can write

$$G_+ = G_0 + G_1,$$

where all components of G_0 intersect T and $G_1|_T = 0$. Pick any component E of G_0 . Then $\text{ord}_E(G_-) = 0$. We have

$$\begin{aligned} & \sigma_E(K_X + \Delta + \mathbf{M}_X) \\ &= \sigma_E(f^*(K_X + \Delta + \mathbf{M}_X)) + \text{ord}_E(G_-) && (\text{ord}_E(G_-) = 0) \\ &\geq \sigma_E(f^*(K_X + \Delta + \mathbf{M}_X) + G_-) && ([31, Remark 2.3 (1)]) \\ &= \sigma_E(f'^*(K_{X'} + \Delta' + \mathbf{M}'_{X'}) + G_+) \\ &= \sigma_E(f'^*(K_{X'} + \Delta' + \mathbf{M}'_{X'})) + \text{ord}_E(G_+) && ([31, Remark 2.3 (3)]), \end{aligned}$$

therefore $\sigma_P(K_X + \Delta + \mathbf{M}_X) > 0$. By the second condition of Lemma 3.5, we see that $a(E, X, \Delta + \mathbf{M}_X) \geq 1$, hence we have

$$a(E, X', \Delta' + \mathbf{M}'_{X'}) > a(E, X, \Delta + \mathbf{M}_X) \geq 1.$$

By the same argument as in [31, Proof of Lemma 2.5], we see that $E|_T$ is exceptional over S' . Therefore, $G_0|_T$ is exceptional over S' . This implies that

$$\text{coeff}_Q(G_0|_T) = 0$$

for every prime divisor Q on S' . Since we have $G_1|_T = 0$ and

$$f|_T^*((K_X + \Delta + \mathbf{M}_X)|_S) - f'|_T^*((K_{X'} + \Delta' + \mathbf{M}'_{X'})|_{S'}) = (G_0 + G_1)|_T - G_-|_T,$$

the inequality

$$a(Q, S', \Delta_{S'} + \mathbf{N}_{S'}) - a(Q, S, \Delta_S + \mathbf{N}_S) \leq 0$$

holds for every prime divisor Q on S' . Therefore, Lemma 3.5 holds. □

Lemma 3.6 (cf. [22, Lemma 5.3], [31, Lemma 2.25]). *Let (X, Δ, \mathbf{M}) be a generalized lc pair such that \mathbf{M} is a finite $\mathbb{R}_{>0}$ -linear combination of b -nef \mathbb{Q} - b -Cartier \mathbb{Q} - b -divisors. Let (Y, Γ, \mathbf{M}) be a generalized lc pair with a projective birational morphism $f: Y \rightarrow X$. Suppose that $K_X + \Delta + \mathbf{M}_X$ is pseudo-effective and all prime divisors D on Y satisfy*

$$0 \leq a(D, Y, \Gamma + \mathbf{M}_Y) - a(D, X, \Delta + \mathbf{M}_X) \leq \sigma_D(K_X + \Delta + \mathbf{M}_X).$$

Then (X, Δ, \mathbf{M}) has a minimal model (resp. a good minimal model) if and only if (Y, Γ, \mathbf{M}) has a minimal model (resp. a good minimal model).

Proof. Let $g: W \rightarrow Y$ be a log resolution of (Y, Γ) such that $f \circ g: W \rightarrow X$ is a log resolution of (X, Δ) . Let $(W, \Delta_W, \mathbf{M})$ and $(W, \Gamma_W, \mathbf{M})$ be generalized log birational models of (X, Δ, \mathbf{M}) and (Y, Γ, \mathbf{M}) as in Definition 2.10, respectively. We may write

$$K_W + \Delta_W + \mathbf{M}_W = g^* f^*(K_X + \Delta + \mathbf{M}_X) + E_W$$

for some $E_W \geq 0$ which is exceptional over X .

By [31, Remark 2.3 (3)], all prime divisors P on W satisfy

$$\sigma_P(K_W + \Delta_W + \mathbf{M}_W) = \sigma_P(K_X + \Delta + \mathbf{M}_X) + \text{coeff}_P(E_W).$$

If P is not exceptional over X , then we have

$$\text{coeff}_P(E_W) = 0, \quad \text{coeff}_P(\Delta_W) = \text{coeff}_{f(g(P))}(\Delta), \quad \text{and} \quad \text{coeff}_P(\Gamma_W) = \text{coeff}_{g(P)}(\Gamma).$$

If P is exceptional over X but not exceptional over Y , then we have

$$\text{coeff}_P(E_W) = a(P, X, \Delta + \mathbf{M}_X), \quad \text{coeff}_P(\Delta_W) = 1, \quad \text{and} \quad \text{coeff}_P(\Gamma_W) = \text{coeff}_{g(P)}(\Gamma).$$

If P is exceptional over Y , then we have

$$\text{coeff}_P(E_W) = a(P, X, \Delta + \mathbf{M}_X), \quad \text{coeff}_P(\Delta_W) = 1, \quad \text{and} \quad \text{coeff}_P(\Gamma_W) = 1.$$

In any case, by simple computations using the hypothesis about the relations between generalized log discrepancies, we see that

$$0 \leq a(P, W, \Gamma_W + \mathbf{M}_W) - a(P, W, \Delta_W + \mathbf{M}_W) \leq \sigma_P(K_W + \Delta_W + \mathbf{M}_W).$$

Thus, we see that $(W, \Delta_W, \mathbf{M})$ and $(W, \Gamma_W, \mathbf{M})$ satisfy the hypothesis of Lemma 3.6. By Lemma 3.1, we may replace (X, Δ, \mathbf{M}) and (Y, Γ, \mathbf{M}) with $(W, \Delta_W, \mathbf{M})$ and $(W, \Gamma_W, \mathbf{M})$, respectively. By the replacement, we may assume that $X = Y$ and $(X, 0)$ is a \mathbb{Q} -factorial klt pair.

We may carry out [22, Proof of Lemma 5.3] in the framework of generalized pairs. It is because we may use the argument of the length of extremal rays [25, Proposition 3.17] and the result of the termination of MMP [25, Theorem 4.1]. By the same argument as in [22, Proof of Lemma 5.3], we see that Lemma 3.6 holds. □

Lemma 3.7 (cf. [29, Lemma 2.15]). *Let (X, Δ, \mathbf{M}) be a generalized lc pair such that \mathbf{M} is a finite $\mathbb{R}_{>0}$ -linear combination of b-nef \mathbb{Q} -b-Cartier \mathbb{Q} -b-divisors. Let (Y, Γ, \mathbf{M}) be a generalized lc pair with a projective birational morphism $f: Y \rightarrow X$. Suppose that we can write*

$$K_Y + \Gamma + \mathbf{M}_Y = f^*(K_X + \Delta + \mathbf{M}_X) + E$$

such that E is effective and f -exceptional. Then (X, Δ, \mathbf{M}) has a minimal model (resp. a good minimal model) if and only if (Y, Γ, \mathbf{M}) has a minimal model (resp. a good minimal model).

Proof. Replacing (Y, Γ, \mathbf{M}) by a \mathbb{Q} -factorial generalized dlt model, we may assume that (Y, Γ, \mathbf{M}) is a \mathbb{Q} -factorial generalized dlt pair. Running a $(K_Y + \Gamma + \mathbf{M}_Y)$ -MMP over X and replacing (Y, Γ, \mathbf{M}) by a crepant generalized dlt model of (X, Δ, \mathbf{M}) , we may assume $E = 0$. Then Lemma 3.7 follows from Lemma 3.6. □

Lemma 3.8 (cf. [31, Lemma 2.22]). *Let $(X, \Delta, \mathbf{M})/Z$ be a generalized lc pair such that Z is quasi-projective. Let S be a subvariety of X . We denote the morphism $X \rightarrow Z$ by π . Let*

$$(X, \Delta, \mathbf{M}) =: (X_0, \Delta_0, \mathbf{M}) \dashrightarrow (X_1, \Delta_1, \mathbf{M}) \dashrightarrow \dots \dashrightarrow (X_i, \Delta_i, \mathbf{M}) \dashrightarrow \dots$$

be a sequence of steps of a $(K_X + \Delta + \mathbf{M}_X)$ -MMP over Z with scaling of some \mathbb{R} -divisor $A \geq 0$. We define

$$\lambda_i = \inf \{ \mu \in \mathbb{R}_{\geq 0} \mid K_{X_i} + \Delta_i + \mathbf{M}_{X_i} + \mu A_i \text{ is nef over } Z \}.$$

Suppose that each step of the $(K_X + \Delta + \mathbf{M}_X)$ -MMP is an isomorphism on a neighborhood of S and $\lim_{i \rightarrow \infty} \lambda_i = 0$. Then, for any π -ample \mathbb{R} -divisor H on X and any closed point $x \in S$, there exists $E \geq 0$ such that $E \sim_{\mathbb{R}, Z} K_X + \Delta + \mathbf{M}_X + H$ and $\text{Supp} E \not\ni x$. In particular, if Z is a point, then $\sigma_P(K_X + \Delta + \mathbf{M}_X) = 0$ for every prime divisor P over X such that $c_X(P)$ intersects S .

Proof. The argument in [31, Proof of Lemma 2.22] works with no changes. Note that [31, Proof of Lemma 2.22] does not use any result about the abundance conjecture or the existence of flips for lc pairs. □

Lemma 3.9 (cf. [31, Lemma 2.26]). *Let (X, Δ, \mathbf{M}) and $(X', \Delta', \mathbf{M}')$ be generalized lc pairs with a birational map $X \dashrightarrow X'$ such that \mathbf{M} is a finite $\mathbb{R}_{>0}$ -linear combination of b-nef \mathbb{Q} -b-Cartier \mathbb{Q} -b-divisors and $\mathbf{M} = \mathbf{M}'$ by $X \dashrightarrow X'$. Suppose in addition that*

- $a(P, X, \Delta + \mathbf{M}_X) \leq a(P, X', \Delta' + \mathbf{M}'_{X'})$ for all prime divisors P on X , and
- $a(P', X', \Delta' + \mathbf{M}'_{X'}) \leq a(P', X, \Delta + \mathbf{M}_X)$ for all prime divisors P' on X' .

Then $K_X + \Delta + \mathbf{M}_X$ is abundant if and only if $K_{X'} + \Delta' + \mathbf{M}'_{X'}$ is abundant. Furthermore, (X, Δ, \mathbf{M}) has a minimal model (resp. a good minimal model) if and only if $(X', \Delta', \mathbf{M}')$ has a minimal model (resp. a good minimal model).

Proof. We closely follow [31, Proof of Lemma 2.26]. Let $f: Y \rightarrow X$ and $f': Y \rightarrow X'$ be a common log resolution of $(X, \Delta) \dashrightarrow (X', \Delta')$ such that \mathbf{M} descends to Y . We define an \mathbb{R} -divisor Γ on Y by

$$\Gamma := - \sum_D \min\{a(D, X, \Delta + \mathbf{M}_X) - 1, a(D, X', \Delta' + \mathbf{M}'_{X'}) - 1, 0\} D,$$

where D runs over prime divisors on Y . Then Γ is an effective snc \mathbb{R} -divisor, (Y, Γ, \mathbf{M}) is generalized lc and there exist an f -exceptional \mathbb{R} -divisor $E \geq 0$ and an f' -exceptional \mathbb{R} -divisor $E' \geq 0$ such that

$$E + f^*(K_X + \Delta + \mathbf{M}_X) = K_Y + \Gamma + \mathbf{M}_Y = f'^*(K_{X'} + \Delta' + \mathbf{M}'_{X'}) + E'.$$

Lemma 3.9 follows from the relation and Lemma 3.7. □

Lemma 3.10 (cf. [29, Proposition 3.3]). *Let (X, Δ, \mathbf{M}) be a generalized lc pair such that \mathbf{M} is a finite $\mathbb{R}_{>0}$ -linear combination of b -nef \mathbb{Q} - b -Cartier \mathbb{Q} - b -divisors. Let $\pi : X \rightarrow Z$ be a contraction to a normal projective variety Z . Suppose that*

- $\kappa_t(X/Z, K_X + \Delta + \mathbf{M}_X) = \kappa_\sigma(X/Z, K_X + \Delta + \mathbf{M}_X) = 0$,
- *any generalized lc center of (X, Δ, \mathbf{M}) dominates Z and*
- $\kappa_\sigma(X, K_X + \Delta + \mathbf{M}_X) = \dim Z$.

Then (X, Δ, \mathbf{M}) has a good minimal model.

Proof. The argument in [29, Proof of Proposition 3.3] works with no changes because we can use the generalized canonical bundle formula [26] instead of the canonical bundle formula [19]. So we only outline the proof.

By taking a log resolution of (X, Δ) and applying weak semistable reduction ([2]), we may assume that $(X, 0)$ is \mathbb{Q} -factorial klt and all fibers of π have the same dimensions.

We run a $(K_X + \Delta + \mathbf{M}_X)$ -MMP over Z with scaling of an ample divisor. Applying the negativity lemma for very exceptional divisors ([3, Section 3]) and replacing (X, Δ, \mathbf{M}) , we may assume $K_X + \Delta + \mathbf{M}_X \sim_{\mathbb{R}, Z} 0$ (but we lose the property of being equidimensional of $X \rightarrow Z$). We used the first condition of Lemma 3.10 for the reduction.

By generalized canonical bundle formula [26, Theorem 1.2] and the second condition of Lemma 3.10, there is a generalized klt pair $(Z, \Delta_Z, \mathbf{N})$ such that

$$K_X + \Delta + \mathbf{M}_X \sim_{\mathbb{R}} \pi^*(K_Z + \Delta_Z + \mathbf{N}_Z).$$

By the third condition of Lemma 3.10, we see that $K_Z + \Delta_Z + \mathbf{N}_Z$ is big, so $(Z, \Delta_Z, \mathbf{N})$ has a good minimal model $(Z', \Delta_{Z'}, \mathbf{N})$ by [6]. Since $(Z, \Delta_Z, \mathbf{N})$ is generalized klt, the birational map $Z \dashrightarrow Z'$ is a birational contraction. Hence, we can find an open subset $U' \subset Z'$ such that $\text{codim}_{Z'}(Z' \setminus U') \geq 2$ and $Z' \dashrightarrow Z$ is an isomorphism on U' .

Let $f : Y \rightarrow X$ be a log resolution of (X, Δ) such that \mathbf{M} descends to Y and the map $\pi_Y : Y \dashrightarrow Z'$ is a morphism. Let (Y, Γ, \mathbf{M}) be a generalized log birational model of (X, Δ, \mathbf{M}) as in Definition 2.10. Then

$$K_Y + \Gamma + \mathbf{M}_Y \sim_{\mathbb{R}} \pi_Y^*(K_{Z'} + \Delta_{Z'} + \mathbf{N}_{Z'}) + E + F$$

for some $E \geq 0$ and $F \geq 0$ such that E is exceptional over X and $\pi_Y(\text{Supp} F) \subset Z' \setminus U'$. Then (Y, Γ, \mathbf{M}) has a good minimal model over U' .

Running a $(K_Y + \Gamma + \mathbf{M}_Y)$ -MMP over Z' with scaling of an ample divisor, we get a birational contraction $\phi : Y \dashrightarrow Y'$ over Z' with the morphism $\pi' : Y' \rightarrow Z'$ such that

$$\phi_*(K_Y + \Gamma + \mathbf{M}_Y) \sim_{\mathbb{R}} \pi'^*(K_{Z'} + \Delta_{Z'} + \mathbf{N}_{Z'}) + \phi_*E + \phi_*F$$

is the limit of movable divisors over Z' and $\phi_*E|_{\pi'^{-1}(U')} = 0$. Since $\text{codim}_{Z'}(Z' \setminus U') \geq 2$, applying the negativity lemma for very exceptional divisors ([3, Section 3]) to $\phi_*E + \phi_*F$, we have $\phi_*E + \phi_*F = 0$. Thus, we have

$$\phi_*(K_Y + \Gamma + \mathbf{M}_Y) \sim_{\mathbb{R}} \pi'^*(K_{Z'} + \Delta_{Z'} + \mathbf{N}_{Z'}),$$

and the right-hand side is semiample. From the fact, we see that (Y, Γ, \mathbf{M}) has a good minimal model. So (X, Δ, \mathbf{M}) has a good minimal model. □

3.2. Generalized abundance

In this subsection, we study the property of being log abundant for special generalized lc pairs under MMP.

Theorem 3.11. *Let (X, Δ) be a projective lc pair, and let $\pi: X \rightarrow Z$ be a projective surjective morphism to a normal projective variety Z . Suppose that there is an effective \mathbb{R} -Cartier divisor C on X such that*

- *the pair $(X, \Delta + tC)$ is lc for some $t > 0$, and*
- *$K_X + \Delta + C \sim_{\mathbb{R}, Z} 0$.*

*Let A_Z be a big and semiample \mathbb{R} -divisor on Z such that the pullback of A_Z to $\pi(S)^\nu$ is big for any lc center S of (X, Δ) , where $\pi(S)^\nu$ is the normalization of $\pi(S)$. Then, $K_X + \Delta + \pi^*A_Z$ is abundant.*

We will use Lemma 3.12 below to prove Theorem 3.11.

Lemma 3.12. *Assume Theorem 3.11 for all projective lc pairs of dimension at most $n - 1$. Let (X, Δ) be a projective lc pair, let $\pi: X \rightarrow Z$ be a morphism and let C and A_Z be \mathbb{R} -Cartier divisors as in Theorem 3.11 such that $\dim X \leq n - 1$. Let $f: Y \rightarrow X$ be a log resolution of (X, Δ) , and let $\Gamma \geq 0$ be an \mathbb{R} -divisor on Y such that (Y, Γ) is an lc pair and the effective part of the divisor $K_Y + \Gamma - f^*(K_X + \Delta)$ is f -exceptional. Then, $K_Y + \Gamma + f^*\pi^*A_Z$ is abundant.*

Proof. The argument in [33, Proof of Lemma 5.2] or [32, Proof of Lemma 3.6] works with no changes. □

Proof of Theorem 3.11. The argument of [33, Proof of Theorem 5.4] works with no changes. We only outline the proof. We prove Theorem 3.11 by induction on $\dim X$.

We put $A = \pi^*A_Z$. We may assume that $K_X + \Delta + A$ is pseudo-effective and π is a contraction. By replacing A with a general one, we may also assume that $(X, \Delta + A)$ is lc, Δ and A have no common components and all lc centers of $(X, \Delta + A)$ are lc centers of (X, Δ) . By taking a dlt blowup of $(X, \Delta + A)$, we may assume that $(X, \Delta + A)$ is \mathbb{Q} -factorial dlt.

If $(X, \Delta + A)$ is a klt pair, then we follow [33, Proof of Lemma 5.3], and we see that $K_X + \Delta + A$ is abundant. More generally, if $K_X + \Delta - \epsilon[\Delta] + A$ is pseudo-effective for some $\epsilon > 0$, then the argument in [33, Step 1 in the proof of Theorem 5.4] enables us to reduce to the klt case, hence we see that $K_X + \Delta + A$ is abundant. From these facts, we may assume that $K_X + \Delta - \epsilon[\Delta] + A$ is not pseudo-effective for every $\epsilon > 0$. Then there is a component S of $[\Delta]$ such that $K_X + \Delta - \epsilon S + A$ is not pseudo-effective for every $\epsilon > 0$.

For all $\epsilon' > 0$, we run a $(K_X + \Delta - \epsilon'S + A)$ -MMP and get a birational contraction $X \dashrightarrow X'$ to a Mori fiber space $X' \rightarrow \bar{Z}$. Let Δ', S' and A' be the birational transforms of Δ, S and A on X' , respectively. By taking $\epsilon' > 0$ sufficiently small with the aid of the ACC for lc thresholds ([21, Theorem 1.1]) and the ACC for numerically trivial pairs ([21, Theorem 1.5]), we see that $(X', \Delta' + A')$ is lc and we have the relation $K_{X'} + \Delta' + A' \sim_{\mathbb{R}, \bar{Z}} 0$.

Let $\phi: Y \rightarrow X$ and $Y \rightarrow X'$ be a common log resolution of the birational map $(X, \Delta) \dashrightarrow (X', \Delta')$. Putting $A_Y = \phi^*A$, we can write

$$K_Y + \Gamma + A_Y = \phi^*(K_X + \Delta + A) + E, \tag{†}$$

where $\Gamma \geq 0$ and $E \geq 0$ have no common components. We run a $(K_Y + \Gamma + A_Y)$ -MMP over \bar{Z} with scaling of an ample divisor. As in [33, Step 3 in the proof of Theorem 5.4], an appropriate $(K_Y + \Gamma + A_Y)$ -MMP terminates with a good minimal model $(Y', \Gamma' + A_{Y'})$ over \bar{Z} . Let $Y' \rightarrow Z'$ be the contraction over \bar{Z} induced by $K_{Y'} + \Gamma' + A_{Y'}$. Then the induced morphism $Z' \rightarrow \bar{Z}$ is birational. We have the following diagram.

$$\begin{array}{ccc} Y & \dashrightarrow & Y' \\ \phi \downarrow & & \downarrow \\ X & & Z' \\ \pi \downarrow & & \\ Z & & \end{array}$$

We put $T = \phi_*^{-1}S$, and let T' be the birational transform of T on Y' . By construction, T' dominates Z' . We define projective dlt pairs

$$(S, \Delta_S), \quad (T, \Gamma_T), \quad \text{and} \quad (T', \Gamma_{T'})$$

with the divisorial adjunctions $K_S + \Delta_S = (K_X + \Delta)|_S$, $K_T + \Gamma_T = (K_Y + \Gamma)|_T$ and $K_{T'} + \Gamma_{T'} = (K_{Y'} + \Gamma')|_{T'}$, respectively. We put

$$A_S = A|_S, \quad A_T = A_Y|_T, \quad \text{and} \quad A_{T'} = A_{Y'}|_{T'}.$$

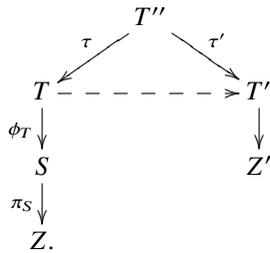
By construction, it is sufficient to prove that $K_{Y'} + \Gamma' + A_{Y'}$ is abundant. Since we have $K_{Y'} + \Gamma' + A_{Y'} \sim_{\mathbb{R}, Z'} 0$ and T' dominates Z' , it is sufficient to prove that

$$K_{T'} + \Gamma_{T'} + A_{T'} \sim_{\mathbb{R}} (K_{Y'} + \Gamma' + A_{Y'})|_{T'}$$

is abundant. We take a common log resolution $\tau: T'' \rightarrow T$ and $\tau': T'' \rightarrow T'$ of the birational map $(T, \Gamma_T) \dashrightarrow (T', \Gamma_{T'})$. Replacing A with a general member of $|A|_{\mathbb{R}}$, we may assume that

- $A_T \geq 0$ and $A_{T'} \geq 0$,
- $\tau^* A_T \leq \tau'^{-1} A_{T'}$ (see [33, Lemma 2.1]) and
- $\tau': T'' \rightarrow T'$ is a log resolution of $(T', \Gamma_{T'} + A_{T'})$ (see [33, Lemma 2.2]).

Let $\pi_S: S \rightarrow Z$ be the restriction of $\pi: X \rightarrow Z$ to S , and let $\phi_T: T \rightarrow S$ be the birational morphism induced by $\phi: Y \rightarrow X$. Now we have the following diagram.



By restricting equation (†) to T , we get

$$K_T + \Gamma_T + A_T = \phi_T^*(K_S + \Delta_S + A_S) + E|_T$$

such that Γ_T and $E|_T$ are effective. By [33, Lemma 2.4], we see that $\Gamma_T + A_T$ and $E|_T$ have no common components and $E|_T$ is ϕ_T -exceptional. We put $A_{T''} = \tau^* A_T$. Then $A_{T''} \leq \tau'^{-1} A_{T'}$. We have

$$K_{T''} + \Psi_{T''} + A_{T''} = \tau'^*(K_{T'} + \Gamma_{T'} + A_{T'}) + E_{T''}$$

for some $\Psi_{T''} \geq 0$ and $E_{T''} \geq 0$ such that $(T'', \Psi_{T''} + A_{T''})$ is a log smooth lc pair and $\Psi_{T''} + A_{T''}$ and $E_{T''}$ have no common components.

We may write

$$K_{T''} + \Psi_{T''} + A_{T''} = \tau^* \phi_T^*(K_S + \Delta_S + A_S) + M - N,$$

with $M \geq 0$ and $N \geq 0$ having no common components. Pick any component Q of M . If Q is not exceptional over S , then we have $a(Q, T'', \Psi_{T''} + A_{T''}) < a(Q, S, \Delta_S + A_S) \leq 1$. We apply [33, Step 6

in the proof of Theorem 5.4], and we get the following contradiction

$$\begin{aligned}
 a(Q, S, \Delta_S + A_S) &= \min\{1, a(Q, S, \Delta_S + A_S)\} && (a(Q, S, \Delta_S + A_S) \leq 1) \\
 &= \min\{1, a(Q, T, \Gamma_T + A_T)\} && (E|_T \text{ is } \phi_T\text{-exceptional}) \\
 &\leq \min\{1, a(Q, T', \Gamma_{T'} + A_{T'})\} && ([13, Lemma 4.2.10]) \\
 &= a(Q, T'', \Psi_{T''} + A_{T''}) && ([33, Lemma 2.3]) \\
 &< a(Q, S, \Delta_S + A_S).
 \end{aligned}$$

From this, we see that M is exceptional over S .

By applying the induction hypothesis of Theorem 3.11 to $(S, \Delta_S) \rightarrow \pi(S)^\vee$, $C|_S$ and $A_Z|_{\pi(S)^\vee}$, we see that $K_S + \Delta_S + A_S$ is abundant. By applying Lemma 3.12 to $(S, \Delta_S) \rightarrow \pi(S)^\vee$ and $(T'', \Psi_{T''}) \rightarrow S$, we see that $K_{T''} + \Psi_{T''} + A_{T''}$ is abundant. Then $K_{T'} + \Gamma_{T'} + A_{T'}$ is abundant, hence $K_X + \Delta + A$ is abundant. □

Corollary 3.13 (cf. [32, Theorem 4.1]). *Let (X, Δ) be a projective lc pair and $M \geq 0$ a semiample \mathbb{R} -divisor on X such that M is log big with respect to (X, Δ) and $(X, \Delta + M)$ is an lc pair whose lc centers coincide with those of (X, Δ) . Then, for any sequence of steps of a $(K_X + \Delta + M)$ -MMP starting the lc pair $(X, \Delta + M)$*

$$(X, \Delta + M) \dashrightarrow (X', \Delta' + M'),$$

the divisor $K_{X'} + \Delta' + M'$ is log abundant with respect to $(X', \Delta' + M')$.

Proof. We can apply Lemma 3.12, and the argument in [32, Proof of Theorem 4.1] works with no changes. □

3.3. Minimal model program

In this subsection, we study the termination of MMP for generalized lc pairs that preserves the property of being log abundant.

Theorem 3.14 (cf. [31, Theorem 3.4]). *Let (X, Δ, \mathbf{M}) be a generalized dlt pair such that \mathbf{M} is a finite $\mathbb{R}_{>0}$ -linear combination of b -nef \mathbb{Q} - b -Cartier \mathbb{Q} - b -divisors. Suppose that*

- $K_X + \Delta + \mathbf{M}_X$ is pseudo-effective and abundant,
- for any generalized lc center S of (X, Δ, \mathbf{M}) , the restriction $(K_X + \Delta + \mathbf{M}_X)|_S$ is nef, and
- for any prime divisor P over X such that $c_X(P)$ intersects a generalized lc center of (X, Δ, \mathbf{M}) and $a(P, X, \Delta + \mathbf{M}_X) < 1$, we have $\sigma_P(K_X + \Delta + \mathbf{M}_X) = 0$.

Then (X, Δ, \mathbf{M}) has a minimal model.

Proof. We closely follow [31, Subsection 3.1]. We prove Theorem 3.14 in several steps.

Step 1. In this step, we replace (X, Δ, \mathbf{M}) with a crepant \mathbb{Q} -factorial generalized dlt model which has good properties. We follow [31, Proof of Proposition 3.2].

Since $K_X + \Delta + \mathbf{M}_X$ is pseudo-effective and abundant, there is an effective \mathbb{R} -divisor D on X such that $K_X + \Delta + \mathbf{M}_X \sim_{\mathbb{R}} D$. We take the Iitaka fibration $X \dashrightarrow V$ associated to D . Then we have

$$\dim V = \kappa_\sigma(X, K_X + \Delta + \mathbf{M}_X).$$

We take a log resolution $\bar{f}: \bar{X} \rightarrow X$ of (X, Δ) such that \mathbf{M} descends to \bar{X} and the induced map $\bar{X} \dashrightarrow V$ is a morphism. Then we have

$$K_{\bar{X}} + \bar{\Delta} + \mathbf{M}_{\bar{X}} = \bar{f}^*(K_X + \Delta + \mathbf{M}_X) + \bar{E}$$

for some $\bar{\Delta} \geq 0$ and $\bar{E} \geq 0$ which have no common components. By construction and Lemma 2.14, we have

$$(i) \quad \kappa_\sigma(\bar{X}, K_{\bar{X}} + \bar{\Delta} + \mathbf{M}_{\bar{X}}) = \dim V \text{ and } \kappa_\sigma(\bar{X}/V, K_{\bar{X}} + \bar{\Delta} + \mathbf{M}_{\bar{X}}) = 0.$$

Moreover, $K_{\bar{X}} + \bar{\Delta} + \mathbf{M}_{\bar{X}}$ is \mathbb{R} -linearly equivalent to the sum of an effective \mathbb{R} -divisor and the pullback of an ample divisor on V . Therefore, we can find an effective \mathbb{R} -divisor

$$\bar{D} \sim_{\mathbb{R}} K_{\bar{X}} + \bar{\Delta} + \mathbf{M}_{\bar{X}}$$

such that $\text{Supp} \bar{D}$ contains all generalized lc centers of $(\bar{X}, \bar{\Delta}, \mathbf{M})$ which are vertical over V . By applying [28, Lemma 2.10] to the morphism $(\bar{X}, \bar{\Delta}) \rightarrow V$ and replacing $(\bar{X}, \bar{\Delta})$, we may assume that

$$(ii) \quad \bar{\Delta} = \bar{\Delta}' + \bar{\Delta}'' \text{ such that } \bar{\Delta}' \text{ is effective, } \bar{\Delta}'' = 0 \text{ or } \bar{\Delta}'' \text{ is reduced and vertical over } V, \text{ and all generalized lc centers of } (\bar{X}, \bar{\Delta} - s\bar{\Delta}'', \mathbf{M}) \text{ dominate } V \text{ for all } s \in (0, 1].$$

By taking a log resolution of $(\bar{X}, \bar{\Delta} + \bar{D})$ and replacing $(\bar{X}, \bar{\Delta})$ and \bar{D} , we may assume that $(\bar{X}, \bar{\Delta} + \bar{D})$ is log smooth. Note that (ii) is preserved after this replacement. Since $\text{Supp} \bar{\Delta}''$ is a union of generalized lc centers of $(\bar{X}, \bar{\Delta}, \mathbf{M})$ which are vertical over V , we see that $\text{Supp} \bar{D} \supset \text{Supp} \bar{\Delta}''$. By decomposing \bar{D} appropriately, we obtain effective \mathbb{R} -divisors \bar{G} and \bar{H} such that

$$(iii) \quad K_{\bar{X}} + \bar{\Delta} + \mathbf{M}_{\bar{X}} \sim_{\mathbb{R}} \bar{G} + \bar{H},$$

$$(iv) \quad \text{Supp} \bar{\Delta}'' \subset \text{Supp} \bar{G} \subset \text{Supp} [\bar{\Delta}] \text{ and}$$

$$(v) \quad \text{no component of } \bar{H} \text{ is a component of } [\bar{\Delta}] \text{ and } (\bar{X}, \bar{\Delta} + \bar{H}) \text{ is log smooth.}$$

We fix a real number $t_0 \in (0, 1)$ such that $\bar{\Delta} - t_0 \bar{G} \geq 0$. We pick $t \in (0, t_0]$, and we consider $(\bar{X}, \bar{\Delta} - t\bar{G}, \mathbf{M})$. Then the morphism

$$(\bar{X}, \bar{\Delta} - t\bar{G}, \mathbf{M}) \rightarrow V$$

satisfies the following three conditions.

- Any generalized lc center of $(\bar{X}, \bar{\Delta} - t\bar{G}, \mathbf{M})$ dominates V ,
- $\kappa_\sigma(\bar{X}, K_{\bar{X}} + \bar{\Delta} - t\bar{G} + \mathbf{M}_{\bar{X}}) = \dim V$ and
- $\kappa_t(\bar{X}/V, K_{\bar{X}} + \bar{\Delta} - t\bar{G} + \mathbf{M}_{\bar{X}}) = \kappa_\sigma(\bar{X}/V, K_{\bar{X}} + \bar{\Delta} - t\bar{G} + \mathbf{M}_{\bar{X}}) = 0$.

Indeed, the first condition follows from (iv) and (ii), and the other two conditions follow from (iii), (i) and [33, Remark 2.8 (1)]. By Lemma 3.10, we see that $(\bar{X}, \bar{\Delta} - t\bar{G}, \mathbf{M})$ has a good minimal model for all $t \in (0, t_0]$.

Running a $(K_{\bar{X}} + \bar{\Delta} + \mathbf{M}_{\bar{X}})$ -MMP over X , we get a crepant generalized dlt model

$$\tilde{f}: (\tilde{X}, \tilde{\Delta}, \mathbf{M}) \rightarrow (X, \Delta, \mathbf{M}).$$

Let \tilde{G} (resp. \tilde{H}) be the birational transform of \bar{G} (resp. \bar{H}) on \tilde{X} . Then

$$K_{\tilde{X}} + \tilde{\Delta} + \mathbf{M}_{\tilde{X}} \sim_{\mathbb{R}} \tilde{G} + \tilde{H}$$

by (iii). By (v) and replacing t_0 , we may assume that $(\tilde{X}, \tilde{\Delta} + t_0 \tilde{H}, \mathbf{M})$ is a \mathbb{Q} -factorial generalized dlt pair, and we may further assume that $\tilde{X} \dashrightarrow \tilde{X}$ is a sequence of steps of a $(K_{\tilde{X}} + \tilde{\Delta} + t\tilde{H} + \mathbf{M}_{\tilde{X}})$ -MMP and a sequence of steps of a $(K_{\tilde{X}} + \tilde{\Delta} - t\tilde{G} + \mathbf{M}_{\tilde{X}})$ -MMP for all $t \in (0, t_0]$. Then it follows that $(\tilde{X}, \tilde{\Delta} + t_0 \tilde{H}, \mathbf{M})$ is a \mathbb{Q} -factorial generalized dlt pair, and $(\tilde{X}, \tilde{\Delta} - t\tilde{G}, \mathbf{M})$ has a good minimal model for all $t \in (0, t_0]$ since $(\bar{X}, \bar{\Delta} - t\bar{G}, \mathbf{M})$ has a good minimal model.

Applying [31, Lemma 2.4] to $K_{\tilde{X}} + \tilde{\Delta} + \mathbf{M}_{\tilde{X}}$ and \tilde{H} and replacing t_0 , we may assume that

$$\text{Supp} N_\sigma(K_{\tilde{X}} + \tilde{\Delta} + t\tilde{H} + \mathbf{M}_{\tilde{X}})$$

is independent of the choice of $t \in (0, t_0]$.

Since $K_{\tilde{X}} + \tilde{\Delta} + \mathbf{M}_{\tilde{X}} = \tilde{f}^*(K_X + \Delta + \mathbf{M}_X)$, it is easy to check that $(\tilde{X}, \tilde{\Delta}, \mathbf{M})$ satisfies all the conditions of Theorem 3.14. Replacing (X, Δ, \mathbf{M}) by $(\tilde{X}, \tilde{\Delta}, \mathbf{M})$, we may assume that X is \mathbb{Q} -factorial and there exist effective \mathbb{R} -divisors G and H on X such that

- $K_X + \Delta + \mathbf{M}_X \sim_{\mathbb{R}} G + H$,
- $\text{Supp}G \subset \text{Supp}[\Delta]$ and
- there exists a real number $t_0 > 0$ such that the following properties hold for all $t \in (0, t_0]$:
 - $(X, \Delta + tH, \mathbf{M})$ is generalized dlt and $\text{Supp}N_{\sigma}(K_X + \Delta + tH + \mathbf{M}_X)$ does not depend on t , and
 - $(X, \Delta - tG, \mathbf{M})$ has a good minimal model.

Step 2. We follow [31, Proof of Proposition 3.3].

Pick any real number $t \in (0, t_0]$. Since $(X, \Delta - \frac{t}{1+t}G, \mathbf{M})$ has a good minimal model and

$$K_X + \Delta + tH + \mathbf{M}_X \sim_{\mathbb{R}} (1+t) \left(K_X + \Delta - \frac{t}{1+t}G + \mathbf{M}_X \right),$$

there is a sequence of steps of a $(K_X + \Delta + tH + \mathbf{M}_X)$ -MMP to a good minimal model

$$\phi_t : (X, \Delta + tH, \mathbf{M}) \dashrightarrow (X_t, \Delta_t + tH_t, \mathbf{M}).$$

Because $\text{Supp}N_{\sigma}(K_X + \Delta + tH + \mathbf{M}_X)$ is independent of $t \in (0, t_0]$, prime divisors contracted by ϕ_t are independent of $t \in (0, t_0]$. Therefore, putting

$$X_0 = X_{t_0}, \quad \Delta_0 = \Delta_{t_0}, \quad \text{and} \quad H_0 = H_{t_0},$$

then X_0 and X_t are isomorphic in codimension one for all $t \in (0, t_0]$. From this fact, we see that $(X_t, \Delta_t + tH_t, \mathbf{M})$ is a good minimal model of $(X_0, \Delta_0 + tH_0, \mathbf{M})$.

By applying [29, Proof of Lemma 2.14] and [31, Proof of Proposition 3.3], we get a sequence of steps of a $(K_{X_0} + \Delta_0 + \mathbf{M}_{X_0})$ -MMP with scaling of t_0H_0

$$(X_0, \Delta_0, \mathbf{M}) \dashrightarrow (X_1, \Delta_1, \mathbf{M}) \dashrightarrow \cdots \dashrightarrow (X_i, \Delta_i, \mathbf{M}) \dashrightarrow \cdots$$

such that

- if we define $t_i = \inf \{ \mu \in \mathbb{R}_{\geq 0} \mid K_{X_i} + \Delta_i + \mathbf{M}_{X_i} + \mu H_i \text{ is nef} \}$ for each i , then $\lim_{i \rightarrow \infty} t_i = 0$,
- for all $t \in [t_i, t_{i-1}] \cap \mathbb{R}_{>0}$, the generalized pair $(X_i, \Delta_i + tH_i, \mathbf{M})$ is a good minimal model of both $(X, \Delta + tH, \mathbf{M})$ and $(X_0, \Delta_0 + tH_0, \mathbf{M})$, and
- the MMP occurs only in $\text{Supp}[\Delta_0]$.

By the argument as in [31, Step 1 in the proof of Theorem 3.4], it is sufficient to prove the termination of the $(K_{X_0} + \Delta_0 + \mathbf{M}_{X_0})$ -MMP.

Step 3. We follow [31, Step 1 in the proof of Theorem 3.4].

For every i and generalized lc center S_i of $(X_i, \Delta_i, \mathbf{M})$, we define a generalized dlt pair $(S_i, \Delta_{S_i}, \mathbf{N})$ by applying divisorial adjunction for generalized pairs repeatedly. We put $H_{S_i} = H_i|_{S_i}$. Similarly, for every generalized lc center S of (X, Δ, \mathbf{M}) , we define a generalized dlt pair $(S, \Delta_S, \mathbf{N})$ by applying divisorial adjunction for generalized pairs repeatedly. We put $H_S = H|_S$.

There exists $m_0 \gg 0$ such that, for every $i \geq m_0$, the birational map $X_{m_0} \dashrightarrow X_i$ is an isomorphism on a neighborhood of the generic points of all generalized lc centers of $(X_{m_0}, \Delta_{m_0}, \mathbf{M})$.

We will prove the termination of the $(K_{X_0} + \Delta_0 + \mathbf{M}_{X_0})$ -MMP by applying the special termination for MMP for generalized dlt pairs (cf. [25, Subsection 4.2]). More precisely, by induction on $l \in \mathbb{Z}_{\geq 0}$, we will prove that the MMP terminates on a neighborhood of all l -dimensional generalized lc centers of $(X_{m_0}, \Delta_{m_0}, \mathbf{M})$ for all l . Pick $l \in \mathbb{Z}_{\geq 0}$. By the induction hypothesis and the standard argument of the special termination, we can find $m \gg m_0$ such that, for every $i \geq m$ and every generalized lc center S_m of $(X_m, \Delta_m, \mathbf{M})$ with $\dim S_m \leq l$, the induced birational map $S_m \dashrightarrow S_i$ to the corresponding generalized

lc center S_i of $(X_i, \Delta_i, \mathbf{M})$ is small and the birational transform of Δ_{S_m} (resp. $H_{S_m}, \mathbf{N}_{S_m}$) to S_i is equal to Δ_{S_i} (resp. $H_{S_i}, \mathbf{N}_{S_i}$).

To prove the special termination, it is sufficient to prove the existence of a minimal model of $(S_m, \Delta_{S_m}, \mathbf{N})$ for every generalized lc center S_m of $(X_m, \Delta_m, \mathbf{M})$.

Step 4. We follow [31, Step 2 in the proof of Theorem 3.4].

From now on, we fix a generalized lc center S_m of $(X_m, \Delta_m, \mathbf{M})$. Unless otherwise stated, we assume that all integers i appearing in the rest of the proof satisfy $i \geq m$. For all i , let S_i be the generalized lc center of $(X_i, \Delta_i, \mathbf{M})$ such that $X_m \dashrightarrow X_i$ induces a birational map $S_m \dashrightarrow S_i$. By construction of the birational map $X \dashrightarrow X_i$, we can find a generalized lc center S of (X, Δ, \mathbf{M}) such that $X \dashrightarrow X_i$ induces a birational map $S \dashrightarrow S_i$. Using the S , we will define the following variety, divisor and inequalities.

- (a) A birational morphism $\psi: T \rightarrow S_m$ from a projective \mathbb{Q} -factorial variety T such that, for every prime divisor \bar{D} on S , if we have $a(\bar{D}, S_m, \Delta_{S_m} + \mathbf{N}_{S_m}) < a(\bar{D}, S, \Delta_S + \mathbf{N}_S)$, then \bar{D} is a ψ -exceptional divisor on T ,
- (b) An effective \mathbb{R} -divisor Ψ on T defined by $\Psi = -\sum_D (a(D, S, \Delta_S + \mathbf{N}_S) - 1)D$, where D runs over all prime divisors on T ,
- (c) The inequality $a(Q, S, (\Delta_S + t_i H_S) + \mathbf{N}_S) \leq a(Q, S_i, (\Delta_{S_i} + t_i H_{S_i}) + \mathbf{N}_{S_i})$ for all i and all prime divisors Q over S and
- (d) The inequality $a(Q', S_m, \Delta_{S_m} + \mathbf{N}_{S_m}) \leq a(Q', T, \Psi + \mathbf{N}_T)$ for all prime divisors Q' over S_m , in particular, the generalized pair (T, Ψ, \mathbf{N}) is generalized lc.

The generalized pair $(X_i, \Delta_i + t_i H_i, \mathbf{M})$ is a good minimal model of $(X, \Delta + t_i H, \mathbf{M})$, hence the negativity lemma (see also [13, Proof of Lemma 4.2.10]) implies the inequality

$$a(Q, S, (\Delta_S + t_i H_S) + \mathbf{N}_S) \leq a(Q, S_i, (\Delta_{S_i} + t_i H_{S_i}) + \mathbf{N}_{S_i})$$

for all prime divisors Q over S . We have (c).

In this paragraph, we prove that the equality

$$a(\tilde{D}, S_m, \Delta_{S_m} + \mathbf{N}_{S_m}) = a(\tilde{D}, S, \Delta_S + \mathbf{N}_S) \tag{*}$$

holds for all prime divisors \tilde{D} on S_m . Note that \tilde{D} is a prime divisor on S_i for every i since the birational map $S_m \dashrightarrow S_i$ is small. For every i , we may apply Lemma 3.5 to the birational map $(X, \Delta, \mathbf{M}) \dashrightarrow (X_i, \Delta_i, \mathbf{M})$ and the generalized pairs $(S, \Delta_S, \mathbf{N})$ and $(S_i, \Delta_{S_i}, \mathbf{N})$ because $(X, \Delta, \mathbf{M}) \dashrightarrow (X_i, \Delta_i, \mathbf{M})$ obviously satisfies the first condition of Lemma 3.5, and the second condition of Lemma 3.5 follows from the third condition of Theorem 3.14. Thus, we have

$$a(\tilde{D}, S_i, \Delta_{S_i} + \mathbf{N}_{S_i}) \leq a(\tilde{D}, S, \Delta_S + \mathbf{N}_S),$$

and therefore we have

$$a(\tilde{D}, S_i, (\Delta_{S_i} + t_i H_{S_i}) + \mathbf{N}_{S_i}) \leq a(\tilde{D}, S, \Delta_S + \mathbf{N}_S)$$

for all prime divisors \tilde{D} on S_m . By this inequality and (c), we have

$$a(\tilde{D}, S, (\Delta_S + t_i H_S) + \mathbf{N}_S) \leq a(\tilde{D}, S_i, (\Delta_{S_i} + t_i H_{S_i}) + \mathbf{N}_{S_i}) \leq a(\tilde{D}, S, \Delta_S + \mathbf{N}_S).$$

We have $a(\tilde{D}, S_i, (\Delta_{S_i} + t_i H_{S_i}) + \mathbf{N}_{S_i}) = a(\tilde{D}, S_m, (\Delta_{S_m} + t_i H_{S_m}) + \mathbf{N}_{S_m})$ for all i since \tilde{D} appears as a prime divisor on S_m and S_i . Since $\lim_{i \rightarrow \infty} t_i = 0$, we get equation (*) for all prime divisors \tilde{D} on S_m by taking the limit $i \rightarrow \infty$.

We put

$$\mathcal{C} = \left\{ \bar{D} \mid \begin{array}{l} \bar{D} \text{ is a prime divisor on } S \text{ such that} \\ a(\bar{D}, S_m, \Delta_{S_m} + \mathbf{N}_{S_m}) < a(\bar{D}, S, \Delta_S + \mathbf{N}_S) \end{array} \right\}.$$

By equation (★), all elements of \mathcal{C} are exceptional over S_m . By a basic property of generalized log discrepancies and (c), we have

$$a(\bar{D}, S, (\Delta_S + t_m H_S) + \mathbf{N}_S) \leq a(\bar{D}, S_m, \Delta_{S_m} + \mathbf{N}_{S_m}).$$

This implies

$$a(\bar{D}, S, (\Delta_S + t_m H_S) + \mathbf{N}_S) \leq a(\bar{D}, S_m, \Delta_{S_m} + \mathbf{N}_{S_m}) < a(\bar{D}, S, \Delta_S + \mathbf{N}_S)$$

for all $\bar{D} \in \mathcal{C}$. Since every element of \mathcal{C} is a prime divisor on S , we see that all elements of \mathcal{C} are components of H_S . Thus, \mathcal{C} is a finite set, and $a(\bar{D}, S_m, \Delta_{S_m} + \mathbf{N}_{S_m}) < 1$ for all $\bar{D} \in \mathcal{C}$ because $a(\bar{D}, S, \Delta_S + \mathbf{N}_S) \leq 1$. From these facts, for every $\bar{D} \in \mathcal{C}$ we get the relation

$$\begin{aligned} 0 &\leq a(\bar{D}, S, (\Delta_S + t_0 H_S) + \mathbf{N}_S) \\ &< a(\bar{D}, S, (\Delta_S + t_m H_S) + \mathbf{N}_S) \quad (\bar{D} \text{ is a component of } H_S \text{ and } m \gg 0) \\ &\leq a(\bar{D}, S_m, (\Delta_{S_m} + t_m H_{S_m}) + \mathbf{N}_{S_m}) \quad (\text{the inequality (c)}) \\ &\leq a(\bar{D}, S_m, \Delta_{S_m} + \mathbf{N}_{S_m}) \\ &< a(\bar{D}, S, \Delta_S + \mathbf{N}_S) \leq 1 \quad (\text{definition of } \mathcal{C}). \end{aligned}$$

In this way, we have $0 < a(\bar{D}, S_m, \Delta_{S_m} + \mathbf{N}_{S_m}) < 1$. By Lemma 3.4, there exists a projective birational morphism $\psi: T \rightarrow S_m$ such that T is \mathbb{Q} -factorial and ψ^{-1} exactly extracts elements of \mathcal{C} . We have constructed the desired birational morphism as in (a).

Let D be a prime divisor on T . When D is ψ -exceptional, by the definitions of \mathcal{C} and ψ we have

$$a(D, S_m, \Delta_{S_m} + \mathbf{N}_{S_m}) < a(D, S, \Delta_S + \mathbf{N}_S) \leq 1.$$

When D is not ψ -exceptional, from equation (★) we see that

$$a(D, S, \Delta_S + \mathbf{N}_S) = a(D, S_m, \Delta_{S_m} + \mathbf{N}_{S_m}) \leq 1.$$

Thus, the relation

$$a(D, S_m, \Delta_{S_m} + \mathbf{N}_{S_m}) \leq a(D, S, \Delta_S + \mathbf{N}_S) \leq 1. \tag{★★}$$

holds for all prime divisors D on T . Note that only finitely many prime divisors D on T satisfy $a(D, S, \Delta_S + \mathbf{N}_S) < 1$. Therefore, we may define an \mathbb{R} -divisor $\Psi \geq 0$ on T by

$$\Psi = - \sum_D (a(D, S, \Delta_S + \mathbf{N}_S) - 1) D,$$

where D runs over all prime divisors on T . This is the divisor stated in (b).

Finally, we prove (d). Since T is \mathbb{Q} -factorial, $K_T + \Psi + \mathbf{N}_T$ is \mathbb{R} -Cartier. By equation (★★), we obtain

$$K_T + \Psi + \mathbf{N}_T \leq \psi^*(K_{S_m} + \Delta_{S_m} + \mathbf{N}_{S_m}).$$

From this, we have

$$0 \leq a(Q', S_m, \Delta_{S_m} + \mathbf{N}_{S_m}) \leq a(Q', T, \Psi + \mathbf{N}_T)$$

for any prime divisor Q' over S_m . This shows (d).

Step 5. The goal of this step is to show that (T, Ψ, \mathbf{N}) has a minimal model. We follow [31, Step 3 in the proof of Theorem 3.4].

By the second condition of Theorem 3.14, the generalized pair $(S, \Delta_S, \mathbf{N})$ is a minimal model of $(S, \Delta_S, \mathbf{N})$ itself. By applying [31, Step 3 in the proof of Theorem 3.4] with minor changes, we see that the following property holds.

- Let \tilde{Q} be a prime divisor over S . Then the following two statements hold:
 - If \tilde{Q} is a divisor on S , then $a(\tilde{Q}, S, \Delta_S + \mathbf{N}_S) \leq a(\tilde{Q}, T, \Psi + \mathbf{N}_T)$, and
 - If \tilde{Q} is a divisor on T , then $a(\tilde{Q}, T, \Psi + \mathbf{N}_T) \leq a(\tilde{Q}, S, \Delta_S + \mathbf{N}_S)$.

By applying Lemma 3.9 to the map $(T, \Psi, \mathbf{N}) \dashrightarrow (S, \Delta_S, \mathbf{N})$, we see that (T, Ψ, \mathbf{N}) has a minimal model.

Step 6. We follow [31, Step 4 in the proof of Theorem 3.4]. With this step, we complete the proof of the special termination of the $(K_{X_0} + \Delta_0 + \mathbf{M}_{X_0})$ -MMP in Step 3.

We recall that $S_m \dashrightarrow S_i$ is small, $\Delta_{S_i} + t_i H_{S_i}$ is equal to the birational transform of $\Delta_{S_m} + t_i H_{S_m}$ on S_i , and the divisor

$$K_{S_i} + \Delta_{S_i} + t_i H_{S_i} + \mathbf{N}_{S_i} \sim_{\mathbb{R}} (K_{X_i} + \Delta_i + t_i H_i + \mathbf{M}_{X_i})|_{S_i}$$

is nef. So $(S_i, \Delta_{S_i} + t_i H_{S_i}, \mathbf{N})$ is a weak generalized lc model of $(S_m, \Delta_{S_m} + t_i H_{S_m}, \mathbf{N})$ for every i .

We pick an arbitrary prime divisor D on T . By using the generalized pair analogue of [31, Remark 2.9 (1)], we see that

$$\begin{aligned} & \sigma_D(K_{S_m} + \Delta_{S_m} + t_i H_{S_m} + \mathbf{N}_{S_m}) \\ &= a(D, S_i, (\Delta_{S_i} + t_i H_{S_i}) + \mathbf{N}_{S_i}) - a(D, S_m, (\Delta_{S_m} + t_i H_{S_m}) + \mathbf{N}_{S_m}). \end{aligned}$$

From the relation, we have

$$\begin{aligned} & \sigma_D(K_{S_m} + \Delta_{S_m} + t_i H_{S_m} + \mathbf{N}_{S_m}) \\ &= a(D, S_i, (\Delta_{S_i} + t_i H_{S_i}) + \mathbf{N}_{S_i}) - a(D, S_m, (\Delta_{S_m} + t_i H_{S_m}) + \mathbf{N}_{S_m}) \\ &\geq a(D, S, (\Delta_S + t_i H_S) + \mathbf{N}_S) - a(D, S_m, (\Delta_{S_m} + t_i H_{S_m}) + \mathbf{N}_{S_m}) \quad ((c) \text{ in Step 4}). \end{aligned}$$

By (b) in Step 4, the equality $a(D, S, \Delta_S + \mathbf{N}_S) = a(D, T, \Psi + \mathbf{N}_T)$ holds. From these relations and the fact that $\lim_{i \rightarrow \infty} t_i = 0$, we obtain

$$\begin{aligned} & \sigma_D(K_{S_m} + \Delta_{S_m} + \mathbf{N}_{S_m}) \\ &= \lim_{i \rightarrow \infty} \sigma_D(K_{S_m} + \Delta_{S_m} + t_i H_{S_m} + \mathbf{N}_{S_m}) \quad ([31, Remark 2.3 (2)]) \\ &\geq \lim_{i \rightarrow \infty} (a(D, S, (\Delta_S + t_i H_S) + \mathbf{N}_S) - a(D, S_m, (\Delta_{S_m} + t_i H_{S_m}) + \mathbf{N}_{S_m})) \\ &= a(D, S, \Delta_S + \mathbf{N}_S) - a(D, S_m, \Delta_{S_m} + \mathbf{N}_{S_m}) \\ &= a(D, T, \Psi + \mathbf{N}_T) - a(D, S_m, \Delta_{S_m} + \mathbf{N}_{S_m}) \end{aligned}$$

for all prime divisors D on T . By this relation and (d) in Step 4, we obtain

$$0 \leq a(D, T, \Psi + \mathbf{N}_T) - a(D, S_m, \Delta_{S_m} + \mathbf{N}_{S_m}) \leq \sigma_D(K_{S_m} + \Delta_{S_m} + \mathbf{N}_{S_m}).$$

Therefore, we may apply Lemma 3.6 to $(S_m, \Delta_{S_m}, \mathbf{N})$ and (T, Ψ, \mathbf{N}) . Because (T, Ψ, \mathbf{N}) has a minimal model, $(S_m, \Delta_{S_m}, \mathbf{N})$ has a minimal model.

By arguments in the last paragraph of Step 3, we complete the special termination.

Since the $(K_{X_0} + \Delta_0 + \mathbf{M}_{X_0})$ -MMP occurs only in $\text{Supp}[\Delta_0]$ (see Step 2), we see that the $(K_{X_0} + \Delta_0 + \mathbf{M}_{X_0})$ -MMP terminates, so (X, Δ, \mathbf{M}) has a minimal model. \square

Theorem 3.15 (cf. [31, Theorem 3.5]). *Let (X, Δ, \mathbf{M}) be a generalized lc pair such that \mathbf{M} is a finite $\mathbb{R}_{>0}$ -linear combination of b -nef \mathbb{Q} - b -Cartier \mathbb{Q} - b -divisors. Let A be an \mathbb{R} -divisor on X such that $K_X + \Delta + \mathbf{M}_X + A$ is nef.*

Then, no infinite sequence of steps of a $(K_X + \Delta + \mathbf{M}_X)$ -MMP with scaling of A

$$(X, \Delta, \mathbf{M}) =: (X_0, \Delta_0, \mathbf{M}) \dashrightarrow (X_1, \Delta_1, \mathbf{M}) \dashrightarrow \cdots \dashrightarrow (X_i, \Delta_i, \mathbf{M}) \dashrightarrow \cdots$$

satisfies the following properties.

- *If we define $\lambda_i = \inf \{ \mu \in \mathbb{R}_{\geq 0} \mid K_{X_i} + \Delta_i + \mathbf{M}_{X_i} + \mu A_i \text{ is nef} \}$ for each $i \geq 0$, then $\lim_{i \rightarrow \infty} \lambda_i = 0$, and*
- *There are infinitely many i such that $K_{X_i} + \Delta_i + \mathbf{M}_{X_i}$ is log abundant with respect to $(X_i, \Delta_i, \mathbf{M})$.*

Proof. The argument of [31, Proof of Theorem 3.5] works without any change by using Lemma 3.8, the special termination for generalized dlt pairs ([25, Subsection 4.2]), and Theorem 3.14 instead of [31, Lemma 2.22], [13] and [31, Theorem 3.5], respectively. □

Theorem 3.16. *Let (X, Δ) be a projective lc pair, and let M be a finite $\mathbb{R}_{>0}$ -linear combination of nef \mathbb{Q} -divisors on X . Suppose that $K_X + \Delta + M$ is pseudo-effective and M is log big with respect to (X, Δ) . We put $\mathbf{M} = \overline{M}$. Then there is a sequence of steps of a $(K_X + \Delta + M)$ -MMP starting with the generalized lc pair (X, Δ, \mathbf{M})*

$$(X, \Delta, \mathbf{M}) \dashrightarrow \cdots \dashrightarrow (X_i, \Delta_i, \mathbf{M}) \dashrightarrow \cdots$$

that terminates with a good minimal model.

Proof. We first reduce the theorem to the case where $K_X + \Delta + lM$ is nef for some positive real number $l > 2$. We may write $M = \sum_j r_j M^{(j)}$, where $M^{(j)}$ are nef Cartier divisors on X and $r_j > 0$. We fix a real number $l > 2$ such that $lr_j > 2 \cdot \dim X$ for all j . Pick an ample \mathbb{R} -divisor A on X such that $K_X + \Delta + lM + A$ is nef. Since M is nef, by [33, Theorem 1.5], the generalized lc pair $(X, \Delta, lM + t\overline{A})$ has a good minimal model for every real number $t > 0$. Applying the argument in [29, Proof of Lemma 2.14] (see [33, Proof of Theorem 1.7]) with the aid of the length of extremal rays ([15, Section 18] or [17, Theorem 4.6.2]), we may construct a $(K_X + \Delta + lM)$ -MMP with scaling of A

$$(X, \Delta, l\mathbf{M}) \dashrightarrow \cdots \dashrightarrow (X_i, \Delta_i, l\mathbf{M}) \dashrightarrow \cdots$$

such that if we define

$$\lambda_i = \inf \{ \mu \in \mathbb{R}_{\geq 0} \mid K_{X_i} + \Delta_i + l\mathbf{M}_{X_i} + \mu A_i \text{ is nef} \}$$

for each i , then $\lim_{i \rightarrow \infty} \lambda_i = 0$. By the choice of l and the length of extremal rays (see [15, Section 18] or [17, Theorem 4.6.2]), the birational transforms $M_i^{(j)}$ of $M^{(j)}$ on X_i are numerically trivial with respect to the extremal contraction of the MMP. This fact implies the following properties:

- $M_i^{(j)}$ is nef and Cartier for every i and j , and
- the sequence of birational maps $X \dashrightarrow \cdots \dashrightarrow X_i \dashrightarrow \cdots$ is a sequence of steps of a $(K_X + \Delta + M)$ -MMP.

By the first property, $\mathbf{M}_{X_i} = \sum_j r_j M_i^{(j)}$ is a finite $\mathbb{R}_{>0}$ -linear combination of nef Cartier divisors on X_i , and the log bigness of M shows that \mathbf{M}_{X_i} is log big with respect to (X_i, Δ_i) for every i . By Lemma 2.15, we see that $K_{X_i} + \Delta_i + l\mathbf{M}_{X_i}$ is log big with respect to (X_i, Δ_i) for every i . By Theorem 3.15, the $(K_X + \Delta + lM)$ -MMP terminates with a minimal model $(X_m, \Delta_m, l\mathbf{M})$. Then the birational map

$$X \dashrightarrow X_m$$

is a sequence of steps of a $(K_X + \Delta + M)$ -MMP. To prove Theorem 3.16, it is sufficient to prove the existence of a $(K_{X_m} + \Delta_m + \mathbf{M}_{X_m})$ -MMP that terminates with a good minimal model. In this way, replacing (X, Δ, \mathbf{M}) with $(X_m, \Delta_m, \mathbf{M})$, we may assume that $K_X + \Delta + lM$ is nef for some $l > 2$.

Replacing l by $2l$, we may assume that $K_X + \Delta + lM$ is log big with respect to (X, Δ) . By the argument of Shokurov polytope (cf. [25, Subsection 3.3]) and [12], we see that $K_X + \Delta + lM$ is semiample. We fix an effective \mathbb{R} -divisor

$$B \sim_{\mathbb{R}} K_X + \Delta + lM$$

such that $(X, \Delta + B)$ is lc. Then

$$l(K_X + \Delta + M) = (l - 1)(K_X + \Delta) + (K_X + \Delta + lM) \sim_{\mathbb{R}} (l - 1)\left(K_X + \Delta + \frac{1}{l-1}B\right).$$

By Corollary 3.13, $K_X + \Delta + \frac{1}{l-1}B$ is log abundant with respect to $(X, \Delta + \frac{1}{l-1}B)$ and every $(K_X + \Delta + \frac{1}{l-1}B)$ -MMP preserves the property of being log abundant. Then the argument in [31, Proof of Corollary 3.9] implies the existence of a sequence of steps of a $(K_X + \Delta + \frac{1}{l-1}B)$ -MMP that terminates with a good minimal model.

Since all $(K_X + \Delta + \frac{1}{l-1}B)$ -MMP are $(K_X + \Delta + M)$ -MMP, there exists a sequence of steps of a $(K_X + \Delta + M)$ -MMP that terminates with a good minimal model. In this way, we see that Theorem 3.16 holds. \square

Proof of Theorem 1.1. If $K_X + \Delta + M$ is pseudo-effective, then Theorem 1.1 follows from Theorem 3.16. If $K_X + \Delta + M$ is not pseudo-effective, then take an ample \mathbb{R} -divisor A on X and a real number $\epsilon > 0$ such that $K_X + \Delta + M + A$ is nef and $K_X + \Delta + M + \epsilon A$ is not pseudo-effective. Since $M + \epsilon A$ is ample, by [33, Theorem 1.7] there is $X \dashrightarrow X'$ a sequence of steps of a $(K_X + \Delta + \epsilon A + M)$ -MMP with scaling of A that terminates with a Mori fiber space. Then $X \dashrightarrow X'$ is a sequence of steps of a $(K_X + \Delta + M)$ -MMP with scaling of A terminating with a Mori fiber space. \square

The following result was mentioned in [32].

Theorem 3.17. *Let (X, B, \mathbf{M}) be a generalized lc pair such that \mathbf{M} is a finite $\mathbb{R}_{>0}$ -linear combination of b -nef \mathbb{Q} - b -Cartier \mathbb{Q} - b -divisors. Let A be an effective ample \mathbb{R} -divisor on X such that $(X, B + A, \mathbf{M})$ is a generalized lc pair and generalized lc centers of $(X, B + A, \mathbf{M})$ are generalized lc centers of (X, B, \mathbf{M}) . Let (Y, Γ, \mathbf{M}) be a \mathbb{Q} -factorial generalized dlt model of $(X, B + A, \mathbf{M})$. Then, every $(K_Y + \Gamma + \mathbf{M}_Y)$ -MMP with scaling of an ample divisor*

$$(Y, \Gamma, \mathbf{M}) =: (Y_0, \Gamma_0, \mathbf{M}) \dashrightarrow \cdots \dashrightarrow (Y_i, \Gamma_i, \mathbf{M}) \dashrightarrow \cdots$$

terminates with a minimal model.

Proof. By [32, Theorem 1.3], we see that $K_{Y_i} + \Gamma_i + \mathbf{M}_{Y_i}$ is log abundant with respect to $(Y_i, \Gamma_i, \mathbf{M})$ for all i . Hence, the theorem follows from Theorem 3.15. \square

4. Effectivity of Iitaka fibration

In this section, we prove Theorem 1.2 and Theorem 1.3.

Theorem 4.1. *Let d and p be positive integers, and let $\Phi \subset \mathbb{Q}$ be a DCC set. Then, there exists a positive integer m , depending only on d, p and Φ , satisfying the following. Let (X, Δ, \mathbf{M}) be a generalized dlt pair such that*

- $\dim X = d$,
- the coefficients of Δ belong to Φ ,
- $p\mathbf{M}$ is b -Cartier,
- $K_X + \Delta + \mathbf{M}_X \equiv 0$ and

- there is a log resolution $f: \tilde{X} \rightarrow X$ of (X, Δ) such that
 - f is an isomorphism over an open subset $U \subset X$ containing all the generic points of generalized lc centers of (X, Δ, \mathbf{M}) ,
 - \mathbf{M} descends to \tilde{X} and
 - writing $K_{\tilde{X}} + \tilde{\Delta} + \mathbf{M}_{\tilde{X}} = f^*(K_X + \Delta + \mathbf{M}_X) + \tilde{E}$, where $\tilde{\Delta} \geq 0$ and $\tilde{E} \geq 0$ have no common components, then $\mathbf{M}_{\tilde{X}}$ is log big with respect to $(\tilde{X}, \tilde{\Delta})$.

Then $H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta + \mathbf{M}_X) \rfloor)) \neq 0$. In particular, $m(K_X + \Delta + \mathbf{M}_X) \sim 0$.

Proof. We prove the theorem by induction on the dimension of X .

By the global ACC [7, Theorem 1.6], we can find a finite set $\Phi' \subset \Phi$, depending only on d, p and Φ , such that the coefficients of Δ belong to Φ' . By [4, Theorem 1.10], there exists a positive integer m' , depending only on d, p and Φ' , such that if (X, Δ, \mathbf{M}) is a generalized klt pair, then $H^0(X, \mathcal{O}_X(\lfloor m'(K_X + \Delta + \mathbf{M}_X) \rfloor)) \neq 0$. Let s be the minimum positive integer such that $sa \in \mathbb{Z}$ for every $a \in \Phi'$. If (X, Δ, \mathbf{M}) is not generalized klt, then we may find a component S of $\lfloor \Delta \rfloor$. Since $sp\mathbf{M}_{\tilde{X}}$ is Cartier and $spa \in \mathbb{Z}$ for every $a \in \Phi'$, by the projective case of Lemma 2.16 the morphism

$$H^0(X, \mathcal{O}_X(\lfloor lsp(K_X + \Delta + \mathbf{M}_X) \rfloor)) \longrightarrow H^0(S, \mathcal{O}_S(\lfloor lsp(K_S + \Delta_S + \mathbf{N}_S) \rfloor))$$

is surjective for every positive integer l , where the divisor $K_S + \Delta_S + \mathbf{N}_S$ is the generalized log canonical divisor of the generalized pair $(S, \Delta_S, \mathbf{N})$ defined with divisorial adjunction for (X, Δ, \mathbf{M}) . By Remark 2.8, there exists a DCC set $\Omega \subset \mathbb{Q}$, depending only on d, p and Φ' , such that all the coefficients of Δ_S belong to Ω . By the induction hypothesis of Theorem 4.1, there is a positive integer m'' , depending only on $d - 1, p$ and Ω , such that $H^0(S, \mathcal{O}_S(\lfloor m''(K_S + \Delta_S + \mathbf{N}_S) \rfloor)) \neq 0$. Then $H^0(X, \mathcal{O}_X(\lfloor m''sp(K_X + \Delta + \mathbf{M}_X) \rfloor)) \neq 0$ by the above surjection. Hence, $m := m'm''sp$ is the desired positive integer. \square

We will use the following result about the generalized canonical bundle formula by Filipazzi–Moraga [10].

Lemma 4.2 (cf. [10, Theorem 1.5], see also [4, Proposition 6.3]). *Let d and p be positive integers. Let $\mathfrak{R} \subset [0, 1]$ be a finite set of rational numbers and $\Phi(\mathfrak{R})$ the hyperstandard set associated to \mathfrak{R} . Then, there exist positive integers n and q and a DCC set $\Omega \subset \mathbb{Q}$, depending only on d, p and \mathfrak{R} , satisfying the following. Let $\pi: X \rightarrow Z$ be a contraction of normal projective varieties, and let (X, Δ, \mathbf{M}) be a generalized klt pair such that*

- $\dim X = d$,
- the coefficients of Δ belong to $\Phi(\mathfrak{R})$,
- $p\mathbf{M}$ is b -Cartier,
- $K_X + \Delta + \mathbf{M}_X \sim_{\mathbb{Q}, Z} 0$ and
- X is of Fano type over an open subset U of Z ,

Then $n(K_X + \Delta + \mathbf{M}_X) \sim \pi^*D$ for some \mathbb{Q} -Cartier divisor D on Z and there exists a generalized lc pair $(Z, \Delta_Z, \mathbf{N})$ constructed with generalized canonical bundle formula such that

- $K_Z + \Delta_Z + \mathbf{N}_Z = \frac{1}{n}D$,
- the coefficients of Δ_Z belong to Ω and
- $q\mathbf{N}$ is b -Cartier.

Remark 4.3. The existence of n satisfying $n(K_X + \Delta + \mathbf{M}_X) \sim n\pi^*(K_Z + \Delta_Z + \mathbf{N}_Z)$ in Lemma 4.2 was proved in [10, Proof of Theorem 1.5] (see [10, Proof of Lemma 5.4] or [4, Proof of Proposition 6.3]). If $\dim Z = 0$, then the existence of n is equivalent to the existence of n -complements, which was proved by Birkar [4, Theorem 1.10].

Theorem 4.4. *Let d and p be positive integers, and let $\Phi \subset \mathbb{Q}$ be a DCC set. Then, there exist positive integers n and q and a DCC set $\Omega \subset \mathbb{Q}$, depending only on d, p and Φ , satisfying the following. Let*

$\pi: X \rightarrow Z$ be a contraction of normal projective varieties, and let (X, Δ, \mathbf{M}) be a generalized lc pair such that

- $\dim X = d$,
- the coefficients of Δ belong to Φ ,
- \mathbf{M} is b -big/ Z and $p\mathbf{M}$ is b -Cartier,
- $K_X + \Delta + \mathbf{M}_X \sim_{\mathbb{Q}, Z} 0$ and
- all generalized lc centers of (X, Δ, \mathbf{M}) are vertical over Z .

Then $n(K_X + \Delta + \mathbf{M}_X) \sim \pi^*D$ for some \mathbb{Q} -Cartier divisor D on Z and there exists a generalized lc pair $(Z, \Delta_Z, \mathbf{N})$ constructed with generalized canonical bundle formula such that

- $K_Z + \Delta_Z + \mathbf{N}_Z = \frac{1}{n}D$,
- the coefficients of Δ_Z belong to Ω and
- $q\mathbf{N}$ is b -Cartier.

Proof. We first show the existence of Ω . Let $\mathcal{J}(d, \Phi, p) \subset \mathbb{R}$ be the set of the generalized lc thresholds of effective Cartier divisors with respect to generalized lc pairs $(Y, \Gamma, \mathbf{L})/W$ such that $\dim Y = d$, the coefficients of Γ belong to Φ and $p\mathbf{L}$ is b -Cartier. Then we have $\mathcal{J}(d, \Phi, p) \subset \mathbb{Q}$ since $\Phi \subset \mathbb{Q}$, and $\mathcal{J}(d, \Phi, p)$ is an ACC set by [7, Theorem 1.5]. Let P be a prime divisor on Z . Then there is an open subset $U \subset Z$ containing the generic point of P such that the generalized lc threshold of π^*P with respect to (X, Δ, \mathbf{M}) over the generic point of P is equal to the generalized lc threshold of $\pi^*P|_{\pi^{-1}(U)}$ with respect to $(\pi^{-1}(U), \Delta|_{\pi^{-1}(U)}, \mathbf{M}|_{\pi^{-1}(U)})/U$. From this and the construction of the discriminant part of generalized canonical bundle formula, we see that the set

$$\Omega := \{ 1 - t \mid t \in \mathcal{J}(d, \Phi, p) \}$$

is the desired DCC set.

From now on, we prove the existence of n and q of Theorem 4.4. Replacing (X, Δ, \mathbf{M}) by a \mathbb{Q} -factorial generalized dlt model, we may assume that $(X, 0)$ is a \mathbb{Q} -factorial klt pair. Let Δ^h be the horizontal part of Δ . Restricting (X, Δ, \mathbf{M}) to the general fiber of π and applying [7, Theorem 1.6], we can find a finite set $\Phi' \subset \Phi$, which depends only on d, p and Φ , such that the coefficients of Δ^h belong to Φ' .

Since \mathbf{M} is b -big/ Z and $(X, \Delta^h, \mathbf{M})$ is generalized klt $(X, \Delta^h, \mathbf{M})$ has a good minimal model $(X', \Delta'^h, \mathbf{M})$ over Z by [6]. Let $\pi': X' \rightarrow Z'$ be the contraction over Z induced by $K_{X'} + \Delta'^h + \mathbf{M}_{X'}$. Then $Z' \rightarrow Z$ is birational. Since \mathbf{M} is b -big/ Z and $(X', \Delta'^h, \mathbf{M})$ is generalized klt, it follows that X' is of Fano type over Z' . By applying Lemma 4.2 to $(X', \Delta'^h, \mathbf{M}) \rightarrow Z'$, we get a generalized lc pair $(Z', \Delta_{Z'}, \mathbf{N}')$ on Z' such that

$$n(K_{X'} + \Delta'^h + \mathbf{M}_{X'}) \sim n\pi'^*(K_{Z'} + \Delta_{Z'} + \mathbf{N}'_{Z'})$$

and $q\mathbf{N}'$ is b -Cartier, where n and q depend only on d, Φ' and p .

By construction, there are open subsets $V \subset Z$ and $V' \subset Z'$ such that the restriction of $(X, \Delta, \mathbf{M}) \rightarrow Z$ over V coincides with the restriction of $(X', \Delta'^h, \mathbf{M}) \rightarrow Z'$ over V' . By an argument similar to [4, 3.4 (2)], we see that the moduli part of the generalized canonical bundle formula for $\pi: (X, \Delta, \mathbf{M}) \rightarrow Z$ depends only on the generic fiber of π . This implies that \mathbf{N}' is the moduli part of the generalized canonical bundle formula for $(X, \Delta, \mathbf{M}) \rightarrow Z$. Thus, the integer q satisfy the condition of Theorem 4.4.

We can easily check that n also satisfies the condition of Theorem 4.4. Thus, n and q are the desired positive integers. □

Theorem 4.5. *Let d and p be positive integers, and let $\Phi \subset \mathbb{Q}$ be a DCC set. Then, there exist positive integers n and q and a DCC set $\Omega \subset \mathbb{Q}$, depending only on d, p and Φ , satisfying the following.*

Let $\pi: X \rightarrow Z$ be a contraction of normal projective varieties, and let (X, Δ, \mathbf{M}) be a generalized dlt pair such that

- $\dim X = d$,
- the coefficients of Δ belong to Φ ,
- $p\mathbf{M}$ is b-Cartier,
- $K_X + \Delta + \mathbf{M}_X \sim_{\mathbb{Q}, Z} 0$ and
- there is a log resolution $f: \tilde{X} \rightarrow X$ of (X, Δ) such that
 - f is an isomorphism over an open subset $U \subset X$ containing all the generic points of generalized lc centers of (X, Δ, \mathbf{M}) ,
 - \mathbf{M} descends to \tilde{X} and
 - writing $K_{\tilde{X}} + \tilde{\Delta} + \mathbf{M}_{\tilde{X}} = f^*(K_X + \Delta + \mathbf{M}_X) + \tilde{E}$, where $\tilde{\Delta} \geq 0$ and $\tilde{E} \geq 0$ have no common components, then $\mathbf{M}_{\tilde{X}}$ is log big with respect to $(\tilde{X}, \tilde{\Delta})$.

Then $n(K_X + \Delta + \mathbf{M}_X) \sim \pi^*D$ for some \mathbb{Q} -Cartier divisor D on Z , and there exists a generalized lc pair $(Z, \Delta_Z, \mathbf{N})$ such that

- $K_Z + \Delta_Z + \mathbf{N}_Z = \frac{1}{n}D$,
- the coefficients of Δ_Z belong to Ω and
- $q\mathbf{N}$ is b-Cartier.

Proof. We prove the theorem by induction on the dimension of X .

If all generalized lc centers of (X, Δ, \mathbf{M}) are vertical over Z , then Theorem 4.5 directly follows from Theorem 4.4. Hence, we may assume that there is a component S of $[\Delta]$ dominating Z .

Applying Theorem 4.1 to the general fiber of π , we can find a positive integer n , depending only on d, p and Φ , such that $n(K_X + \Delta + \mathbf{M}_X) \sim \pi^*D$ for some \mathbb{Q} -Cartier divisor D on Z .

Let m be a positive integer, which may depend on (X, Δ, \mathbf{M}) , such that $m\Delta$ is a Weil divisor and both $m\mathbf{M}_{\tilde{X}}$ and mD are Cartier. Then Lemma 2.16 implies that the morphism $\pi_*\mathcal{O}_X \otimes \mathcal{O}_Z(mD) \rightarrow \pi_{S*}\mathcal{O}_S \otimes \mathcal{O}_Z(mD)$ is surjective, where $\pi_S = \pi|_S$. Thus, π_S is a contraction.

Let $(S, \Delta_S, \mathbf{L})$ be a generalized dlt pair constructed with divisorial adjunction for generalized pairs. Then we have

$$n(K_S + \Delta_S + \mathbf{L}_S) \sim \pi_S^*D.$$

By Remark 2.8, there is a DCC set $\Psi \subset \mathbb{Q}$, depending only on d, p and Φ , such that the coefficients of Δ_S belong to Ψ . We apply the induction hypothesis of Theorem 4.5 to $\pi_S: (S, \Delta_S, \mathbf{L}) \rightarrow Z$. We can find positive integers n' and q' and a DCC set $\Omega \subset \mathbb{Q}$, depending only on d, p and Ψ , such that

$$n'(K_S + \Delta_S + \mathbf{L}_S) \sim \pi_S^*D'$$

for some \mathbb{Q} -Cartier divisor D' on Z and there exists a generalized lc pair $(Z, \Delta'_Z, \mathbf{N}')$ such that

- $K_Z + \Delta'_Z + \mathbf{N}'_Z = \frac{1}{n'}D'$,
- the coefficients of Δ'_Z belong to Ω and
- $q'\mathbf{N}'$ is b-Cartier.

Let \mathbf{D} and \mathbf{D}' be the b-divisors defined by $\mathbf{D} = \overline{D}$ and $\mathbf{D}' = \overline{D}'$, respectively. We define a generalized lc pair $(Z, \Delta_Z, \mathbf{N})$ by putting

$$\Delta_Z = \Delta'_Z \quad \text{and} \quad \mathbf{N} = \mathbf{N}' - \frac{1}{n'}\mathbf{D}' + \frac{1}{n}\mathbf{D}.$$

We will prove that $nn'q'\mathbf{N}'$ is b-Cartier. Let $\overline{Z} \rightarrow Z$ be a resolution of Z such that \mathbf{N}' descends to \overline{Z} . Let $\overline{S} \rightarrow S$ be a resolution of S such that the induced map $\pi_{\overline{S}}: \overline{S} \rightarrow \overline{Z}$ is a morphism. By the relation

$$nn'\pi_S^*\left(\frac{1}{n}D\right) \sim nn'(K_S + \Delta_S + \mathbf{L}_S) \sim nn'\pi_S^*\left(\frac{1}{n'}D'\right)$$

and the definitions of \mathbf{N}' and \mathbf{N} , it follows that $nn'q'\pi_S^*\mathbf{N}_{\overline{Z}} \sim nn'q'\pi_S^*\mathbf{N}'_{\overline{Z}}$. Since $nn'q'\mathbf{N}'_{\overline{Z}}$ is Cartier, by [10, Proposition 5.3] we see that $nn'q'\mathbf{N}_{\overline{Z}}$ is Cartier.

We put $q = nn'q'$. The generalized lc pair $(Z, \Delta_Z, \mathbf{N})$ satisfies $K_Z + \Delta_Z + \mathbf{N}_Z = \frac{1}{n}D$, the coefficients of Δ_Z belong to Ω and $q\mathbf{N}$ is b-Cartier. From the above discussion, the integers n and q and the DCC set Ω satisfy the condition of Theorem 4.5. □

We are ready to prove Theorem 1.2 and Theorem 1.3.

Proof of Theorem 1.3. Let $\phi: X \dashrightarrow X'$ be finite steps of a $(K_X + \Delta + M)$ -MMP such that $\phi_*(K_X + \Delta + M)$ is semiample, and let $\pi: X' \rightarrow Z$ be the contraction induced by $\phi_*(K_X + \Delta + M)$. We put $\Delta' = \phi_*\Delta$ and $M' = \phi_*M$. We consider the generalized dlt pair $(X', \Delta', \overline{M})$.

Let $U' \subset X'$ be an open subset such that $(U', \Delta'|_{U'})$ is log smooth, U' contains all the generic points of generalized lc centers of $(X', \Delta', \overline{M})$ and $X' \dashrightarrow X$ is an isomorphism on U' . Since M is nef and log big with respect to (X, Δ) , by taking an appropriate resolution of the graph of ϕ , we get a common log resolution $f: \tilde{X} \rightarrow X$ and $f': \tilde{X} \rightarrow X'$ of $(X, \Delta) \dashrightarrow (X', \Delta')$ such that f' and U' satisfy the final condition of Theorem 4.5.

Since $K_X + \Delta + M$ is not big, we have $\dim Z < \dim X'$. Let F be the general fiber of π , and put $\tilde{F} = f'^{-1}(F)$. We recall the hypothesis that $M = \sum_i \mu_i M_i$ and M_i are log big with respect to (X, Δ) . Since $\dim F > 0$, we see that $(f^*M_i)|_{\tilde{F}}$ is not numerically trivial for all i . Applying the global ACC [7, Theorem 1.6] to the restriction of $(X', \Delta', \overline{M})$ to F , we see that all μ_i belong to a finite set $\Phi' \subset \Phi$ depending only on d and Φ . Therefore, there exists a positive integer p , depending only on d and Φ , such that $p\mu_i$ are integers. Then $p\overline{M}$ is b-Cartier.

Obviously, the coefficients of Δ' belong to Φ , hence we can apply Theorem 4.5 to $(X', \Delta', \overline{M}) \rightarrow Z$. By Theorem 4.5, there exist positive integers n and q and a DCC set $\Omega \subset \mathbb{Q}$, depending only on d, p and Φ , such that

$$n(K_{X'} + \Delta' + M') \sim \pi^*D$$

for some \mathbb{Q} -Cartier divisor D on Z and there is a generalized lc pair $(Z, \Delta_Z, \mathbf{N})$ such that

- $K_Z + \Delta_Z + \mathbf{N}_Z = \frac{1}{n}D$,
- the coefficients of Δ_Z belong to Ω and
- $q\mathbf{N}$ is b-Cartier.

Then n, q and Ω depend only on d and Φ because p depends only on d and Φ . Since we have

$$\begin{aligned} H^0(X, \mathcal{O}_X(\lfloor l n(K_X + \Delta + M) \rfloor)) &\simeq H^0(X', \mathcal{O}_{X'}(\lfloor l n(K_{X'} + \Delta' + M') \rfloor)) \\ &\simeq H^0(X', \mathcal{O}_{X'}(\lfloor l \pi^* D \rfloor)) \\ &\simeq H^0(Z, \mathcal{O}_Z(\lfloor l n(K_Z + \Delta_Z + \mathbf{N}_Z) \rfloor)) \end{aligned}$$

for every positive integer l , we get the desired n, p and Ω . □

Proof of Theorem 1.2. The theorem follows from Theorem 1.3 and [7, Theorem 1.3]. □

Example 4.6. We fix positive integers d and p . We consider the category \mathcal{C} whose objects are generalized dlt pairs $(X, \Delta, \overline{M})$ such that $\dim X = d$, Δ is a Weil divisor, pM is Cartier, M is nef and log big with respect to (X, Δ) and $K_X + \Delta + M \sim_{\mathbb{Q}} 0$. We show that even if $d = 2$ and $p = 1$, some set

$$D \subset \{X \mid (X, \Delta, \overline{M}) \in \mathcal{C} \text{ for some } \Delta \text{ and } M\}$$

is unbounded.

We put $V = \mathbb{P}^1$. We fix a very ample Cartier divisor H_V such that $\mathcal{O}_V(H_V) = \mathcal{O}_V(1)$. We construct a \mathbb{P}^1 -bundle

$$X_n := \mathbb{P}_V(\mathcal{O}_V \oplus \mathcal{O}_V(-nH_V)) \xrightarrow{f} V.$$

Let Δ_n be the unique section corresponding to $\mathcal{O}_{X_n}(1)$. Define $M_n := \Delta_n + (n+2)f^*H_V$. By construction, we have $K_{X_n} + \Delta_n + M_n \sim 0$. We can easily check that (X_n, Δ_n) is a dlt pair and M_n is an ample Cartier divisor. We also see that $(-K_{X_n} \cdot A_n) \geq n$ for every n and every ample Cartier divisor A_n on X_n . Indeed, we may write $A_n \sim a_n\Delta_n + b_nf^*H_V$ for some $a_n \geq 1$ and $b_n \geq 1$ (cf. [27, III, Exercise 12.5]). Then

$$(-K_{X_n} \cdot A_n) = (\Delta_n + M_n) \cdot A_n > (nf^*H_V \cdot A_n) \geq (nf^*H_V \cdot a_n\Delta_n) \geq n.$$

This fact shows that the set $\{X_n\}_{n \geq 1}$ is unbounded.

5. Boundedness results

In this section, we prove some boundedness results. Before we start the main parts of this section, we recall the notion of the Iitaka volume for \mathbb{Q} -Cartier divisors and we prove Theorem 1.4.

Definition 5.1 (Iitaka volume, [39, Definition 1.1]). Let X be a normal projective variety, and let D be a \mathbb{Q} -Cartier divisor on X such that the Iitaka dimension $\kappa(X, D)$ is nonnegative. Then the *Iitaka volume* of D , denoted by $\text{Ivol}(D)$, is defined by

$$\text{Ivol}(D) := \limsup_{m \rightarrow \infty} \frac{\dim H^0(X, \mathcal{O}_X(\lfloor mD \rfloor))}{m^{\kappa(X, D)}/\kappa(X, D)!}.$$

Proof of Theorem 1.4. The theorem follows from Theorem 1.3 and [5, Theorem 1.3]. With notations as in Theorem 1.3, the divisor $K_Z + \Delta_Z + \mathbf{N}_Z$ is ample and \mathbb{Q} -Cartier. Therefore, $\text{Ivol}(K_X + \Delta + M) = (K_Z + \Delta_Z + \mathbf{N}_Z)^{\dim Z} \in \mathbb{Q}_{>0}$. □

From now on, we prove some boundedness results, and we prove Theorems 1.6 and Theorem 1.7.

5.1. Boundedness of complements

In this subsection, we prove Theorem 1.6.

Lemma 5.2. *Let $\mathfrak{R} \subset [0, 1]$ be a finite set of rational numbers, and let $\Phi(\mathfrak{R})$ be the hyperstandard set associated to \mathfrak{R} . Let q be a positive integer such that $qa \in \mathbb{Z}$ for every $a \in \mathfrak{R}$. Then $b - \{-qmb\} \geq 0$ for every $b \in \Phi(\mathfrak{R})$ and every positive integer m .*

Proof. We may write $b = 1 - \frac{a}{r}$ for some $a \in \mathfrak{R}$ and $r \in \mathbb{Z}_{>0}$. Then

$$b - \{-qmb\} = 1 - \frac{a}{r} - \left\{-qm\left(1 - \frac{a}{r}\right)\right\} = 1 - \frac{a}{r} - \left\{\frac{qma}{r}\right\}.$$

Since $qa \in \mathbb{Z}$ and $m \in \mathbb{Z}$, we have $\left\{\frac{qma}{r}\right\} \leq \frac{r-1}{r}$. Since $a \leq 1$, we have

$$1 - \frac{a}{r} - \left\{\frac{qma}{r}\right\} \geq 1 - \frac{1}{r} - \frac{r-1}{r} \geq 0.$$

Therefore, $b - \{-qmb\} \geq 0$. □

Lemma 5.3. *Let (X, Δ) be a subpair such that X is projective, and let D be a Cartier divisor on X . Let n be a positive integer, and let V_1, \dots, V_l be subvarieties of X . Suppose that, for each $1 \leq i \leq l$, there is $D_i \in |D|$ such that $(X, \Delta + \frac{1}{n}D_i)$ is sub-lc on a neighborhood of V_i . Then, for every general member $D' \in |D|$ the subpair $(X, \Delta + \frac{1}{n}D')$ is sub-lc on a neighborhood of $\bigcup_{i=1}^l V_i$.*

Proof. We take a resolution $f: Y \rightarrow X$ of the linear system $|D|$, i.e., $f^*|D| = |M| + F$, where M is a base point free Cartier divisor and F is the fixed divisor. Replacing X by Y and replacing Δ, D and V_1, \dots, V_l accordingly, we may assume that the movable part of $|D|$ is base point free. By the hypothesis, F satisfies the property that $(X, \Delta + \frac{1}{n}F)$ is sub-lc on a neighborhood of $\bigcup_{i=1}^l V_i$. Thus, Lemma 5.3 holds true. \square

The following theorem is the main result of this subsection.

Theorem 5.4. *Let d be a positive integer. Let $\mathfrak{R} \subset [0, 1]$ be a finite set of rational numbers and $\Phi(\mathfrak{R})$ the hyperstandard set associated to \mathfrak{R} . Then there exists a positive integer n , depending only on d and \mathfrak{R} , satisfying the following. Let (X, Δ) be a projective dlt pair such that*

- $\dim X = d$,
- the coefficients of Δ belong to $\Phi(\mathfrak{R})$ and
- $-(K_X + \Delta)$ is nef and log big.

Then there is an effective \mathbb{Q} -divisor Ξ such that $(X, \Delta + \Xi)$ is lc and $n(K_X + \Delta + \Xi) \sim 0$.

Proof. We prove the theorem by induction on the dimension of X .

If (X, Δ) is klt, then the existence of n is proved by Birkar [4]. So we may assume that (X, Δ) is not klt. Let p be the minimum positive integer such that $pa \in \mathbb{Z}$ for all $a \in \mathfrak{R}$.

Since (X, Δ) is dlt, there is a log resolution $f: \tilde{X} \rightarrow X$ of (X, Δ) such that f is an isomorphism over an open subset $U \subset X$ containing all the generic points of lc centers of (X, Δ) . We put $\tilde{D} = f^*(K_X + \Delta)$. We put $\tilde{S} = f_*^{-1}S$ for any component S of $[\Delta]$. We prove Theorem 5.4 in two steps.

Step 1. In this step, we prove that the morphism

$$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\lfloor -mp\tilde{D} \rfloor)) \longrightarrow H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(\lfloor -mp\tilde{D} \rfloor|_{\tilde{S}}))$$

defined with Definition 2.9 is surjective for every component S of $[\Delta]$ and every positive integer m . We closely follow the proof of Lemma 2.16.

We can write

$$K_{\tilde{X}} + \tilde{\Delta} = f^*(K_X + \Delta) + \tilde{E},$$

where $\tilde{\Delta} \geq 0$ and $\tilde{E} \geq 0$ have no common components. Then $-\tilde{D}$ is nef and log big with respect to $(\tilde{X}, \tilde{\Delta})$, and we can write

$$K_{\tilde{X}} + \tilde{\Delta} - (mp + 1)\tilde{D} = -mp\tilde{D} + \tilde{E}.$$

We can write $-mp\tilde{D} = \lfloor -mp\tilde{D} \rfloor + \{-mpf_*^{-1}\Delta\} + \tilde{E}'$ with an effective f -exceptional \mathbb{Q} -divisor \tilde{E}' . We put

$$\tilde{\Delta}' = \tilde{\Delta} - \{-mpf_*^{-1}\Delta\} - \tilde{S}.$$

By Lemma 5.2, we have $\tilde{\Delta}' \geq 0$ and

$$K_{\tilde{X}} + \tilde{S} + \tilde{\Delta}' - (mp + 1)\tilde{D} = \lfloor -mp\tilde{D} \rfloor + \tilde{E} + \tilde{E}'.$$

We define a Weil divisor \tilde{F} on \tilde{X} so that

$$\text{coeff}_P(\tilde{F}) = \begin{cases} 0 & (\text{coeff}_P(\tilde{\Delta}' - \{(\tilde{E} + \tilde{E}')\}) \geq 0) \\ 1 & (\text{coeff}_P(\tilde{\Delta}' - \{(\tilde{E} + \tilde{E}')\}) < 0) \end{cases}$$

for every prime divisor P . We put

$$\tilde{B} = \tilde{\Delta}' - \{(\tilde{E} + \tilde{E}')\} + \tilde{F} \quad \text{and} \quad \tilde{G} = \lfloor (\tilde{E} + \tilde{E}') \rfloor + \tilde{F}.$$

Then we have

$$K_{\tilde{X}} + \tilde{S} + \tilde{B} - (mp + 1)\tilde{D} = \lfloor -mp\tilde{D} \rfloor + \tilde{G}.$$

By the definition, we can check that \tilde{B} is a boundary divisor, $\lfloor \tilde{B} \rfloor \leq \lfloor \tilde{\Delta}' \rfloor$ and \tilde{G} is an effective f -exceptional Weil divisor. Then $-\tilde{D}$ is nef and log big with respect to $(\tilde{X}, \tilde{S} + \tilde{B})$. By the Kodaira type vanishing theorem [12, Lemma 1.5], we see that

$$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\lfloor -mp\tilde{D} \rfloor + \tilde{G})) \longrightarrow H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}((\lfloor -mp\tilde{D} \rfloor + \tilde{G})|_{\tilde{S}}))$$

is surjective. Since \tilde{G} is f -exceptional, there are natural isomorphisms

$$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\lfloor -mp\tilde{D} \rfloor)) \simeq H^0(X, \mathcal{O}_X(\lfloor -mp(K_X + \Delta) \rfloor)) \simeq H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\lfloor -mp\tilde{D} \rfloor + \tilde{G})).$$

From the following diagram

$$\begin{array}{ccc} H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\lfloor -mp\tilde{D} \rfloor + \tilde{G})) & \twoheadrightarrow & H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}((\lfloor -mp\tilde{D} \rfloor + \tilde{G})|_{\tilde{S}})) \\ \uparrow \simeq & & \uparrow \\ H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\lfloor -mp\tilde{D} \rfloor)) & \longrightarrow & H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(\lfloor -mp\tilde{D} \rfloor|_{\tilde{S}})), \end{array}$$

the lower horizontal morphism is surjective. We finish this step.

For each component S of $\lfloor \Delta \rfloor$, we define a dlt pair (S, Δ_S) by divisorial adjunction $K_S + \Delta_S = (K_X + \Delta)|_S$. By Remark 2.8, there exists a finite set $\mathfrak{S} \subset [0, 1]$ of rational numbers, depending only on d and \mathfrak{R} , such that all the coefficients of Δ_S belong to the hyperstandard set associated to \mathfrak{S} . By the induction hypothesis of Theorem 5.4, there is an n' , depending only on $d - 1$ and \mathfrak{S} , such that, for every (S, Δ_S) where S is a component of $\lfloor \Delta \rfloor$, there is

$$\Delta_S^+ \geq \Delta_S$$

such that (S, Δ_S^+) is lc and $n'(K_S + \Delta_S^+) \sim 0$.

Step 2. We put $n = n'p$. From now on, we prove that the n satisfies the property of Theorem 5.4.

Pick a component S of $\lfloor \Delta \rfloor$, put $\tilde{S} = f_*^{-1}S$ and take a \mathbb{Q} -divisor $\Delta_S^+ \geq \Delta_S$ on S such that (S, Δ_S^+) is lc and $n'(K_S + \Delta_S^+) \sim 0$. We put

$$C_S = n(\Delta_S^+ - \Delta_S) \sim -n(K_S + \Delta_S).$$

We put $C_{\tilde{S}} = f_S^*C_S$, where $f_S = f|_{\tilde{S}}: \tilde{S} \rightarrow S$. Then $-n\tilde{D}|_{\tilde{S}} \sim C_{\tilde{S}}$. We put $\tilde{R} = \{-n\tilde{D}\}$. Then

$$-n\tilde{D}|_{\tilde{S}} = \lfloor -n\tilde{D} \rfloor|_{\tilde{S}} + \tilde{R}|_{\tilde{S}},$$

and $\tilde{R}|_{\tilde{S}}$ is well-defined as a \mathbb{Q} -divisor on \tilde{S} . We have $\lfloor \tilde{R} \rfloor|_{\tilde{S}} = 0$ since $\text{Supp}(\tilde{R} + \tilde{S})$ is snc. Therefore, we have

$$\lfloor -n\tilde{D} \rfloor|_{\tilde{S}} = \lfloor -n\tilde{D} \rfloor|_{\tilde{S}} \quad \text{and} \quad \tilde{R}|_{\tilde{S}} = \{-n\tilde{D}\}|_{\tilde{S}}.$$

Therefore, we have

$$\lfloor -n\tilde{D} \rfloor|_{\tilde{S}} \sim \lfloor C_{\tilde{S}} \rfloor \quad \text{and} \quad \tilde{R}|_{\tilde{S}} = \{C_{\tilde{S}}\}.$$

By Step 1, we can find an effective Weil divisor $\tilde{C} \sim \lfloor -n\tilde{D} \rfloor$ such that $\tilde{C}|_{\tilde{S}} = \lfloor C_{\tilde{S}} \rfloor$. We put $C = f_*(\tilde{C} + \tilde{R})$. Since $\tilde{C} + \tilde{R} \sim -n\tilde{D}$ and $\tilde{D} = f^*(K_X + \Delta)$, we can easily check $C \sim -n(K_X + \Delta)$ and $\tilde{C} + \tilde{R} = f^*C$.

By definition of C , we see that

$$C|_S = f_{\tilde{S}*}((\tilde{C} + \tilde{R})|_{\tilde{S}}) = f_{\tilde{S}*}C_{\tilde{S}} = C_S = n(\Delta_S^+ - \Delta_S).$$

From this and inversion of adjunction [35], the pair $(X, \Delta + \frac{1}{n}C)$ is lc on a neighborhood of S . We define $\tilde{\Gamma}$ by $K_{\tilde{X}} + \tilde{\Gamma} = f^*(K_X + \Delta)$. Since $f^*C = \tilde{C} + \tilde{R}$, we see that the subpair $(\tilde{X}, \tilde{\Gamma} + \frac{1}{n}(\tilde{C} + \tilde{R}))$ is sub-lc on a neighborhood of $f^{-1}(S)$.

In this way, for every component S of $[\Delta]$, we may find $\tilde{C} \in |[-n\tilde{D}]|$ such that the subpair $(\tilde{X}, \tilde{\Gamma} + \frac{1}{n}\tilde{R} + \frac{1}{n}\tilde{C})$ is sub-lc on a neighborhood of $f^{-1}(S)$. By Lemma 5.3, there is

$$\tilde{M} \in |[-n\tilde{D}]|$$

such that $(\tilde{X}, \tilde{\Gamma} + \frac{1}{n}\tilde{R} + \frac{1}{n}\tilde{M})$ is sub-lc on a neighborhood of $f^{-1}([\Delta])$. By construction, we have $\tilde{M} + \tilde{R} \sim -nf^*(K_X + \Delta)$. By putting

$$M = f_*(\tilde{M} + \tilde{R}),$$

we have $M \sim -n(K_X + \Delta)$ and the pair $(X, \Delta + \frac{1}{n}M)$ is lc on a neighborhood of $[\Delta]$.

Finally, if $(X, \Delta + \frac{1}{n}M)$ is not lc, then there is $\epsilon \in (0, \frac{1}{n})$ such that the non-klt locus of $(X, \Delta + (\frac{1}{n} - \epsilon)M)$ has at least two connected components. Then we get a contradiction by the connectedness principle for non-klt locus [37, 17.4 Theorem]. Thus, $(X, \Delta + \frac{1}{n}M)$ is lc, and therefore $\Xi := \frac{1}{n}M$ is the desired divisor. □

Proof of Theorem 1.6. We will freely use the notations in Theorem 1.6. We may assume that $K_X + \Delta + M$ is not big. By the argument similar to the proof of Theorem 1.3, the global ACC [7, Theorem 1.6] implies the existence of a positive integer m , depending only on d and Φ , such that $m\Delta'_F$ is a Weil divisor and mM is Cartier. By Theorem 4.1, we have $n_1(K_F + \Delta'_F + M'_F) \sim 0$ for some positive integer n_1 which depends only on d and m . If there exists $n_2 \in \mathbb{Z}_{>0}$ depending only on d and m such that $(F, \Delta'_F + \frac{1}{n_2}B'_F)$ is lc for some $B'_F \in |-n_2(K_F + \Delta'_F)|$, then $n := n_1n_2$ is the desired positive integer. In this way, we only need to prove the existence of the above $n_2 \in \mathbb{Z}_{>0}$.

We run a $(K_X + \Delta + 3dmM)$ -MMP with scaling of an ample divisor. By the length of extremal rays ([15, Section 18] or [17, Theorem 4.6.2]) and the same argument as in the proof of Theorem 3.16, we get a sequence of steps of a $(K_X + \Delta + 3dmM)$ -MMP to a good minimal model

$$(X, \Delta, 3dm\overline{M}) \dashrightarrow (\tilde{X}, \tilde{\Delta}, 3dm\overline{M})$$

such that

- $X \dashrightarrow \tilde{X}$ is a sequence of steps of a $(K_X + \Delta + M)$ -MMP, and
- defining \tilde{M} to be the birational transform of M on \tilde{X} , then $m\tilde{M}$ is a nef Cartier divisor on \tilde{X} that is log big with respect to $(\tilde{X}, \tilde{\Delta})$.

With the argument in [29, Proof of Lemma 2.14], we may construct a sequence of steps of a $(K_{\tilde{X}} + \tilde{\Delta} + \tilde{M})$ -MMP with scaling of $(3dm - 1)\tilde{M}$ terminating with a good minimal model

$$(\tilde{X}, \tilde{\Delta}, \overline{M}) \dashrightarrow (\tilde{X}', \tilde{\Delta}', \overline{M}).$$

Let \tilde{M}' be the birational transform of M on \tilde{X}' . By construction, $K_{\tilde{X}'} + \tilde{\Delta}' + (1+t)\tilde{M}'$ is nef for some $t > 0$. Moreover, taking a common resolution of $\tilde{X} \dashrightarrow \tilde{X}'$ and using the negativity lemma, we see that \tilde{M}' is log big with respect to $(\tilde{X}', \tilde{\Delta}')$. Let $\tilde{\pi}: \tilde{X}' \rightarrow Z$ be the contraction induced by $K_{\tilde{X}'} + \tilde{\Delta}' + \tilde{M}'$, and let \tilde{F} be the general fiber of $\tilde{\pi}$. We put $\tilde{\Delta}'_{\tilde{F}} = \tilde{\Delta}'|_{\tilde{F}}$ and $\tilde{M}'_{\tilde{F}} = \tilde{M}'|_{\tilde{F}}$.

Since $K_{\tilde{X}'} + \tilde{\Delta}' + \tilde{M}' \sim_{\mathbb{Q},Z} 0$ and $K_{\tilde{X}'} + \tilde{\Delta}' + (1+t)\tilde{M}'$ is nef for some $t > 0$, we see that the divisor

$$\tilde{M}' \sim_{\mathbb{Q},Z} -(K_{\tilde{X}'} + \tilde{\Delta}')$$

is nef and log big over Z with respect to $(\tilde{X}', \tilde{\Delta}')$. By Theorem 5.4, there exists a positive integer n_2 , depending only on d and m , such that $(\tilde{F}, \tilde{\Delta}'_{\tilde{F}} + \frac{1}{n_2}\tilde{B}'_{\tilde{F}})$ is an lc pair for some $\tilde{B}'_{\tilde{F}} \in |-n_2(K_{\tilde{F}} + \tilde{\Delta}'_{\tilde{F}})|$. Because $X \dashrightarrow X'$ and $X \dashrightarrow \tilde{X}'$ are sequences of steps of $(K_X + \Delta + M)$ -MMP to good minimal models, the induced birational map $X' \dashrightarrow \tilde{X}'$ is isomorphic in codimension one and therefore so is $F \dashrightarrow \tilde{F}$. Then $(F, \Delta'_F + \frac{1}{n_2}B'_F)$ is lc for some $B'_F \in |-n_2(K_F + \Delta'_F)|$. Hence, $n := n_1n_2$ satisfies the conditions of Theorem 1.6. □

The following example shows that the condition of the log bigness of the nef divisors in Theorem 1.2 and Theorem 1.6 cannot be relaxed to the bigness.

Example 5.5. Let (X, Δ) be a projective log smooth dlt pair such that Δ is a reduced divisor, $H^1(X, \mathcal{O}_X) \neq 0$ and $-(K_X + \Delta)$ is nef and big. For example, take an elliptic curve E with a very ample divisor H and define

$$X := \mathbb{P}_E(\mathcal{O}_E \oplus \mathcal{O}_E(-H)).$$

For each $n \geq 1$, let D_n be a Cartier divisor on X such that $nD_n \sim 0$ and $lD_n \not\sim 0$ for all $0 < l < n$. We define $L_n := -(K_X + \Delta) + D_n$, and we consider the set $\{(X, \Delta, \overline{L_n})\}_{n \geq 1}$. If the log bigness of M_i in Theorem 1.2 or Theorem 1.6 can be relaxed to the bigness, then there is a positive integer m such that $mD_n = m(K_X + \Delta + L_n) \sim 0$ for all n , which contradicts the definition of D_n . Therefore, the log bigness of M_i in Theorem 1.2 and Theorem 1.6 cannot be relaxed to the bigness.

5.2. Effective finite generation

In this subsection, we prove Theorem 1.7.

We will first prove the effective base point free theorem for special generalized dlt pairs (Theorem 5.8). For the proof, we need the base point free theorem for quasi-log schemes [17, Theorem 6.5.1] (see also [1, Theorem 5.1]).

Definition 5.6 (Quasi-log scheme, [17, Definition 6.2.2], see also [1, Definition 4.1]). A quasi-log scheme is a scheme X endowed with an \mathbb{R} -Cartier divisor (or \mathbb{R} -line bundle) ω on X , a closed subscheme $X_{-\infty} \subseteq X$ and a finite collection $\{C\}$ of reduced and irreducible subschemes of X such that there is a proper morphism $f: (Y, B_Y) \rightarrow X$ from a globally embedded simple normal crossing pair satisfying the following properties:

- $f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$,
- the natural map $\mathcal{O}_X \rightarrow f_*\mathcal{O}_Y([\!-\!(B_Y^{<1})\!])$ induces an isomorphism

$$\mathcal{I}_{X_{-\infty}} \xrightarrow{\cong} f_*\mathcal{O}_Y([\!-\!(B_Y^{<1})\!] - \lfloor B_Y^{>1} \rfloor),$$

where $\mathcal{I}_{X_{-\infty}}$ is the defining ideal sheaf of $X_{-\infty}$, and

- the collection of reduced and irreducible subschemes $\{C\}$ coincides with the images of the strata of (Y, B_Y) that are not included in $X_{-\infty}$.

We simply write $[X, \omega]$ to denote the above data

$$(X, \omega, f: (Y, B_Y) \rightarrow X)$$

if there is no risk of confusion. We sometimes use $\text{Nqlc}(X, \omega)$ to denote $X_{-\infty}$.

Theorem 5.7 [15, Theorem 6.5.1]. *Let $[X, \omega]$ be a quasi-log scheme and $\pi: X \rightarrow S$ a projective morphism to a scheme S . Let D be a π -nef Cartier divisor on X such that $pD - \omega$ is π -ample for some positive integer p and $\mathcal{O}_{X_{-\infty}}(mD|_{X_{-\infty}})$ is $\pi|_{X_{-\infty}}$ -generated for all $m \gg 0$. Then there is a positive integer m_0 such that $\mathcal{O}_X(mD)$ is π -generated for every integer m such that $m \geq m_0$.*

Theorem 5.8. *Let d and p be positive integers. Then there exists a positive integer m , depending only on d and p , satisfying the following. Let (X, Δ, \mathbf{M}) be a generalized dlt pair such that*

- $\dim X = d$,
- \mathbf{M} is a \mathbb{Q} -b-divisor and $p(K_X + \Delta + \mathbf{M}_X)$ is nef and Cartier and
- there is a log resolution $f: \tilde{X} \rightarrow X$ of (X, Δ) such that
 - f is an isomorphism over an open subset $U \subset X$ containing all the generic points of generalized lc centers of (X, Δ, \mathbf{M}) ,
 - \mathbf{M} descends to \tilde{X} and
 - writing $K_{\tilde{X}} + \tilde{\Delta} + \mathbf{M}_{\tilde{X}} = f^*(K_X + \Delta + \mathbf{M}_X) + \tilde{E}$, where $\tilde{\Delta} \geq 0$ and $\tilde{E} \geq 0$ have no common components, then $\mathbf{M}_{\tilde{X}}$ is log big with respect to $(\tilde{X}, \tilde{\Delta})$.

Then $m(K_X + \Delta + \mathbf{M}_X)$ is base point free.

Proof. We prove it by induction on the dimension of X .

If (X, Δ, \mathbf{M}) is generalized klt, then we may find a \mathbb{Q} -divisor $E \geq 0$ on X such that $\mathbf{M}_X - E$ is ample and $(X, \Delta + E)$ is klt. Then Theorem 5.8 follows from the effective base point free theorem for klt pairs [36]. Thus, we may assume that (X, Δ, \mathbf{M}) is not generalized klt.

Step 1. In this step, we define the structure of a quasi-log scheme on X .

Let A be an ample \mathbb{Q} -divisor on X . Since $\mathbf{M}_{\tilde{X}}$ is nef and big, rescaling A if necessary, we can find

$$\tilde{R} := \tilde{R}_1 + \tilde{R}_2 \in |\mathbf{M}_{\tilde{X}} - f^*A|_{\mathbb{Q}}$$

such that \tilde{R}_1 is ample and $\tilde{R}_2 \geq 0$. We take a log resolution $g: Y \rightarrow \tilde{X}$ of $(\tilde{X}, \tilde{\Delta} + \tilde{E} + \tilde{R})$, where $\tilde{\Delta}$ and \tilde{E} are as in Theorem 5.8, such that $-E_Y$ is g -ample for a g -exceptional \mathbb{Q} -divisor $E_Y \geq 0$. Rescaling E_Y , we may assume that $-E_Y + g^*\tilde{R}_1$ is ample. We put $h = f \circ g: Y \rightarrow X$. Then, for every $0 < t \ll 1$ we have

$$\begin{aligned} g^*\mathbf{M}_{\tilde{X}} - th^*A &\sim_{\mathbb{Q}} (1-t)g^*\mathbf{M}_{\tilde{X}} + tg^*\tilde{R} \\ &= (1-t)g^*\mathbf{M}_{\tilde{X}} + t(g^*\tilde{R}_1 - E_Y) + tE_Y + tg^*\tilde{R}_2. \end{aligned}$$

We define Δ_Y by $K_Y + \Delta_Y = g^*(K_{\tilde{X}} + \tilde{\Delta} - \tilde{E})$. By choosing $t > 0$ sufficiently small, we can find

$$R_Y \sim_{\mathbb{Q}} g^*\mathbf{M}_{\tilde{X}} - th^*A$$

such that

- R_Y is the sum of an effective ample \mathbb{Q} -divisor and an effective \mathbb{Q} -divisor,
- $(Y, \Delta_Y + R_Y)$ is log smooth and
- $\lfloor (\Delta_Y + R_Y) \rfloor = \lfloor \Delta_Y \rfloor$.

By perturbation of the coefficients of $\Delta_Y + R_Y$, we may assume that no coefficient of $\Delta_Y + R_Y$ is one. We put

$$B_Y = \Delta_Y + R_Y \quad \text{and} \quad B = h_*B_Y.$$

Then $K_Y + B_Y \sim_{\mathbb{Q}} (K_Y + \Delta_Y) + g^*\mathbf{M}_{\tilde{X}} - th^*A$, and therefore we have

$$K_Y + B_Y \sim_{\mathbb{Q}} g^*(K_{\tilde{X}} + \tilde{\Delta} - \tilde{E}) + g^*\mathbf{M}_{\tilde{X}} - h^*(tA) = h^*(K_X + \Delta + \mathbf{M}_X - tA).$$

From this, we have

$$K_Y + B_Y = h^*(K_X + B) \quad \text{and} \quad K_X + \Delta + \mathbf{M}_X \sim_{\mathbb{Q}} K_X + B + tA. \tag{I}$$

Recalling that no coefficient of B_Y is one, we have

$$\lceil (-B_Y^{<1}) \rceil - \lfloor B_Y^{>1} \rfloor = -\lfloor B_Y \rfloor = -\lfloor \Delta_Y \rfloor$$

and the right-hand side is the sum of $-g^* \lfloor \tilde{\Delta} \rfloor$ and an effective h -exceptional divisor. From this, we can check that

$$h_* \mathcal{O}_Y(\lceil (-B_Y^{<1}) \rceil - \lfloor B_Y^{>1} \rfloor) = \mathcal{O}_X(-\lfloor \Delta \rfloor). \tag{II}$$

As in (I) and (II), we defined an effective \mathbb{Q} -divisor B on X , a log resolution $h: Y \rightarrow X$ of (X, B) and an \mathbb{Q} -divisor B_Y defined with $K_Y + B_Y = h^*(K_X + B)$ such that

- $(K_X + \Delta + \mathbf{M}_X) - (K_X + B)$ is ample, and
- no coefficient of B_Y is one and $h_* \mathcal{O}_Y(\lceil (-B_Y^{<1}) \rceil - \lfloor B_Y^{>1} \rfloor) = \mathcal{O}_X(-\lfloor \Delta \rfloor)$.

From now on, we put $L = K_X + \Delta + \mathbf{M}_X$.

Step 2. In this step, we prove the existence of a positive integer m' , depending only on d and p , such that $\mathcal{O}_X(m'L)$ is generated by global sections on a neighborhood of $\lfloor \Delta \rfloor$.

Pick any component S of $\lfloor \Delta \rfloor$, and put $\tilde{S} = f_*^{-1}S$. Let $(S, \Delta_S, \mathbf{M}')$ be the generalized dlt pair defined with divisorial adjunction. We have

$$L|_S \sim_{\mathbb{Q}} K_S + \Delta_S + \mathbf{M}'_S.$$

We define \mathbf{M}'' to be a \mathbb{Q} -b-divisor on S such that $\mathbf{M}''_S = L|_S - (K_S + \Delta_S + \mathbf{M}'_S)$ and \mathbf{M}'' descends to S . Then the generalized dlt pair

$$(S, \Delta_S, \mathbf{M}' + \mathbf{M}'')$$

satisfies the condition that $p(K_S + \Delta_S + \mathbf{M}'_S + \mathbf{M}''_S) = pL|_S$ is Cartier. It is easy to see that we may apply the induction hypothesis of Theorem 5.8 to $(S, \Delta_S, \mathbf{M}' + \mathbf{M}'')$. Since $K_S + \Delta_S + \mathbf{M}'_S + \mathbf{M}''_S = pL|_S$, applying the induction hypothesis of Theorem 5.8 to $(S, \Delta_S, \mathbf{M}' + \mathbf{M}'')$, there exists a positive integer m'' , depending only on d and p , such that $m''L|_S$ is base point free.

By our assumption of $\tilde{\Delta}$ and \tilde{E} , we see that $\tilde{\Delta} + \lceil \tilde{E} \rceil - \tilde{E}$ is a boundary \mathbb{Q} -divisor such that $\lfloor \tilde{\Delta} \rfloor = \lfloor (\tilde{\Delta} + \lceil \tilde{E} \rceil - \tilde{E}) \rfloor$ and

$$K_{\tilde{X}} + \tilde{\Delta} + \lceil \tilde{E} \rceil - \tilde{E} + \mathbf{M}_{\tilde{X}} + (m''p - 1)f^*L = m''pf^*L + \lceil \tilde{E} \rceil.$$

Then $\mathbf{M}_{\tilde{X}} + (m''p - 1)f^*L$ is nef and log big with respect to $(\tilde{X}, \tilde{\Delta} + \lceil \tilde{E} \rceil - \tilde{E})$. From the fact and the Kodaira type vanishing theorem [12, Lemma 1.5], we have

$$H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(m''pf^*L + \lceil \tilde{E} \rceil - \tilde{S})) = 0.$$

By the same argument as in [12] or the proof of Lemma 2.16, we obtain the diagram.

$$\begin{array}{ccc} H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(m''pf^*L + \lceil \tilde{E} \rceil)) & \twoheadrightarrow & H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}((m''pf^*L + \lceil \tilde{E} \rceil)|_{\tilde{S}})) \\ \uparrow \simeq & & \uparrow \\ H^0(X, \mathcal{O}_X(m''pL)) & \longrightarrow & H^0(S, \mathcal{O}_S(m''pL|_S)). \end{array}$$

Therefore, the base locus of $m''pL$ does not intersect S . Since S is a component of $[\Delta]$ and $m''p$ does not depend on S , we see that $m' := m''p$ is the desired positive integer.

Step 3. In this step, we construct a contraction $\pi : X \rightarrow Z$ induced by $m'L$ such that $m'L \sim \pi^*D_Z$ for some ample Cartier divisor D_Z on Z .

By Step 1, we get a quasi-log scheme $(X, K_X + B, h : (Y, B_Y) \rightarrow X)$ such that

$$m'L - (K_X + B) = (m' - 1)L + (K_X + \Delta + \mathbf{M}_X) - (K_X + B)$$

is ample. By Step 2 and $[\Delta] = \text{Nqlc}(X, K_X + B)$, we see that

$$\mathcal{O}_{\text{Nqlc}(X, K_X + B)}(m'L|_{\text{Nqlc}(X, K_X + B)})$$

is globally generated. Since X is projective, applying Theorem 5.7 to $[X, K_X + B]$ and $m'L$, we get a contraction $\pi : X \rightarrow Z$ induced by $m'L$ and a positive integer m_0 such that both $m_0m'L$ and $(m_0 + 1)m'L$ are the pullbacks of Cartier divisors on Z . Therefore, $m'L \sim \pi^*D_Z$ for some Cartier divisor D_Z on Z . By construction, D_Z is ample.

Step 4. From this step, we prove the existence of m in Theorem 5.8. In this step, we treat the case when there is a component S' of $[\Delta]$ dominating Z .

By Lemma 2.16, the restriction $\pi_{S'} := \pi|_{S'} : S' \rightarrow Z$ is a contraction. By Step 3, we have $\pi_{S'^*}\mathcal{O}_{S'}(m'L|_{S'}) \simeq \mathcal{O}_Z(D_Z)$. By Step 2, it follows that $m'L|_{S'}$ is base point free. Then D_Z is base point free, and therefore $m'L$ is base point free.

Step 5. Finally, we treat the case when $[\Delta]$ is vertical over Z .

We put $L' = m'L$. For each positive integer l , let $Z_l \subset Z$ be the base locus of $|lD_Z|$. By the relation $L' \sim \pi^*D_Z$, we see that $\pi^{-1}(Z_l)$ is disjoint from $[\Delta]$ for all l .

In the rest of the proof, we closely follow [14, Proof of Theorem 1.1]. We recall the property that the non-klt locus of (X, B) and the non-lc locus of (X, B) are the same and they coincide with $[B] = [\Delta]$ (see Step 1). We put

$$A'_l = \frac{3}{2}lL' - (K_X + B)$$

for each positive integer l . By Step 1, it follows that A'_l is ample for all l . Fix an arbitrary positive integer l , and suppose $Z_l \neq \emptyset$. Let C_0, C_1, \dots, C_d be general members of $|lD_Z|$, and we put

$$C = \pi^*\left(\frac{1}{2}C_0 + C_1 + \dots + C_d\right).$$

Then $C \sim_{\mathbb{Q}} (d + \frac{1}{2})l\pi^*D_Z$, and the non-lc locus of $(X, B + C)$ is the disjoint union of $\pi^{-1}(Z_l)$ and $[B]$.

Pick an arbitrary irreducible component V of $\pi^{-1}(Z_l)$ such that $\pi(V)$ has the maximal dimension. Then V is disjoint from $[B]$. Let $\tilde{h} : \tilde{Y} \rightarrow X$ be a log resolution of $(X, B + C)$ such that $\tilde{h}^{-1}(\pi^{-1}(\pi(V)))$ is an snc divisor. We may write

$$K_{\tilde{Y}} = \tilde{h}^*(K_X + B) + \sum_i e_i \tilde{E}_i$$

for some $e_i \in \mathbb{Q}$. Let γ be the lc threshold of C with respect to (X, B) over the generic point of $\pi(V)$. It follows that $\gamma \in (0, 1) \cap \mathbb{Q}$ because (X, B) is klt on $X \setminus [B]$ and $\pi(V)$ is disjoint from $\pi([B])$. We may

write $\gamma\tilde{h}^*C = \sum_i \gamma_i \tilde{E}_i$ for some $\gamma_i \in \mathbb{Q}_{\geq 0}$. Then we have

$$\begin{aligned} \tilde{h}^*\pi^*((d+2)lD_Z) &\sim_{\mathbb{Q}} \frac{3}{2}\tilde{h}^*(lL') + \tilde{h}^*\pi^*((d+\frac{1}{2})lD_Z) \\ &\sim_{\mathbb{Q}} \tilde{h}^*(K_X + B + A'_1) + \gamma\tilde{h}^*C + (1-\gamma)\tilde{h}^*C \\ &\sim_{\mathbb{Q}} K_{\tilde{Y}} + \sum_i (\gamma_i - e_i)\tilde{E}_i + \tilde{h}^*A'_1 + (1-\gamma)\tilde{h}^*C. \end{aligned}$$

We can write

$$\sum_i [(\gamma_i - e_i)]\tilde{E}_i = \tilde{F} + \tilde{G}_1 + \tilde{G}_2 - \tilde{H},$$

where $\tilde{F}, \tilde{G}_1, \tilde{G}_2$, and \tilde{H} are all effective and they have no common components with each other such that

- the image of every component of \tilde{F} by $\pi \circ \tilde{h}$ is $\pi(V)$,
- the image of every component of \tilde{G}_1 by $\pi \circ \tilde{h}$ does not contain $\pi(V)$,
- the image of every component of \tilde{G}_2 by $\pi \circ \tilde{h}$ contains $\pi(V)$ but does not coincide with $\pi(V)$ and
- \tilde{H} is \tilde{h} -exceptional.

Note that $\tilde{h}_*^{-1}[B] \subset \text{Supp}\tilde{G}_1$, \tilde{F} is a nonzero reduced divisor by the definition of γ , and \tilde{G}_2 is reduced because every irreducible component of $(\pi \circ \tilde{h})(\tilde{G}_2)$ intersects $\pi(V)$ and $(\tilde{Y}, \sum_i (\gamma_i - e_i)\tilde{E}_i)$ is sub-lc over a neighborhood of the generic point of $\pi(V)$. By taking blowups along lc centers of (\tilde{Y}, \tilde{G}_2) if necessary, we may assume that any lc center T of (\tilde{Y}, \tilde{G}_2) does not satisfy $(\pi \circ \tilde{h})(T) \subset \pi(V)$.

For each $j \in \mathbb{Z}_{\geq 0}$, we define

$$\tilde{N}_j := \tilde{h}^*\pi^*((j+d+2)lD_Z) + \tilde{H} - \tilde{G}_1,$$

and consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{\tilde{Y}}(\tilde{N}_j - \tilde{F}) \longrightarrow \mathcal{O}_{\tilde{Y}}(\tilde{N}_j) \longrightarrow \mathcal{O}_{\tilde{F}}(\tilde{N}_j|_{\tilde{F}}) \longrightarrow 0.$$

By construction, we have

$$\tilde{N}_j - \tilde{F} \sim_{\mathbb{Q}} K_{\tilde{Y}} + \sum_i \{\gamma_i - e_i\}\tilde{E}_i + \tilde{G}_2 + \tilde{h}^*A'_1 + (1-\gamma)\tilde{h}^*C + j\tilde{h}^*\pi^*D_Z, \tag{♣}$$

and no lc center T of the lc pair $(\tilde{Y}, \sum_i \{\gamma_i - e_i\}\tilde{E}_i + \tilde{G}_2)$ satisfies $(\pi \circ \tilde{h})(T) \subset \pi(V)$. Here, the log canonicity of the pair $(\tilde{Y}, \sum_i \{\gamma_i - e_i\}\tilde{E}_i + \tilde{G}_2)$ follows from the facts that $(\tilde{Y}, \text{Supp}(\sum_i \tilde{E}_i + \tilde{G}_2))$ is log smooth, \tilde{G}_2 is reduced and \tilde{G}_2 and $\sum_i \{\gamma_i - e_i\}\tilde{E}_i$ have no common components. By [15, Theorem 6.3 (i)], the connecting morphism

$$(\pi \circ \tilde{h})_*\mathcal{O}_{\tilde{F}}(\tilde{N}_j|_{\tilde{F}}) \longrightarrow R^1(\pi \circ \tilde{h})_*\mathcal{O}_{\tilde{Y}}(\tilde{N}_j - \tilde{F})$$

is a zero map. Thus, we get the following exact sequence

$$0 \longrightarrow (\pi \circ \tilde{h})_*\mathcal{O}_{\tilde{Y}}(\tilde{N}_j - \tilde{F}) \longrightarrow (\pi \circ \tilde{h})_*\mathcal{O}_{\tilde{Y}}(\tilde{N}_j) \longrightarrow (\pi \circ \tilde{h})_*\mathcal{O}_{\tilde{F}}(\tilde{N}_j|_{\tilde{F}}) \longrightarrow 0.$$

Since A'_1 is ample, there is an $\tilde{A} \sim_{\mathbb{Q}} \tilde{h}^*A'_1$ such that $(\tilde{Y}, \sum_i \{\gamma_i - e_i\}\tilde{E}_i + \tilde{G}_2 + \tilde{A})$ is a log smooth lc pair. Because $(1-\gamma)\tilde{h}^*C + j\tilde{h}^*\pi^*D_Z$ is the pullback of an ample \mathbb{Q} -divisor on Z , we have

$$H^1(Z, (\pi \circ \tilde{h})_*\mathcal{O}_{\tilde{Y}}(\tilde{N}_j - \tilde{F})) = 0$$

by equation (♣) and [15, Theorem 6.3 (ii)]. From this, we get a surjective morphism

$$H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(\tilde{N}_j)) \longrightarrow H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(\tilde{N}_j|_{\tilde{F}})). \tag{♣}$$

From the definitions of γ and \tilde{F} , it follows that no component of \tilde{F} is a component of $\sum_i \{\gamma_i - e_i\} \tilde{E}_i$. By equation (♣) and the construction of \tilde{A} , we have

$$\tilde{N}_j|_{\tilde{F}} \sim_{\mathbb{Q}} K_{\tilde{F}} + \left(\sum_i \{\gamma_i - e_i\} \tilde{E}_i + \tilde{G}_2 + \tilde{A} \right) \Big|_{\tilde{F}} + \left((1 - \gamma) \tilde{h}^* C + j \tilde{h}^* \pi^* D_Z \right) \Big|_{\tilde{F}}$$

and $(\tilde{F}, (\sum_i \{\gamma_i - e_i\} \tilde{E}_i + \tilde{G}_2 + \tilde{A})|_{\tilde{F}})$ is a simple normal crossing pair. By the vanishing theorem for simple normal crossing pairs [14, Theorem 3.2 (b)], we see that

$$H^q(\pi(V), (\pi \circ \tilde{h})_* \mathcal{O}_{\tilde{F}}(\tilde{N}_j|_{\tilde{F}})) = 0$$

for all $q > 0$. Thus, we have

$$\begin{aligned} \dim H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(\tilde{N}_j|_{\tilde{F}})) &= \dim H^0(\pi(V), (\pi \circ \tilde{h})_* \mathcal{O}_{\tilde{F}}(\tilde{N}_j|_{\tilde{F}})) \\ &= \chi(\pi(V), (\pi \circ \tilde{h})_* \mathcal{O}_{\tilde{F}}(\tilde{N}_j|_{\tilde{F}})). \end{aligned}$$

By definition of \tilde{N}_j , we have

$$(\pi \circ \tilde{h})_* \mathcal{O}_{\tilde{F}}(\tilde{N}_j|_{\tilde{F}}) = (\pi \circ \tilde{h})_* \mathcal{O}_{\tilde{F}}(\tilde{H}|_{\tilde{F}} - \tilde{G}_1|_{\tilde{F}}) \otimes \mathcal{O}_{\pi(V)}((j + d + 2)lD_Z|_{\pi(V)}),$$

and $(\pi \circ \tilde{h})_* \mathcal{O}_{\tilde{F}}(\tilde{H}|_{\tilde{F}} - \tilde{G}_1|_{\tilde{F}})$ is a nonzero sheaf because $\tilde{G}_1|_{\tilde{F}}$ is zero at the generic point of $\pi(V)$ (see the definition of \tilde{G}_1). Therefore, $\dim H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(\tilde{N}_j|_{\tilde{F}}))$ is a nonzero polynomial of degree at most d with one variable j . Then there is $0 \leq j' \leq d$ such that

$$H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(\tilde{N}_{j'}|_{\tilde{F}})) \neq 0,$$

so $H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(\tilde{N}_{j'}))$ has an element not vanishing on \tilde{F} by equation (♣).

Since $\tilde{N}_{j'} = \tilde{h}^* \pi^* ((j' + d + 2)lD_Z) + \tilde{H} - \tilde{G}_1$ and $\text{Supp} \tilde{G}_1 \not\supset \tilde{F}$, we see that

$$H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(\tilde{h}^* \pi^* ((j' + d + 2)lD_Z) + \tilde{H}))$$

has an element not vanishing on \tilde{F} . Since \tilde{H} is \tilde{h} -exceptional, it follows that

$$H^0(X, \mathcal{O}_X(\pi^* ((j' + d + 2)lD_Z)))$$

has an element not vanishing on $\tilde{h}(\tilde{F})$. This implies that the base locus of $(j' + d + 2)lD_Z$ does not contain $\pi(V)$.

Recall that V is an arbitrary irreducible component of $\pi^{-1}(Z_l)$ such that $\pi(V)$ has the maximal dimension. From the above argument, it follows that $Z_{(2d+2)! \cdot l}$ does not contain any irreducible component of Z_l of the maximal dimension. Thus, we have

$$\dim Z_{(2d+2)! \cdot l} < \dim Z_l$$

for all $l \in \mathbb{Z}_{>0}$. Therefore, mL is base point free, where m is defined to be $((2d + 2)!)^d \cdot m'$.

We finish the proof of Theorem 5.8. □

Remark 5.9. By arguing inductively, for fixed integers d and p and every generalized dlt pair (X, Δ, \mathbf{M}) as in Theorem 5.8, we can check that $((2d + 2)!)^{d^2} p(K_X + \Delta + \mathbf{M}_X)$ is base point free. However, compared to [14, Remark 1.2], it looks far from optimal.

Lemma 5.10. *Let d and p be positive integers, and let v be a positive real number. Then there exists a positive integer m , depending only on d, p and v , satisfying the following. Let (X, Δ, \mathbf{M}) be a generalized lc pair such that*

- $\dim X = d$,
- $p\mathbf{M}$ is b -Cartier,
- $p(K_X + \Delta + \mathbf{M}_X)$ is an ample and base point free Cartier divisor and
- $\text{vol}(X, K_X + \Delta + \mathbf{M}_X) \leq v$.

Then $m(K_X + \Delta + \mathbf{M}_X)$ is very ample, and putting $R_l = H^0(X, \mathcal{O}_X(l(K_X + \Delta + \mathbf{M}_X)))$ for every $l \in \mathbb{Z}_{\geq 0}$ then $\bigoplus_{l \in \mathbb{Z}_{\geq 0}} R_{lm}$ is generated by R_m as a graded \mathbb{C} -algebra.

Proof. If Lemma 5.10 does not hold, then there is a sequence of generalized lc pairs $\{(X_i, \Delta_i, \mathbf{M}^i)\}_{i \geq 1}$ as in Lemma 5.10 such that if m_i is the smallest positive integer such that $m_i(K_{X_i} + \Delta_i + \mathbf{M}_{X_i}^i)$ is very ample and $H^0(X_i, \mathcal{O}_{X_i}(m_i(K_{X_i} + \Delta_i + \mathbf{M}_{X_i}^i)))$ generates $\bigoplus_{l \in \mathbb{Z}_{\geq 0}} H^0(X_i, \mathcal{O}_{X_i}(lm_i(K_{X_i} + \Delta_i + \mathbf{M}_{X_i}^i)))$, then $\lim_{i \rightarrow \infty} m_i = \infty$.

Since $p\mathbf{M}^i$ is b -Cartier and $p(K_{X_i} + \Delta_i + \mathbf{M}_{X_i}^i)$ is Cartier, $p\Delta_i$ is a Weil divisor for every i . By the effective birationality for generalized pairs [7, Theorem 1.3], there exists a positive integer p' , depending only on d and p , such that $|pp'(K_{X_i} + \Delta_i + \mathbf{M}_{X_i}^i)|$ defines a birational morphism. Then we may find $G_i \in |pp'(K_{X_i} + \Delta_i + \mathbf{M}_{X_i}^i)|$. Note that

- G_i is Cartier and base point free,
- $\text{vol}(X_i, G_i) \leq p^d p'^d v$ and
- $G_i - (K_{X_i} + \Delta_i)$ is the pushdown of a pseudo-effective \mathbb{Q} -divisor.

By [4, Proposition 4.4] and replacing $\{(X_i, \Delta_i, \mathbf{M}^i)\}_{i \geq 1}$ with a subsequence, we can find a projective morphism $h: V \rightarrow T$ of varieties, a reduced divisor D on V and a positive real number c , depending only on d, p and v , such that, for every i , there is a closed point $t_i \in T$ with the fibers V_{t_i} and D_{t_i} satisfying the following.

- (V_{t_i}, D_{t_i}) is log smooth,
- there is a birational map $\phi_i: X_i \dashrightarrow V_{t_i}$ such that if E_i is the sum of $\phi_{i*}G_i$ and the reduced ϕ_i^{-1} -exceptional divisor, then $\text{Supp} E_i \subset D_{t_i}$ and
- if $f_i: Y \rightarrow X_i$ and $f'_i: Y \rightarrow V_{t_i}$ is a common resolution of ϕ_i , then $f_i^*G_i \sim_{\mathbb{Q}, V_{t_i}} 0$ and all the coefficients of $f'_i f_i^*G_i$ are at most c .

Shrinking T and replacing $\{(X_i, \Delta_i, \mathbf{M}^i)\}_{i \geq 1}$ with a subsequence if necessary, we may assume that the set $\{t_i\}_{i \geq 1}$ is dense in T .

Put $D'_i = f'_i f_i^*G_i$ for each i . Then $f_i^*G_i = f_i^*D'_i$ by the third property. Since G_i is ample, the birational map $\varphi_i := \phi_i^{-1}: V_{t_i} \dashrightarrow X_i$ is a morphism. Taking a log resolution of (V, D) and shrinking T , we may assume that the morphism $h: (V, D) \rightarrow T$ is log smooth and T is affine. Moreover, taking an appropriate étale base change of h , we may assume that the restriction of every component of D to V_t is irreducible for every closed point $t \in T$ with the fiber V_t .

Since $\varphi_i^*G_i$ is an effective Cartier divisor on V_{t_i} such that $\text{Supp} \varphi_i^*G_i \subset \text{Supp} D|_{V_{t_i}}$ and all the coefficients of $\varphi_i^*G_i$ is at most c , the coefficients of $\varphi_i^*G_i$ belong to a finite set depending only on d, p and v . Therefore, replacing $\{(X_i, \Delta_i, \mathbf{M}^i)\}_{i \geq 1}$ by a subsequence, we may find a Cartier divisor \tilde{D} on V such that, for every i , we have $\tilde{D}|_{V_{t_i}} = \varphi_i^*G_i$.

We consider the morphism

$$h^* h_* \mathcal{O}_V(\tilde{D}) \rightarrow \mathcal{O}_V(\tilde{D}).$$

Then there is an open set $U \subset T$ such that, for any $t' \in U$, the morphism

$$h_*\mathcal{O}_V(\tilde{D}) \otimes k(t') \rightarrow H^0(V_{t'}, \mathcal{O}_{V_{t'}}(\tilde{D}|_{V_{t'}}))$$

is an isomorphism, where $k(t')$ is the quotient field of $\mathcal{O}_{T,t'}$ (see [27, III, Theorem 12.8] and [27, III, Corollary 12.9]). Since $\mathcal{O}_{V_{t'}}(\varphi_i^*G_i)$ is globally generated, the morphism

$$h^*h_*\mathcal{O}_V(\tilde{D}) \otimes k(t') \rightarrow \mathcal{O}_V(\tilde{D}) \otimes k(t')$$

is surjective for all $t' \in U \cap \{t_i\}_{i \geq 1}$. Since $\{t_i\}_{i \geq 1}$ is dense in T , we see that $U \cap \{t_i\}_{i \geq 1}$ is not empty. From this fact, shrinking T and replacing $\{(X_i, \Delta_i, \mathbf{M}^i)\}_{i \geq 1}$, we may assume that \tilde{D} is base point free over T .

Let $\tau: V \rightarrow W$ be the contraction over T induced by \tilde{D} . Then there is a Cartier divisor H , which is ample over T , such that $\tilde{D} \sim \tau^*H$. By shrinking T and replacing $\{(X_i, \Delta_i, \mathbf{M}^i)\}_{i \geq 1}$ by a subsequence, we may assume that for every closed point $t \in T$ the fiber W_t is normal. We consider the morphisms $\tau_i := \tau|_{V_{t_i}}: V_{t_i} \rightarrow W_{t_i}$ and $\varphi_i: V_{t_i} \rightarrow X_i$ for each i . Since $\tau_i^*H|_{W_{t_i}} \sim \tilde{D}|_{V_{t_i}} = \varphi_i^*G_i$ and both $H|_{W_{t_i}}$ and G_i are ample, we see that $\dim W_{t_i} = \dim X_i$. This shows that $\tau_i: V_{t_i} \rightarrow W_{t_i}$ is birational, hence τ_i is a contraction. Since $\tau_i^*H|_{W_{t_i}} \sim \varphi_i^*G_i$ and both $H|_{W_{t_i}}$ and G_i are ample, we see that $W_{t_i} \simeq X_i$.

By this discussion, we get

- a projective morphism $W \rightarrow T$ from a normal variety W to an affine variety T , and
- an ample Cartier divisor H

such that, for every i , there exist a closed point $t_i \in T$ and an isomorphism $\psi_i: W_{t_i} \rightarrow X_i$ such that $\psi_i^*G_i \sim H|_{W_{t_i}}$. Shrinking T and replacing $\{(X_i, \Delta_i, \mathbf{M}^i)\}_{i \geq 1}$, we may assume that the morphism $W \rightarrow T$ is flat.

Let m be a positive integer such that

- mH is very ample over T ,
- $H^0(W, \mathcal{O}_W(mH))$ generates $\bigoplus_{l \in \mathbb{Z}_{\geq 0}} H^0(W, \mathcal{O}_W(lmH))$ and
- $H^q(W, \mathcal{O}_W(lH)) = 0$ for all $l \geq m$ and $q > 0$.

Then $H^0(W, \mathcal{O}_W(lmH)) \otimes k(t) \rightarrow H^0(W_t, \mathcal{O}_{W_t}(lmH|_{W_t}))$ is surjective for every closed point $t \in T$. Therefore, for every closed point $t \in T$ and every $l \in \mathbb{Z}_{\geq 0}$, we see that

$$H^0(W_t, \mathcal{O}_{W_t}(mH|_{W_t})) \otimes H^0(W_t, \mathcal{O}_{W_t}(lmH|_{W_t})) \longrightarrow H^0(W_t, \mathcal{O}_{W_t}((1+l)mH|_{W_t}))$$

is surjective. Therefore, we see that mG_i is very ample and $H^0(X_i, \mathcal{O}_{X_i}(mG_i))$ generates the graded ring $\bigoplus_{l \in \mathbb{Z}_{\geq 0}} H^0(X_i, \mathcal{O}_{X_i}(mlG_i))$ for every i . This is a contradiction since we have $mG_i \sim mpp'(K_{X_i} + \Delta_i + \mathbf{M}_{X_i}^i)$ for all i . Thus, Lemma 5.10 holds. \square

Proof of Theorem 1.7. We will freely use the notations as in Theorem 1.7, and we put $\Delta' = \phi_*\Delta$ and $M' = \phi_*M$. Since M is nef and log big with respect to (X, Δ) , we may apply Theorem 5.8 to the generalized dlt pair $(X', \Delta', \overline{M})$. By Theorem 5.8, there exists m' , depending only on d and p , such that $m'(K_{X'} + \Delta' + M')$ is base point free. Let $\pi: X' \rightarrow Z$ be the Stein factorization of the morphism defined with $|m'(K_{X'} + \Delta' + M')|$. Then Z is nothing but $\mathbf{Proj} \bigoplus_{l \in \mathbb{Z}_{\geq 0}} R_l$ in Theorem 1.7. There is an ample and base point free Cartier divisor H_Z on Z such that

$$m'(K_{X'} + \Delta' + M') \sim \pi^*H_Z.$$

Since pM' and $p(K_{X'} + \Delta' + M')$ are Weil divisors, $p\Delta'$ is a Weil divisor. Applying Theorem 4.5 to $\pi: (X', \Delta', \overline{M}) \rightarrow Z$, we can find positive integers n and q , depending only on d and p , such that Z has the structure of a generalized lc pair $(Z, \Delta_Z, \mathbf{N})$ with

- $n(K_{X'} + \Delta' + M') \sim n\pi^*(K_Z + \Delta_Z + \mathbf{N}_Z)$, and
- $q\mathbf{N}$ is b-Cartier.

By the first property, we have

$$nm'\pi^*(K_Z + \Delta_Z + \mathbf{N}_Z) \sim nm'(K_{X'} + \Delta' + M') \sim n\pi^*H_Z.$$

By [10, Proposition 5.3], we see that $nm'(K_Z + \Delta_Z + \mathbf{N}_Z) \sim nH_Z$. In particular, we see that $nm'(K_Z + \Delta_Z + \mathbf{N}_Z)$ is ample and base point free.

Since we have

$$\text{vol}(Z, K_Z + \Delta_Z + \mathbf{N}_Z) = \text{Ivol}(K_X + \Delta + M) \leq v,$$

we may apply Lemma 5.10 to $(Z, \Delta_Z, \mathbf{N})$. There is a positive integer m'' , depending only on d, nm' and v , such that

- $m''(K_Z + \Delta_Z + \mathbf{N}_Z)$ is very ample, and
- putting $R'_l = H^0(Z, \mathcal{O}_Z(lm''(K_Z + \Delta_Z + \mathbf{N}_Z)))$, then R'_l generates $\bigoplus_{l \in \mathbb{Z}_{\geq 0}} R'_l$.

We define $m = nm'm''$. By construction, we can check that

- m depends only on d, p and v ,
- $m(K_Z + \Delta_Z + \mathbf{N}_Z)$ is very ample and
- $H^0(Z, \mathcal{O}_Z(m(K_Z + \Delta_Z + \mathbf{N}_Z)))$ generates $\bigoplus_{l \in \mathbb{Z}_{\geq 0}} H^0(Z, \mathcal{O}_Z(lm(K_Z + \Delta_Z + \mathbf{N}_Z)))$ as a graded \mathbb{C} -algebra.

Now we have

$$\begin{aligned} H^0(X, \mathcal{O}_X(\lfloor lm(K_X + \Delta + M) \rfloor)) &\simeq H^0(X', \mathcal{O}_{X'}(lm(K_{X'} + \Delta' + M'))) \\ &\simeq H^0(Z, \mathcal{O}_Z(lm(K_Z + \Delta_Z + \mathbf{N}_Z))) \end{aligned}$$

for every positive integer l . In this way, we see that $\bigoplus_{l \in \mathbb{Z}_{\geq 0}} R_{lm}$ is generated by R_m as a graded \mathbb{C} -algebra, where $R_{lm} = H^0(X, \mathcal{O}_X(\lfloor lm(K_X + \Delta + M) \rfloor))$. Furthermore, the inequality $\text{vol}(Z, m(K_Z + \Delta_Z + \mathbf{N}_Z)) \leq m^d v$ shows that $Z = \mathbf{Proj} \bigoplus_{l \in \mathbb{Z}_{\geq 0}} R_l$ belongs to a bounded family \mathfrak{F} that depends only on d, p and v . We finish the proof. \square

Finally, we introduce an application of Theorem 5.8.

Theorem 5.11. *Let d and p be positive integers. Then there exists a positive integer m , depending only on d and p , satisfying the following. Let (X, Δ, \mathbf{M}) be a generalized dlt pair such that*

- $\dim X = d$,
- \mathbf{M} is a \mathbb{Q} -b-divisor and $p(K_X + \Delta + \mathbf{M}_X)$ is nef and Cartier, and
- $K_X + \Delta + \mathbf{M}_X$ is log big with respect to (X, Δ, \mathbf{M}) .

Then $m(K_X + \Delta + \mathbf{M}_X)$ is base point free.

Proof. By Theorem 6.1, there is a log resolution $f: \tilde{X} \rightarrow X$ of (X, Δ) such that \mathbf{M} descends to \tilde{X} and f is an isomorphism over an open subset U containing all the generic points of generalized lc centers of (X, Δ, \mathbf{M}) . Set $D = K_X + \Delta + \mathbf{M}_X$, and consider the generalized dlt pair $(X, \Delta, \mathbf{M} + \overline{D})$. We can write

$$K_{\tilde{X}} + \tilde{\Delta} + \mathbf{M}_{\tilde{X}} + f^*D = f^*(K_X + \Delta + \mathbf{M}_X + D) + \tilde{E}$$

with $\tilde{\Delta} \geq 0$ and $\tilde{E} \geq 0$ having no common components. Since D is nef and log big with respect to (X, Δ, \mathbf{M}) and f is an isomorphism over the genetic points of all generalized lc centers of (X, Δ, \mathbf{M}) , we see that $\mathbf{M}_{\tilde{X}} + f^*D$ is nef and log big with respect to $(\tilde{X}, \tilde{\Delta})$. Thus, we may apply Theorem 5.8. There exists m' , depending only on d and p , such that $m'(K_X + \Delta + \mathbf{M}_X + D)$ is base point free. Then $2m'(K_X + \Delta + \mathbf{M}_X)$ is base point free. Hence, $m := 2m'$ is the desired positive integer. \square

Corollary 5.12. *Let d and p be positive integers, and let v be a positive real number. Then there exists a positive integer m , depending only on d, p and v , satisfying the following. Let (X, Δ, \mathbf{M}) be a generalized dlt pair such that*

- $\dim X = d$,
- $p\mathbf{M}$ is b -Cartier and $p(K_X + \Delta + \mathbf{M}_X)$ is Cartier,
- $K_X + \Delta + \mathbf{M}_X$ is ample and
- $\text{vol}(X, K_X + \Delta + \mathbf{M}_X) \leq v$.

Then $m(K_X + \Delta + \mathbf{M}_X)$ is very ample. In particular, (X, Δ, \mathbf{M}) belongs to a bounded family \mathfrak{F} depending only on d, p and v in the sense of Birkar [5].

Proof. By Theorem 5.11, there exists a positive integer n , depending only on d and p , such that $n(K_X + \Delta + \mathbf{M}_X)$ is base point free for every generalized dlt pair (X, Δ, \mathbf{M}) as in Corollary 5.12. Then Corollary 5.12 follows from Lemma 5.10. □

6. Appendix: On definition of generalized dlt pairs

The goal of this appendix is to prove that the definition of generalized dlt pairs by Birkar [4] and that by Han–Li [25] are equivalent when the nef part of a generalized pair is a finite $\mathbb{R}_{>0}$ -linear combination of b -nef \mathbb{Q} - b -Cartier \mathbb{Q} - b -divisors.

Theorem 6.1. *The following three conditions are equivalent for every generalized lc pair $(X, \Delta, \mathbf{M})/Z$ such that \mathbf{M} is a finite $\mathbb{R}_{>0}$ -linear combination of b -nef/ Z \mathbb{Q} - b -Cartier \mathbb{Q} - b -divisors.*

1. *For any generic point η of any generalized lc center of $(X, \Delta, \mathbf{M})/Z$, (X, Δ) is log smooth near η and \mathbf{M} descends to X on a neighborhood of η ([4, 2.13]).*
2. *There is an open subset $U \subset X$ such that $(U, \Delta|_U)$ is log smooth, U contains the generic point of any generalized lc center of $(X, \Delta, \mathbf{M})/Z$ and the generic point of any generalized lc center of $(X, \Delta, \mathbf{M})/Z$ is the generic point of an lc center of $(U, \Delta|_U)$ ([25, Definition 2.3]).*
3. *There is a log resolution $f: \tilde{X} \rightarrow X$ of (X, Δ) and an open subset $V \subset X$ such that \mathbf{M} descends to \tilde{X} , f is an isomorphism over V and V contains the generic point of any generalized lc center of $(X, \Delta, \mathbf{M})/Z$.*

We note that we do not need to assume that X or Z is quasi-projective.

The following result is the key ingredient for the proof.

Lemma 6.2 (cf. [23, Lemma 5.18]). *Let $(X, \Delta, \mathbf{M})/Z$ be a generalized klt pair such that \mathbf{M}_X is \mathbb{R} -Cartier and \mathbf{M} is a finite $\mathbb{R}_{>0}$ -linear combination of b -nef/ Z \mathbb{Q} - b -Cartier \mathbb{Q} - b -divisors. Then there is a projective birational morphism $g: Y \rightarrow X$ such that \mathbf{M} descends to Y and the divisor $F := g^*\mathbf{M}_X - \mathbf{M}_Y$ satisfies $\text{Supp} F = \text{Ex}(g)$.*

Proof. Since \mathbf{M}_X is \mathbb{R} -Cartier, (X, Δ) is a klt pair. Let $f: \tilde{X} \rightarrow X$ be a log resolution of (X, Δ) such that \mathbf{M} descends to \tilde{X} . We can write

$$K_{\tilde{X}} + \tilde{\Delta} = f^*(K_X + \Delta) + \tilde{E}$$

with $\tilde{\Delta} \geq 0$ and $\tilde{E} \geq 0$ having no common components. We may write $\mathbf{M}_{\tilde{X}} = \sum_i \mu_i M_i$, where $\mu_i \in \mathbb{R}_{>0}$ and M_i are Cartier divisors that are nef over Z . We fix α such that $\alpha\mu_i > 2 \cdot \dim X$ for all i .

We fix an open affine covering $\{X_j\}_j$ of X . In particular, all X_j are quasi-projective. We put $\tilde{X}_j = f^{-1}(X_j)$, $\tilde{\Delta}_j = \tilde{\Delta}|_{\tilde{X}_j}$ and $\mathbf{M}^j = \mathbf{M}|_{X_j}$. In the rest of the proof, for any projective birational morphism $W \rightarrow X$ and any open subset $V \subset W$, we will denote $\mathbf{M}_W|_V$ by \mathbf{M}_V .

Since $(\tilde{X}_j, \tilde{\Delta}_j, \alpha\mathbf{M}^j)/X_j$ is generalized klt, by applying [6] we get a sequence of steps of a $(K_{\tilde{X}_j} + \tilde{\Delta}_j + \alpha\mathbf{M}_{\tilde{X}_j}^j)$ -MMP over X_j that terminates with a good minimal model $(\tilde{X}'_j, \tilde{\Delta}'_j, \alpha\mathbf{M}^j)/X_j$

of $(\bar{X}_j, \bar{\Delta}_j, \alpha\mathbf{M}^j)/X_j$. By the length of extremal rays ([15, Section 18] or [17, Theorem 4.6.2]), it follows that the birational transform of $\mathbf{M}_{\bar{X}_j}$ is numerically trivial with respect to the extremal contraction in each step of the MMP. Therefore, we see that

- \mathbf{M}^j descends to \bar{X}'_j , and
- $\bar{X}_j \dashrightarrow \bar{X}'_j$ is a sequence of steps of a $(K_{\bar{X}_j} + \bar{\Delta}_j + (\alpha + t)\mathbf{M}_{\bar{X}_j})$ -MMP over X_j to a minimal model $(\bar{X}'_j, \bar{\Delta}'_j, (\alpha + t)\mathbf{M}^j)/X_j$ for all $t \geq 0$.

Then $K_{\bar{X}'_j} + \bar{\Delta}'_j + (\alpha + t)\mathbf{M}_{\bar{X}'_j}$ is semiample over X_j .

Let $\phi: \bar{X}'_j \rightarrow Y_j$ be the contraction over X_j induced by $K_{\bar{X}'_j} + \bar{\Delta}'_j + 2\alpha\mathbf{M}_{\bar{X}'_j}$, and let $\bar{X}'_j \rightarrow Y'_j$ be the contraction over X_j induced by $K_{\bar{X}'_j} + \bar{\Delta}'_j + \alpha\mathbf{M}_{\bar{X}'_j}$. We pick any curve $\xi \subset \bar{X}'_j$ that is contracted by $\bar{X}'_j \rightarrow X_j$. Since $K_{\bar{X}'_j} + \bar{\Delta}'_j + \alpha\mathbf{M}_{\bar{X}'_j}$ and $\alpha\mathbf{M}_{\bar{X}'_j}$ are nef over X_j , if ξ is contracted by $\phi: \bar{X}'_j \rightarrow Y_j$, then

$$0 = \xi \cdot (K_{\bar{X}'_j} + \bar{\Delta}'_j + 2\alpha\mathbf{M}_{\bar{X}'_j}) \geq \xi \cdot (K_{\bar{X}'_j} + \bar{\Delta}'_j + \alpha\mathbf{M}_{\bar{X}'_j}) \geq 0.$$

Thus, if ξ is contracted by $\phi: \bar{X}'_j \rightarrow Y_j$, then ξ is contracted by $\bar{X}'_j \rightarrow Y'_j$. In particular, the induced birational map $Y_j \dashrightarrow Y'_j$ is a morphism. Then we have

$$\alpha\mathbf{M}_{\bar{X}'_j} = (K_{\bar{X}'_j} + \bar{\Delta}'_j + 2\alpha\mathbf{M}_{\bar{X}'_j}) - (K_{\bar{X}'_j} + \bar{\Delta}'_j + \alpha\mathbf{M}_{\bar{X}'_j}) \sim_{\mathbb{R}, Y_j} 0,$$

so \mathbf{M}_{Y_j} is \mathbb{R} -Cartier and $\mathbf{M}_{\bar{X}'_j} = \phi^*\mathbf{M}_{Y_j}$. From this, we see that \mathbf{M}^j descends to Y_j .

Put $\Delta_{Y_j} = \phi_*\bar{\Delta}'_j$. By construction, the generalized pair $(Y_j, \Delta_{Y_j}, 2\alpha\mathbf{M}^j)/X_j$ is a weak generalized lc model of $(\bar{X}_j, \bar{\Delta}_j, 2\alpha\mathbf{M}^j)/X_j$ such that $K_{Y_j} + \Delta_{Y_j} + 2\alpha\mathbf{M}_{Y_j}$ is ample over X_j . This is a generalized pair analogue of the relative log canonical models for lc pairs. Hence, we may glue Y_j , and we get a projective birational morphism $g: Y \rightarrow X$ and a birational contraction $\psi: \bar{X} \dashrightarrow Y$ over X such that

- \mathbf{M} descends to Y , and
- $K_Y + \psi_*\bar{\Delta} + 2\alpha\mathbf{M}_Y$ is ample over X .

Since \mathbf{M}_X is \mathbb{R} -Cartier, we see that $K_X + \Delta + 2\alpha\mathbf{M}_X$ is \mathbb{R} -Cartier. Then there is an effective g -exceptional \mathbb{R} -divisor E_Y on Y such that

$$K_Y + \psi_*\bar{\Delta} + 2\alpha\mathbf{M}_Y = g^*(K_X + \Delta + 2\alpha\mathbf{M}_X) - E_Y$$

and $\text{Supp}E_Y = \text{Ex}(g)$. We recall the relation $K_{\bar{X}} + \bar{\Delta} = f^*(K_X + \Delta) + \bar{E}$, so we have $K_Y + \psi_*\bar{\Delta} = g^*(K_X + \Delta) + \psi_*\bar{E}$. We put

$$F = \frac{1}{2\alpha}(E_Y + \psi_*\bar{E}).$$

Then F is effective and g -exceptional, $\text{Supp}F \supset \text{Ex}(g)$ and $2\alpha\mathbf{M}_Y = g^*(2\alpha\mathbf{M}_X) - 2\alpha F$. From these facts, we see that $g: Y \rightarrow X$ is the desired morphism. □

From now on, we prove Theorem 6.1.

Proof of Theorem 6.1. Clearly, (3) implies (1), and (1) implies (2). Hence, we only need to prove that (2) implies (3). Let $U \subset X$ be the open subset as in (2). We will prove the existence of a projective birational morphism $g: Y \rightarrow X$ and an open subset $V \subset U$ such that \mathbf{M} descends to Y , g is an isomorphism over V and V contains the generic point of any generalized lc center of $(X, \Delta, \mathbf{M})/Z$. Assuming the existence of such g , then we can construct the desired log resolution $f: \bar{X} \rightarrow X$ and the open subset $V \subset X$ by considering an appropriate log resolution of (X, Δ) factoring through g . Thus, we only need to prove the existence of $g: Y \rightarrow X$ and $V \subset U$ stated above.

In this paragraph, we reduce the problem to the case where K_X is \mathbb{Q} -Cartier and \mathbf{M}_X is \mathbb{R} -Cartier. By [30, Proposition 4.12], with notations as in [30], the pair (X, Δ) of X and Δ is pseudo-lc in the sense of [30, Definition 4.2]. By [30, Theorem 1.2], there exists a projective small birational morphism $h: W \rightarrow X$ from a normal variety W such that putting $\Delta_W = h_*^{-1}\Delta$ then $K_W + \Delta_W$ is an h -ample \mathbb{R} -Cartier divisor on W . Note that [30, Theorem 1.2] can be applied without the quasi-projectivity of the variety. Since h is small, $(K_W + \Delta_W)|_{h^{-1}(U)}$ is the pullback of $(K_X + \Delta)|_U$, hence h is an isomorphism over U . Since h is small, we can check that $(W, \Delta_W, \mathbf{M})/Z$ is a generalized lc pair and $h^{-1}(U)$ contains the generic point of any generalized lc center of $(W, \Delta_W, \mathbf{M})/Z$. Then it is easy to see that $h^{-1}(U)$ satisfies the conditions of (2). Thus, replacing $(X, \Delta, \mathbf{M})/Z$ with $(W, \Delta_W, \mathbf{M})/Z$, we may assume that $K_X + \Delta$ is \mathbb{R} -Cartier. Then \mathbf{M}_X is \mathbb{R} -Cartier. With notations as in [30], the log canonicity of (X, Δ) and [30, Lemma 4.3] imply that the pair $(X, 0)$ is pseudo-lc in the sense of [30, Definition 4.2]. By [30, Theorem 1.2], we get a projective small birational morphism $h': W' \rightarrow X$ from a normal variety W' such that $K_{W'}$ is an h' -ample \mathbb{R} -Cartier divisor on W' . Then $K_{W'}|_{h'^{-1}(U)}$ is the pullback of $K_X|_U$, hence h' is an isomorphism over U . It is easy to see that $(W', h'^{-1}\Delta, \mathbf{M})/Z$ is a generalized lc pair and $h'^{-1}(U)$ satisfies the conditions of (2). Replacing $(X, \Delta, \mathbf{M})/Z$ with $(W', h'^{-1}\Delta, \mathbf{M})/Z$, we may assume that K_X is \mathbb{Q} -Cartier. In this way, we may assume that K_X is \mathbb{Q} -Cartier and \mathbf{M}_X is \mathbb{R} -Cartier.

By the definition of U , it follows that (X, Δ) is dlt. Hence, $(X, 0)$ is klt. By applying Lemma 6.2 to the generalized klt pair $(X, \frac{1}{2}\Delta, \frac{1}{2}\mathbf{M})/Z$, we get a projective birational morphism $g: Y \rightarrow X$ such that $\frac{1}{2}\mathbf{M}$ descends to Y and the divisor $F = \frac{1}{2}g^*\mathbf{M}_X - \frac{1}{2}\mathbf{M}_Y$ satisfies $\text{Supp}F = \text{Ex}(g)$. Let $V_0 \subset X$ be the largest open subset over which g is an isomorphism. Let η be the generic point of a generalized lc center of $(X, \Delta, \mathbf{M})/Z$. Then η is the generic point of an lc center of $(U, \Delta|_U)$, and η is also the generic point of a generalized lc center of $(U, \Delta|_U, \mathbf{M}|_U)/U$ with the identity $U \rightarrow U$. Pick a prime divisor P over U whose center is $\{\eta\} \cap U$ and the log discrepancy $a(P, U, \Delta|_U)$ is zero. Then

$$0 \leq a(P, U, \Delta|_U + \mathbf{M}_U) \leq a(P, U, \Delta|_U) = 0.$$

In particular, we have $a(P, U, \Delta|_U + \mathbf{M}_U) = a(P, U, \Delta|_U)$. Since $\text{Supp}F = \text{Ex}(g)$, the center $c_{g^{-1}(U)}(P)$ of P on $g^{-1}(U)$ is not contained in $\text{Ex}(g)$. Then $c_{g^{-1}(U)}(P)$ intersects $g^{-1}(V_0)$, hence we see that $\eta \in V_0$.

We set $V = V_0 \cap U$. Then g is an isomorphism over V . By the above discussion, V contains the generic point of any generalized lc center of $(X, \Delta, \mathbf{M})/Z$. By the property of U in (2), we see that $(V, \Delta|_V)$ is log smooth. In this way, we get a projective birational morphism $g: Y \rightarrow X$ and an open subset $V \subset U$ such that \mathbf{M} descends to Y , g is an isomorphism over V and V contains the generic point of any generalized lc center of $(X, \Delta, \mathbf{M})/Z$. From this fact, (2) implies (3). We complete the proof. \square

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