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RESEARCH ARTICLE

A presentation of the torus-equivariant quantum K-theory ring of flag manifolds of type A, Part II: quantum double Grothendieck polynomials

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Abstract
In our previous paper, we gave a presentation of the torus-equivariant quantum K-theory ring $QK_H(Fl_{n+1})$ of the (full) flag manifold Fl_{n+1} of type A_n as a quotient of a polynomial ring by an explicit ideal. In this paper, we prove that quantum double Grothendieck polynomials, introduced by Lenart-Maeno, represent the corresponding (opposite) Schubert classes in the quantum K-theory ring $QK_H(Fl_{n+1})$ under this presentation. The main ingredient in our proof is an explicit formula expressing the semi-infinite Schubert class associated to the longest element of the finite Weyl group, which is proved by making use of the general Chevalley formula for the torus-equivariant K-group of the semi-infinite flag manifold associated to $SL_{n+1}(\mathbb{C})$.

Contents

1. Introduction

Let Fl_{n+1} denote the (full) flag manifold G/B of type A_n, where $G = SL_{n+1}(\mathbb{C})$ is the special linear group of rank n, with Borel subgroup B consisting of the upper triangular matrices in $G = SL_{n+1}(\mathbb{C})$ and maximal torus $H \subset B$ consisting of the diagonal matrices in $G = SL_{n+1}(\mathbb{C})$. Let $R(H)$ denote the representation ring of H, which we identify with the group algebra $\mathbb{Z}[P] = \bigoplus_{\nu \in P} \mathbb{Z}\mathbf{e}^{\nu}$ of the weight lattice $P = \sum_{i=1}^{n} \mathbb{Z}\varpi_i$ of $G = SL_{n+1}(\mathbb{C})$, where ϖ_i, $1 \le i \le n$, are the fundamental weights; we set $\varpi_0 := 0$ and $\varpi_{n+1} := 0$ by convention. We denote by $QK_H(Fl_{n+1}) := K_H(Fl_{n+1}) \otimes_{R(H)} R(H)[[Q]]$ the H-equivariant (small) quantum K-theory ring, defined by Givental [Giv] and Lee [Lee], where $K_H(Fl_{n+1}) = \bigoplus_{w \in W} R(H)[\mathcal{O}^w]$ denotes the H-equivariant (ordinary) K-theory ring of Fl_{n+1} with the (opposite) Schubert classes $[\mathcal{O}^w]$ indexed by the elements w of the finite Weyl group $W = S_{n+1}$ of $G = SL_{n+1}(\mathbb{C})$ as a basis over $R(H)$, and where $R(H)[[Q]] = R(H)[[Q_1, \dots, Q_n]]$ denotes the ring of formal power series in the Novikov variables $Q_i := Q^{\alpha_i^\vee}$ corresponding to the simple coroots $\alpha_i^\vee$, $1 \le i \le n$, with coefficients in $R(H)$.

In our previous paper [MaNS], we proved that there exists an $R(H)[[Q]]$-algebra isomorphism

$$\Psi^Q : (R(H)[[Q]])[x_1, \dots, x_n, x_{n+1}]/\mathcal{I}^Q \xrightarrow{\cong} QK_H(Fl_{n+1}), \tag{1.1}$$

where $\mathcal{I}^Q$ is the ideal of $(R(H)[[Q]])[x_1, \dots, x_n, x_{n+1}]$ generated by

$$\sum_{\substack{J \subset [n+1] \\ |J|=l}} \prod_{j \in J,\, j+1 \notin J} (1 - Q_j) \prod_{j \in J} (1 - x_j) - \sum_{\substack{J \subset [n+1] \\ |J|=l}} \mathbf{e}^{\epsilon_J}, \qquad 1 \le l \le n+1,$$

with $[n+1] := \{1, 2, \dots, n+1\}$; here, we understand that $Q_{n+1} = 0$, and for a subset $J \subset [n+1]$, we set $\epsilon_J := \sum_{j \in J} \epsilon_j$, where $\epsilon_j = \varpi_j - \varpi_{j-1}$ for $1 \le j \le n+1$. Also, Ψ^Q maps the residue class of $(1-Q_j)(1-x_j)$ modulo $\mathcal{I}^Q$ to $[\mathcal{O}_{Fl_{n+1}}(-\epsilon_j)]$ for $1 \le j \le n$, and the residue class of $(1-x_{n+1})$ modulo $\mathcal{I}^Q$ to $[\mathcal{O}_{Fl_{n+1}}(-\epsilon_{n+1})]$; note that $-\epsilon_{n+1} = \epsilon_1 + \cdots + \epsilon_n$. Here, for $1 \le j \le n+1$, $\mathcal{O}_{Fl_{n+1}}(-\epsilon_j)$ denotes the quotient (line) bundle $\mathcal{U}_j/\mathcal{U}_{j-1}$ over Fl_{n+1}, where $0 = \mathcal{U}_0 \subset \mathcal{U}_1 \subset \cdots \subset \mathcal{U}_n \subset \mathcal{U}_{n+1} = Fl_{n+1} \times \mathbb{C}^{n+1}$ is the universal, or tautological, flag of subvector bundles of the trivial bundle $Fl_{n+1} \times \mathbb{C}^{n+1}$.

The purpose of this paper is to prove that quantum double Grothendieck polynomials, introduced in [LM, Section 8], represent (opposite) Schubert classes in $QK_H(Fl_{n+1})$ under the presentation above. This result can be thought of as the H-equivariant analog of [LNS, Theorem 51], in which it is proved that quantum Grothendieck polynomials represent (opposite) Schubert classes in the non-equivariant quantum K-theory ring $QK(Fl_n)$. However, the strategy of our proof in the H-equivariant case is quite different from that in the non-equivariant case in that we can reduce the proof to the case of the longest element by using quantum (dual) left Demazure operators introduced in [MNS], which act only on equivariant parameters.

To be more precise, for the longest element $w_\circ \in W = S_{n+1}$, the associated quantum double Grothendieck polynomial $\mathfrak{G}^Q_{w_\circ}(x, y)$ is defined as

$$\mathfrak{G}^Q_{w_\circ}(x, y) = \prod_{k=1}^{n} \left(\sum_{l=0}^{k} (-1)^l (1 - y_{n+1-k})^l F_l^k(x_1, \dots, x_n, x_{n+1}) \right),$$

where

$$F_l^k(x_1,\dots,x_n,x_{n+1}) := \sum_{\substack{J\subset[k]\\ |J|=l}} \prod_{j\in J,\, j+1\notin J}(1-Q_j)\prod_{j\in J}(1-x_j)$$

for $0 \le l \le k \le n$, with $[k] := \{1,2,\dots,k\}$; note that $\mathfrak{G}^Q_{w_\circ}(x,y)$ is an element of $\mathbb{Z}[[Q]][x_1,\dots,x_n,x_{n+1}][(1-y_1)^{\pm1},\dots,(1-y_n)^{\pm1}]$, which can be thought of as a subring of $(\mathbb{Z}[(1-y_1)^{\pm1},\dots,(1-y_n)^{\pm1}][[Q]])[x_1,\dots,x_n,x_{n+1}]$, where we identify $R(H)\cong\mathbb{Z}[\mathbf{e}^{\pm\epsilon_1},\dots,\mathbf{e}^{\pm\epsilon_n}]$ with $\mathbb{Z}[(1-y_1)^{\pm1},\dots,(1-y_n)^{\pm1}]$ by $1-y_j=\mathbf{e}^{-\epsilon_j}$ for $1\le j\le n$, and then identify $(R(H)[[Q]])[x_1,\dots,x_n,x_{n+1}]$ with $(\mathbb{Z}[(1-y_1)^{\pm1},\dots,(1-y_n)^{\pm1}][[Q]])[x_1,\dots,x_n,x_{n+1}]$. For an arbitrary $w\in W=S_{n+1}$, the associated quantum double Grothendieck polynomial $\mathfrak{G}^Q_w(x,y)$ is defined as

$$\mathfrak{G}^Q_w(x,y)=\pi^{(y)}_{ww_\circ}\mathfrak{G}^Q_{w_\circ}(x,y)\in\mathbb{Z}[[Q]][x_1,\dots,x_n,x_{n+1}][(1-y_1)^{\pm1},\dots,(1-y_n)^{\pm1}],$$

where $\pi^{(y)}_{ww_\circ}$ denotes the Demazure operator acting on the y variables (see Section 4.1 for details). Since we can show that the Demazure operators $\pi^{(y)}_i$ for $1\le i\le n$ on the quotient ring $(R(H)[[Q]])[x_1,\dots,x_n,x_{n+1}]/\mathcal{I}^Q$ coincide with the quantum (dual) left Demazure operators $\delta^\vee_i$ for $1\le i\le n$ on $QK_H(Fl_{n+1})$ (see the commutative diagram in (4.5)) under the $R(H)[[Q]]$-algebra isomorphism

$$\Psi^Q : (R(H)[[Q]])[x_1,\dots,x_n,x_{n+1}]/\mathcal{I}^Q \xrightarrow{\cong} QK_H(Fl_{n+1}),$$

we need only consider the case of the longest element $w_\circ\in W=S_{n+1}$; this line of argument is indeed proposed in [MNS, Section 8].

Now, we are ready to state the main result of this paper.

Theorem 1. *Let w be an arbitrary element of $W=S_{n+1}$. Then, under the isomorphism (1.1), the following equality holds in $QK_H(Fl_{n+1})$:*

$$\Psi^Q(\mathfrak{G}^Q_w(x,y) \bmod \mathcal{I}^Q)=[\mathcal{O}^w],$$

where we identify $R(H)\cong\mathbb{Z}[\mathbf{e}^{\pm\epsilon_1},\dots,\mathbf{e}^{\pm\epsilon_n}]$ with $\mathbb{Z}[(1-y_1)^{\pm1},\dots,(1-y_n)^{\pm1}]$, and then $(R(H)[[Q]])[x_1,\dots,x_n,x_{n+1}]$ with $(\mathbb{Z}[(1-y_1)^{\pm1},\dots,(1-y_n)^{\pm1}][[Q]])[x_1,\dots,x_n,x_{n+1}]$ by $1-y_j=\mathbf{e}^{-\epsilon_j}$ for $1\le j\le n$.

Here, we should mention that the result above for the (essential) case of the longest element $w_\circ\in W=S_{n+1}$ is obtained, via the $R(H)$-module isomorphism between $QK_H(Fl_{n+1})$ and the H-equivariant K-group $K_H(\mathbf{Q}_G)$ of the semi-infinite flag manifold $\mathbf{Q}_G$ associated to $G=SL_{n+1}(\mathbb{C})$ (established in [Kat1]), from an explicit formula (Corollary 2.5) expressing the semi-infinite Schubert class $[\mathcal{O}_{\mathbf{Q}_G(w_\circ)}]$ in $K_H(\mathbf{Q}_G)$, with $G=SL_{n+1}(\mathbb{C})$; see Section 3 for details. Also, we prove Corollary 2.5 (or Proposition 2.4) first by applying the general Chevalley formula for $K_H(\mathbf{Q}_G)$ in [LNS] and then by constructing a sign-reversing involution on a certain set of directed paths in the quantum Bruhat graph associated to $W=S_{n+1}$ which appear in the resulting expansion; see Section 5 for details.

This paper is organized as follows. In Section 2, we first fix the basic notation for root systems (in particular, for the root system of type A) and then recall the definition of the quantum Bruhat graph. Also, we recall the definitions of the $(H\times\mathbb{C}^*)$-equivariant K-group $K_{H\times\mathbb{C}^*}(\mathbf{Q}_G)$ and H-equivariant K-group $K_H(\mathbf{Q}_G)$ of the semi-infinite flag manifold $\mathbf{Q}_G$, and state the key proposition (Proposition 2.4), deferring its proof to Section 5. In Section 3, we briefly review the relationship of $K_H(\mathbf{Q}_G)$ with the H-equivariant quantum K-theory ring $QK_H(G/B)$ of the flag manifold G/B and prove our main result (Theorem 1) for the (essential) case of the longest element $w_\circ\in W=S_{n+1}$. In Section 4, we prove our main result (Theorem 1) for an arbitrary $w\in W=S_{n+1}$ through the strategy explained above. Section 5 is entirely devoted to the proof of Proposition 2.4.

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2. An explicit formula expressing the semi-infinite Schubert class associated to $w_\circ$

2.1. Notation for root systems

Let G be a connected, simply-connected, simple algebraic group over $\mathbb{C}$, H a maximal torus of G. We set $\mathfrak{g} := \mathrm{Lie}(G)$ and $\mathfrak{h} := \mathrm{Lie}(H)$. Thus, $\mathfrak{g}$ is a finite-dimensional simple Lie algebra over $\mathbb{C}$, and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. We denote by $\langle \cdot , \cdot \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$ the canonical pairing, where $\mathfrak{h}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$.

Let $\Delta \subset \mathfrak{h}^*$ be the root system of $\mathfrak{g}$, $\Delta^+ \subset \Delta$ the set of positive roots, and $\{\alpha_i\}_{i\in I} \subset \Delta^+$ the set of simple roots. We denote by $\alpha^\vee \in \mathfrak{h}$ the coroot corresponding to $\alpha \in \Delta$. Also, we denote by $\theta \in \Delta^+$ the highest root of Δ and set $\rho := (1/2)\sum_{\alpha\in\Delta^+}\alpha$. The root lattice Q and the coroot lattice $Q^\vee$ of $\mathfrak{g}$ are defined by $Q := \sum_{i\in I}\mathbb{Z}\alpha_i$ and $Q^\vee := \sum_{i\in I}\mathbb{Z}\alpha_i^\vee$. In addition, for $\alpha \in \Delta$, we set

$$\mathrm{sgn}(\alpha) := \begin{cases} 1 & \text{if } \alpha \in \Delta^+, \\ -1 & \text{if } \alpha \in \Delta^-, \end{cases} \qquad |\alpha| := \mathrm{sgn}(\alpha)\alpha \in \Delta^+.$$

For $i \in I$, the weight $\varpi_i \in \mathfrak{h}^*$ given by $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{i,j}$ for all $j \in I$, where $\delta_{i,j}$ denotes the Kronecker delta, is called the i-th fundamental weight. The weight lattice P of $\mathfrak{g}$ is defined by $P := \sum_{i\in I}\mathbb{Z}\varpi_i$. We denote by $\mathbb{Z}[P]$ the group algebra of P, which is the associative algebra generated by formal elements $\mathbf{e}^\nu$, $\nu \in P$, where the product is defined by $\mathbf{e}^\mu\mathbf{e}^\nu := \mathbf{e}^{\mu+\nu}$ for $\mu, \nu \in P$; we identify the group algebra $\mathbb{Z}[P]$ with the representation ring $R(H)$ of H.

A reflection $s_\alpha \in GL(\mathfrak{h}^*)$, $\alpha \in \Delta$, is defined by $s_\alpha\mu := \mu - \langle \mu, \alpha^\vee \rangle \alpha$ for $\mu \in \mathfrak{h}^*$. We write $s_i := s_{\alpha_i}$ for $i \in I$. Then the (finite) Weyl group W of $\mathfrak{g}$ is defined to be the subgroup of $GL(\mathfrak{h}^*)$ generated by $\{s_i\}_{i\in I}$; that is, $W := \langle s_i \mid i \in I \rangle$. For $w \in W$, there exist $i_1, \dots, i_r \in I$ such that $w = s_{i_1}\cdots s_{i_r}$. If r is minimal, then the product $s_{i_1}\cdots s_{i_r}$ is called a reduced expression for w, and r is called the length of w; we denote by $\ell(w)$ the length of w. Note that a reduced expression for w is not unique, but the length $\ell(w)$ is uniquely determined. Also, the affine Weyl group W_{af} of $\mathfrak{g}$ is, by definition, the semi-direct product group $W \ltimes \{t_\xi \mid \xi \in Q^\vee\}$ of W and the abelian group $\{t_\xi \mid \xi \in Q^\vee\} \cong Q^\vee$, where t_ξ denotes the translation in $\mathfrak{h}^*$ corresponding to $\xi \in Q^\vee$.

Definition 2.1. The quantum Bruhat graph of W, denoted by $\mathrm{QBG}(W)$, is the Δ^+-labeled directed graph whose vertices are the elements of W and whose edges are of the following form: $x \xrightarrow{\alpha} y$, with $x, y \in W$ and $\alpha \in \Delta^+$, such that $y = xs_\alpha$ and either of the following holds: (B) $\ell(y) = \ell(x) + 1$, or (Q) $\ell(y) = \ell(x) + 1 - 2\langle \rho, \alpha^\vee \rangle$. An edge satisfying (B) (resp., (Q)) is called a Bruhat edge (resp., quantum edge).

For an edge $x \xrightarrow{\alpha} y$ in $\mathrm{QBG}(W)$, we sometimes write $x \xrightarrow[\mathsf{B}]{\alpha} y$ (resp., $x \xrightarrow[\mathsf{Q}]{\alpha} y$) to indicate that the edge is a Bruhat (resp., quantum) edge.

2.2. The root system of type A

We recall the root system of type A. In the rest of this paper, when G is of type A, we use the notation introduced in this subsection. Also, we set $[m] := \{1, 2, \dots, m\}$ for $m \in \mathbb{Z}_{\geq 0}$.

Assume that G is of type A_n (i.e., $G = SL_{n+1}(\mathbb{C})$). Then $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$, and $\mathfrak{h} := \{h \in \mathfrak{g} \mid h \text{ is a diagonal matrix}\}$ is a Cartan subalgebra of $\mathfrak{g}$. We let $\{\epsilon_k \mid k \in [n+1]\}$ be the standard basis of $\mathbb{Z}^{n+1}$ and realize the weight lattice as $P = \mathbb{Z}^{n+1}/\mathbb{Z}(\epsilon_1 + \cdots + \epsilon_{n+1})$. By abuse of notation, we continue to denote the image of ϵ_k in P by the same symbol. Thus, $\varpi_k := \epsilon_1 + \cdots + \epsilon_k$, for $k \in I = [n]$, are the fundamental weights of $\mathfrak{g}$, and $\epsilon_1 + \cdots + \epsilon_n + \epsilon_{n+1} = 0$. We set $\alpha_i := \epsilon_i - \epsilon_{i+1}$ for $i \in I = [n]$ and $\alpha_{i,j} := \alpha_i + \alpha_{i+1} + \cdots + \alpha_j = \epsilon_i - \epsilon_{j+1}$ for $i, j \in [n]$ with $i \leq j$. Then $\Delta = \{\pm\alpha_{i,j} \mid 1 \leq i \leq j \leq n\}$ forms a root system of $\mathfrak{g}$, with the set of positive roots $\Delta^+ = \{\alpha_{i,j} \mid 1 \leq i \leq j \leq n\}$ and the set of simple roots $\{\alpha_1, \dots, \alpha_n\}$.

The Weyl group of $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$ is, by definition, the subgroup $W = \langle s_1, \dots, s_n \rangle$ of $GL(\mathfrak{h}^*)$, where $s_1, \dots, s_n$ are the simple reflections. For $i, j \in [n+1]$ with $i < j$, we denote by (i, j) the transposition

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of i and j. It is known that the assignment $s_1 \mapsto (1,2)$, $s_2 \mapsto (2,3)$, $\dots$, $s_n \mapsto (n, n+1)$ induces a group isomorphism $W \xrightarrow{\sim} S_{n+1}$, where S_{n+1} denotes the symmetric group of degree $n+1$. By this isomorphism, we regard $x \in W$ as a permutation on $[n+1] = \{1, \dots, n+1\}$; in addition, it follows that $x\epsilon_i = \epsilon_{x(i)}$ for $i \in [n+1]$. Also, the longest element of W, denoted by $w_\circ$, is regarded as the permutation

$$\begin{pmatrix} 1 & 2 & \cdots & n & n+1 \\ n+1 & n & \cdots & 2 & 1 \end{pmatrix};$$

that is, $w_\circ$ is considered as the permutation defined by $w_\circ(k) = n+2-k$ for $k \in [n+1]$.

We know from [Len, Proposition 3.6] the following.

Lemma 2.2. *Let $x \in W \cong S_{n+1}$, and $1 \le a < b \le n+1$.*

(B) *We have a Bruhat edge $x \xrightarrow{(a,b)} x \cdot (a,b)$ in $\mathrm{QBG}(W) = \mathrm{QBG}(S_{n+1})$ if and only if $x(a) < x(b)$, and either $x(c) < x(a)$ or $x(b) < x(c)$ holds for each $a < c < b$.*

(Q) *We have a quantum edge $x \xrightarrow{(a,b)} x \cdot (a,b)$ in $\mathrm{QBG}(W) = \mathrm{QBG}(S_{n+1})$ if and only if $x(a) > x(b)$, and $x(b) < x(c) < x(a)$ for all $a < c < b$.*

2.3. *Equivariant K-groups of semi-infinite flag manifolds*

Let $\mathbf{Q}_G^{\mathrm{rat}}$ denote the semi-infinite flag manifold associated to G, which is a (reduced) ind-scheme of ind-infinite type whose set of $\mathbb{C}$-valued points is $G(\mathbb{C}((z)))/(H(\mathbb{C}) \cdot N(\mathbb{C}((z))))$ (see [KaNS, Kat2] for details), where G is a connected, simply-connected, simple algebraic group over $\mathbb{C}$, $B = HN$ is a Borel subgroup, H is a maximal torus, and N is the unipotent radical of B; note that $\mathbf{Q}_G^{\mathrm{rat}}$ can be thought of as an inductive limit of copies of the (reduced) closed subscheme $\mathbf{Q}_G \subset \prod_{i \in I} \mathbb{P}(L(\varpi_i) \otimes_{\mathbb{C}} \mathbb{C}[[z]])$ of infinite type, introduced in [FM, Section 4.1], where $L(\varpi_i)$ is the irreducible highest weight G-module of highest weight ϖ_i. One has the semi-infinite Schubert (sub)variety $\mathbf{Q}_G(x) \subset \mathbf{Q}_G^{\mathrm{rat}}$ associated to each element x of the affine Weyl group $W_{\mathrm{af}} \cong W \ltimes Q^\vee$, with $W = \langle s_i \mid i \in I \rangle$ the (finite) Weyl group and $Q^\vee = \sum_{i \in I} \mathbb{Z}\alpha_i^\vee$ the coroot lattice of G; note that $\mathbf{Q}_G(x)$ is, by definition, the closure of the orbit under the Iwahori subgroup $\mathbf{I} := (\mathrm{ev}_{z=0})^{-1}(B) \subset G(\mathbb{C}[[z]])$ through the $(H \times \mathbb{C}^*)$-fixed point labeled by $x \in W_{\mathrm{af}}$ in exactly the same way as in [KaNS, Section 4.2] and [O, Section 2.3], and that $\mathbf{Q}_G(x)$ is contained in $\mathbf{Q}_G(e) = \mathbf{Q}_G$ for all $x \in W_{\mathrm{af}}^{\ge 0} := \{x = wt_\xi \in W_{\mathrm{af}} \mid w \in W, \xi \in Q^{\vee,+}\}$, where $Q^{\vee,+} := \sum_{i \in I} \mathbb{Z}_{\ge 0}\alpha_i^\vee \subset Q^\vee$. Also, for each weight $\nu = \sum_{i \in I} m_i \varpi_i \in P$ with $m_i \in \mathbb{Z}$, we have a $(G(\mathbb{C}[[z]]) \rtimes \mathbb{C}^*)$-equivariant line bundle $\mathcal{O}_{\mathbf{Q}_G}(\nu)$ over $\mathbf{Q}_G$, which is given by the restriction of the line bundle $\boxtimes_{i \in I} \mathcal{O}(m_i)$ over $\prod_{i \in I} \mathbb{P}(L(\varpi_i) \otimes_{\mathbb{C}} \mathbb{C}[[z]])$.

The $(H \times \mathbb{C}^*)$-equivariant K-group $K_{H \times \mathbb{C}^*}(\mathbf{Q}_G)$ is a module over $\mathbb{Z}[q, q^{-1}][P]$ (equivariant parameters) and has the semi-infinite Schubert classes $[\mathcal{O}_{\mathbf{Q}_G(x)}]$ associated to $x \in W_{\mathrm{af}}^{\ge 0} \simeq W \times Q^{\vee,+}$ as a topological basis (in the sense of [KaNS, Proposition 5.11]) over $\mathbb{Z}[q, q^{-1}][P]$, where $P = \sum_{i \in I} \mathbb{Z}\varpi_i$ is the weight lattice of G whose group algebra $\mathbb{Z}[P] = \bigoplus_{\nu \in P} \mathbb{Z}\mathbf{e}^\nu$ is identified with the representation ring $R(H)$ of H, and $q \in R(\mathbb{C}^*)$ corresponds to the loop rotation action of $\mathbb{C}^*$. More precisely, the K-group $K_{H \times \mathbb{C}^*}(\mathbf{Q}_G)$ is defined to be the $\mathbb{Z}[q, q^{-1}][P]$-submodule of the equivariant K-group $\widetilde{K}'(\mathbf{Q}_G)$ of $\mathbf{Q}_G$, introduced in [Kat1, Section 1.5], consisting of all those convergent (in the sense of [KaNS, Proposition 5.11]) formal infinite linear combinations of the classes $[\mathcal{O}_{\mathbf{Q}_G(x)}]$, $x \in W_{\mathrm{af}}^{\ge 0}$, of the structure sheaves $\mathcal{O}_{\mathbf{Q}_G(x)}$ of the semi-infinite Schubert varieties $\mathbf{Q}_G(x) \subset \mathbf{Q}_G$ with coefficients $a_x \in \mathbb{Z}[q, q^{-1}][P]$; briefly speaking, convergence holds if the sum $\sum_{x \in W_{\mathrm{af}}^{\ge 0}} |a_x|$ of the absolute values $|a_x|$ lies in $\mathbb{Z}[P]((q^{-1}))$. For each $x \in W_{\mathrm{af}}^{\ge 0}$ and $\nu \in P$, it follows from [KoLN, Theorem 5.16] and (the proof of) [KaNS, Corollary 5.12] that the twisted semi-infinite Schubert class $[\mathcal{O}_{\mathbf{Q}_G(x)}(\nu)] := [\mathcal{O}_{\mathbf{Q}_G(x)}] \otimes [\mathcal{O}_{\mathbf{Q}_G}(\nu)]$, defined by the tensor product in $\widetilde{K}'(\mathbf{Q}_G)$ (see [Kat1, Theorem 1.25]), indeed lies in $K_{H \times \mathbb{C}^*}(\mathbf{Q}_G)$; in particular, we have $[\mathcal{O}_{\mathbf{Q}_G}(\nu)] = [\mathcal{O}_{\mathbf{Q}_G}] \otimes [\mathcal{O}_{\mathbf{Q}_G}(\nu)] \in K_{H \times \mathbb{C}^*}(\mathbf{Q}_G)$ for all $\nu \in P$.

Also, following [Kat1, Section 1.5], we define the H-equivariant K-group $K_H(\mathbf{Q}_G)$ of $\mathbf{Q}_G$ to be the specialization (of coefficients) at $q = 1$ of $\widetilde{K}'(\mathbf{Q}_G)$ (or equivalently, of $K_{H \times \mathbb{C}^*}(\mathbf{Q}_G)$), which turns

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out to be (cf. [Kat1, Lemma 1.22]) the $\mathbb{Z}[P]$-module $\prod_{x\in W_{\mathrm{af}}^{\geq 0}} \mathbb{Z}[P][\mathcal{O}_{\mathbf{Q}_G(x)}]$ (direct product). Namely, the K-group $K_H(\mathbf{Q}_G)$ consists of all infinite linear combinations of the semi-infinite Schubert classes $[\mathcal{O}_{\mathbf{Q}_G(x)}]$, $x \in W_{\mathrm{af}}^{\geq 0}$, with coefficients in $\mathbb{Z}[P]$; the semi-infinite Schubert classes $[\mathcal{O}_{\mathbf{Q}_G(x)}]$, $x \in W_{\mathrm{af}}^{\geq 0}$, form a topological basis of $K_H(\mathbf{Q}_G)$ over $\mathbb{Z}[P]$. Then, for each $\nu \in P$, a $\mathbb{Z}[P]$-linear endomorphism $\bullet \otimes [\mathcal{O}_{\mathbf{Q}_G}(\nu)]$ of $K_H(\mathbf{Q}_G)$ is induced from the $\mathbb{Z}[q,q^{-1}][P]$-linear endomorphism $\bullet \otimes [\mathcal{O}_{\mathbf{Q}_G}(\nu)]$ of $K_{H\times\mathbb{C}^*}(\mathbf{Q}_G)$ by the specialization (of coefficients) at $q = 1$ (see [Kat1, Theorem 1.26]); thus, we have $[\mathcal{O}_{\mathbf{Q}_G}(\nu)] = [\mathcal{O}_{\mathbf{Q}_G}] \otimes [\mathcal{O}_{\mathbf{Q}_G}(\nu)] \in K_H(\mathbf{Q}_G)$ for $\nu \in P$. In addition, for $\xi \in Q^{\vee,+}$, a $\mathbb{Z}[P]$-linear endomorphism t_ξ on $K_H(\mathbf{Q}_G)$, given by $\mathsf{t}_\xi[\mathcal{O}_{\mathbf{Q}_G(x)}] := [\mathcal{O}_{\mathbf{Q}_G(xt_\xi)}]$ for $x \in W_{\mathrm{af}}^{\geq 0}$, is induced by the right action of $Q^\vee \subset H(\mathbb{C}((z)))/H(\mathbb{C})$ on $\mathbf{Q}_G^{\mathrm{rat}}$ (see [Kat1, equation (1.20)]); the $\mathbb{Z}[P]$-linear endomorphism t_ξ of $K_H(\mathbf{Q}_G)$ is also obtained by the specialization at $q = 1$ from the $\mathbb{Z}[q,q^{-1}]$-linear endomorphism of $K_{H\times\mathbb{C}^*}(\mathbf{Q}_G)$ given by the same formula. We know from [Kat1, Theorem 1.26] (cf. [O, Proposition 2.4])

$$(\mathsf{t}_\xi[\mathcal{O}_{\mathbf{Q}_G(x)}]) \otimes [\mathcal{O}_{\mathbf{Q}_G}(\nu)] = \mathsf{t}_\xi([\mathcal{O}_{\mathbf{Q}_G(x)}] \otimes [\mathcal{O}_{\mathbf{Q}_G}(\nu)]) \tag{2.1}$$

for $x \in W_{\mathrm{af}}^{\geq 0}$, $\xi \in Q^{\vee,+}$, and $\nu \in P$; for $j \in I$, we set $\mathsf{t}_j := \mathsf{t}_{\alpha_j^\vee}$. Since the semi-infinite Schubert classes $[\mathcal{O}_{\mathbf{Q}_G(x)}]$, $x \in W_{\mathrm{af}}^{\geq 0}$, form a topological basis of $K_H(\mathbf{Q}_G)$ over $\mathbb{Z}[P]$, it follows that

$$(\mathsf{t}_\xi \bullet) \otimes [\mathcal{O}_{\mathbf{Q}_G}(\nu)] = \mathsf{t}_\xi(\bullet \otimes [\mathcal{O}_{\mathbf{Q}_G}(\nu)]) \tag{2.2}$$

for an arbitrary element $\bullet \in K_H(\mathbf{Q}_G)$ and $\xi \in Q^{\vee,+}$, $\nu \in P$.

Remark 2.3. Let $\mathsf{F}_1, \ldots, \mathsf{F}_N$ be polynomials in t_i, $i \in I$, and let $\nu_1, \ldots, \nu_N \in P$. When we write $(\mathsf{F}_1[\mathcal{O}_{\mathbf{Q}_G}(\nu_1)]) \otimes (\mathsf{F}_2[\mathcal{O}_{\mathbf{Q}_G}(\nu_2)]) \otimes \cdots \otimes (\mathsf{F}_N[\mathcal{O}_{\mathbf{Q}_G}(\nu_N)]) \otimes \bullet$ for $\bullet \in K_H(\mathbf{Q}_G)$, it is always understood to be the element $(\mathsf{F}_1\mathsf{F}_2\cdots\mathsf{F}_N)([\mathcal{O}_{\mathbf{Q}_G}(\nu_1+\nu_2+\cdots+\nu_N)] \otimes \bullet)$ in $K_H(\mathbf{Q}_G)$. Namely, for $\bullet \in K_H(\mathbf{Q}_G)$, we always understand that

$$\begin{aligned}&(\mathsf{F}_1[\mathcal{O}_{\mathbf{Q}_G}(\nu_1)]) \otimes (\mathsf{F}_2[\mathcal{O}_{\mathbf{Q}_G}(\nu_2)]) \otimes \cdots \otimes (\mathsf{F}_N[\mathcal{O}_{\mathbf{Q}_G}(\nu_N)]) \otimes \bullet \\ &\quad = (\mathsf{F}_1\mathsf{F}_2\cdots\mathsf{F}_N)([\mathcal{O}_{\mathbf{Q}_G}(\nu_1+\nu_2+\cdots+\nu_N)] \otimes \bullet).\end{aligned}$$

2.4. *An explicit formula expressing the semi-infinite Schubert class associated to* $w_\circ$

In this subsection, we assume that $G = SL_{n+1}(\mathbb{C})$ and hence $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$, which is of type A_n. For $1 \leq k \leq n$, we set

$$\begin{aligned}w_k &:= (s_1s_2\cdots s_n)(s_1s_2\cdots s_{n-1})\cdots(s_1s_2\cdots s_{k+1})(s_1s_2\cdots s_k) \\ &= (1,2,\ldots,n+1)(1,2,\ldots,n)\cdots(1,2,\ldots,k+2)(1,2,\ldots,k+1) \\ &= \begin{pmatrix} 1 & 2 & \cdots & k-1 & k & k+1 & \cdots & n & n+1 \\ n-k+2 & n-k+3 & \cdots & n & n+1 & n-k+1 & \cdots & 2 & 1 \end{pmatrix},\end{aligned}$$

which is an element of the Weyl group $W = S_{n+1}$ of $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$; we set $w_{n+1} := e$ by convention. Note that $w_1 = w_\circ$, the longest element of W. Also, for a subset J of $[n+1] = \{1,2,\ldots,n+1\}$, we set

$$\epsilon_J := \sum_{j\in J} \epsilon_j. \tag{2.3}$$

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We have $\epsilon_J \in W\varpi_{|J|}$, where $\varpi_0 = \varpi_{n+1} := 0$ by convention. In particular, ϵ_J is a minuscule weight (i.e., $\langle \epsilon_J, \alpha_j^\vee \rangle \in \{-1, 0, 1\}$ for all $1 \le j \le n$). We set

$$\begin{aligned} J^- &:= \left\{1 \le j \le n \mid j \notin J,\ j+1 \in J\right\} \\ &= \left\{1 \le j \le n \mid \langle \epsilon_J, \alpha_j^\vee \rangle < 0\right\} \\ &= \left\{1 \le j \le n \mid \langle \epsilon_J, \alpha_j^\vee \rangle = -1\right\}; \end{aligned} \tag{2.4}$$

recall that $\alpha_j = \epsilon_j - \epsilon_{j+1}$. For $0 \le p \le k \le n+1$, we set

$$\mathfrak{F}_p^k := \sum_{\substack{J \subset [k] \\ |J| = p}} \left(\prod_{j \in J^-} (1 - \mathrm{t}_j) \right) [\mathcal{O}_{\mathbf{Q}_G}(w_\circ \epsilon_J)] \in K_H(\mathbf{Q}_G), \tag{2.5}$$

where $[k] = \left\{1, 2, \dots, k\right\}$; note that $\mathfrak{F}_0^k = 1$. The proof of the following proposition will be given in Section 5.

Proposition 2.4. *For $1 \le k \le n$, the following equality holds in $K_H(\mathbf{Q}_G)$:*

$$\left(\sum_{p=0}^{k} (-1)^p \mathbf{e}^{p\,\epsilon_{n+1-k}} \mathfrak{F}_p^k \right) \otimes [\mathcal{O}_{\mathbf{Q}_G(w_{k+1})}] = [\mathcal{O}_{\mathbf{Q}_G(w_k)}]. \tag{2.6}$$

Noting that $w_1 = w_\circ$ and $w_{n+1} = e$, we obtain the following.

Corollary 2.5. *The following equality holds in $K_H(\mathbf{Q}_G)$:*

$$[\mathcal{O}_{\mathbf{Q}_G(w_\circ)}] = \bigotimes_{k=1}^{n} \left(\sum_{p=0}^{k} (-1)^p \mathbf{e}^{p\,\epsilon_{n+1-k}} \mathfrak{F}_p^k \right). \tag{2.7}$$

3. Proof of the main result for $w = w_\circ$

3.1. Relationship between $K_H(\mathbf{Q}_G)$ and $QK_H(G/B)$

Let G be a connected, simply-connected, simple algebraic group over $\mathbb{C}$, with Borel subgroup $B \subset G$ and maximal torus $H \subset B$. Let $QK_H(G/B) := K_H(G/B) \otimes_{R(H)} R(H)[[Q^{\vee,+}]]$ denote the H-equivariant (small) quantum K-theory ring of the ordinary flag manifold G/B, defined by Givental [Giv] and Lee [Lee], where $R(H)[[Q^{\vee,+}]]$ is the ring of formal power series in the Novikov variables $Q_i = Q^{\alpha_i^\vee}$, $i \in I$, with coefficients in the representation ring $R(H)$ of H; for $\xi = \sum_{i \in I} k_i \alpha_i^\vee \in Q^{\vee,+} = \sum_{i \in I} \mathbb{Z}_{\ge 0} \alpha_i^\vee$, we set $Q^\xi := \prod_{i \in I} Q_i^{k_i} \in R(H)[[Q^{\vee,+}]]$. The quantum K-theory ring $QK_H(G/B)$ is a free module over $R(H)[[Q^{\vee,+}]]$ having the (opposite) Schubert classes $[\mathcal{O}^w]$, $w \in W$, as a basis; also, the quantum multiplication $\star$ in $QK_H(G/B)$ is a deformation of the classical tensor product in $K_H(G/B)$ and is defined in terms of the 2-point and 3-point (genus zero, equivariant) K-theoretic Gromov-Witten invariants; see [Giv] and [Lee] for details.

In [Kat1, Kat3], based on [GL, BF, IMT] (see also [ACT]), Kato established an $R(H)$-module isomorphism Φ from $QK_H(G/B)$ onto the H-equivariant K-group $K_H(\mathbf{Q}_G)$ of the semi-infinite flag manifold $\mathbf{Q}_G$, in which tensor product operation with an arbitrary line bundle class is induced from that in $K_{H \times \mathbb{C}^*}(\mathbf{Q}_G)$ by the specialization at $q = 1$; in our notation, the map Φ sends the (opposite) Schubert class $\mathbf{e}^\mu [\mathcal{O}^w] Q^\xi$ in $QK_H(G/B)$ to the corresponding semi-infinite Schubert class $\mathbf{e}^{-\mu} [\mathcal{O}_{\mathbf{Q}_G(wt_\xi)}]$ in $K_H(\mathbf{Q}_G)$ for $\mu \in P$, $w \in W$ and $\xi \in Q^{\vee,+}$. The isomorphism Φ also respects, in a sense, quantum multiplication $\star$ in $QK_H(G/B)$ and tensor product in $K_H(\mathbf{Q}_G)$. More precisely, one has the commutative diagram

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$$\begin{array}{ccc} QK_H(G/B) & \xrightarrow[\cong]{\Phi} & K_H(\mathbf{Q}_G) \\ {\scriptstyle \bullet\star[\mathcal{O}_{G/B}(-\varpi_i)]}\big\downarrow & & \big\downarrow{\scriptstyle \bullet\otimes[\mathcal{O}_{\mathbf{Q}_G}(w_\circ\varpi_i)]} \\ QK_H(G/B) & \xrightarrow[\cong]{\Phi} & K_H(\mathbf{Q}_G), \end{array} \tag{3.1}$$

where for $\nu \in P$, the line bundle $\mathcal{O}_{G/B}(-\nu)$ over G/B denotes the G-equivariant line bundle constructed as the quotient space $G \times^B \mathbb{C}_\nu$ of the product space $G \times \mathbb{C}_\nu$ by the usual (free) left action of the Borel subgroup B of G corresponding to the positive roots, with $\mathbb{C}_\nu$ the one-dimensional B-module of weight ν. (Here, we warn the reader that the convention of [Kat1] differs from that of [KaNS] and this paper, by the twist coming from the involution $-w_\circ$.) Also, we know from [Kat1] that for $w \in W$ and $\xi \in Q^{\vee,+}$, the equality $\Phi([\mathcal{O}^w]Q^\xi) = \mathrm{t}_\xi\,\Phi([\mathcal{O}^w])$ holds, and hence that for an arbitrary element $\bullet$ of $QK_H(G/B)$ and $\xi \in Q^{\vee,+}$, the equality

$$\Phi(\bullet\, Q^\xi) = \mathrm{t}_\xi\,\Phi(\bullet) \tag{3.2}$$

holds.

We know from [BCMP, Corollary 5.14] that the quantum multiplicative structure over $R(H)[[Q^{\vee,+}]]$ of $QK_H(G/B)$ is completely determined by the quantum multiplication with the line bundle classes $[\mathcal{O}_{G/B}(-\varpi_i)]$, $i \in I$; in fact, by using Nakayama's Lemma (i.e., [E, Corollary 4.8 b]) together with [E, Exercise 7.3], we can show that $QK_H(G/B)$ is generated as an algebra (with quantum multiplication $\star$) over $R(H)[[Q^{\vee,+}]]$ by the line bundle classes $[\mathcal{O}_{G/B}(-\varpi_i)]$, $i \in I$, since $K_H(G/B)$ is known to be generated by the same line bundle classes as an algebra (with tensor product $\otimes$) over $R(H)$ ([Mi]). Therefore, if G is of type A_n (i.e., $G = SL_{n+1}(\mathbb{C})$), then we deduce from [MaNS, Propositions 5.1 and 5.2], together with the comments following them, that $QK_H(Fl_{n+1})$ is generated as an algebra over $R(H)[[Q^{\vee,+}]] = R(H)[[Q]] = R(H)[[Q_1, \dots, Q_n]]$ by the line bundle classes $[\mathcal{O}_{Fl_{n+1}}(-\epsilon_j)]$, $1 \le j \le n$; indeed, for $1 \le j \le n$ and an arbitrary element $\bullet \in QK_H(Fl_{n+1})$, we have

$$\bullet \star [\mathcal{O}_{Fl_{n+1}}(-\varpi_j)] = \bullet \star [\mathcal{O}_{Fl_{n+1}}(-\epsilon_1)] \star \frac{1}{1-Q_1}[\mathcal{O}_{Fl_{n+1}}(-\epsilon_2)] \star \cdots \star \frac{1}{1-Q_{j-1}}[\mathcal{O}_{Fl_{n+1}}(-\epsilon_j)].$$

3.2. An explicit formula expressing the Schubert class associated to $w_\circ$

In this subsection, we assume that G is of type A_n (i.e., $G = SL_{n+1}(\mathbb{C})$). Now, for $0 \le p \le k \le n+1$, we set

$$\mathcal{F}_p^k := \sum_{\substack{J \subset [k] \\ |J|=p}} \prod_{\substack{1 \le j \le k \\ j,\, j+1 \in J}} \frac{1}{1-Q_j} \prod_{j \in J}^{\star} [\mathcal{O}_{Fl_{n+1}}(-\epsilon_j)] \in QK_H(Fl_{n+1}), \tag{3.3}$$

with $[k] = \{1, 2, \dots, k\}$, where $\prod^\star$ denotes the product with respect to the quantum multiplication $\star$; note that $\mathcal{F}_0^k = 1$ for $0 \le k \le n+1$. Also, recall from (2.5) that for $0 \le p \le k \le n+1$, the element $\mathfrak{F}_p^k \in K_H(\mathbf{Q}_G)$ is defined by

$$\mathfrak{F}_p^k = \sum_{\substack{J \subset [k] \\ |J|=p}} \left(\prod_{\substack{1 \le j \le k \\ j \notin J,\, j+1 \in J}} (1 - \mathrm{t}_j) \right) [\mathcal{O}_{\mathbf{Q}_G}(w_\circ \epsilon_J)]. \tag{3.4}$$

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Here, since $-\epsilon_j = \varpi_{j-1} - \varpi_j$ for $1 \le j \le n+1$ ($\varpi_0 = \varpi_{n+1} := 0$ by convention), we see from [MaNS, Proposition 5.2] that

$$\Phi\left(\bullet \star \frac{1}{1-Q_{j-1}}[\mathcal{O}_{Fl_{n+1}}(-\epsilon_j)]\right) = \Phi(\bullet) \otimes [\mathcal{O}_{\mathbf{Q}_G}(w_\circ \epsilon_j)] \tag{3.5}$$

for $1 \le j \le n+1$ and an arbitrary element $\bullet \in QK_H(Fl_{n+1})$, where $G = SL_{n+1}(\mathbb{C})$, and $Q_0 = Q_{n+1} := 0$ by convention.

Proposition 3.1. *For $0 \le p \le k \le n+1$, the following equality holds in $K_H(\mathbf{Q}_G)$:*

$$\Phi(\mathcal{F}_p^k) = \mathfrak{F}_p^k. \tag{3.6}$$

Proof. First note that for a subset $J \subset [k]$, the equality $[\mathcal{O}_{\mathbf{Q}_G}(w_\circ \epsilon_J)] = \bigotimes_{j \in J} [\mathcal{O}_{\mathbf{Q}_G}(w_\circ \epsilon_j)]$ holds since $\epsilon_J = \sum_{j \in J} \epsilon_j$. Therefore, by repeated application of equation (3.5), together with equations (2.2) and (3.2), we deduce that the image $\Phi^{-1}(\mathfrak{F}_p^k)$ of $\mathfrak{F}_p^k$ under the inverse mapping Φ^{-1} of the $R(H)$-module isomorphism Φ is

$$\sum_{\substack{J \subset [k] \\ |J|=p}} \prod_{\substack{1 \le j \le k \\ j \notin J,\, j+1 \in J}} (1-Q_j) \prod_{j \in J} \frac{1}{1-Q_{j-1}} \prod_{j \in J}^{\star} [\mathcal{O}_{Fl_{n+1}}(-\epsilon_j)], \tag{3.7}$$

which is easily seen to be identical to $\mathcal{F}_p^k$. This proves the proposition. □

Also, from Corollary 2.5, we obtain the following, again by repeated application of equation (3.5), together with relations (2.2) and (3.2).

Proposition 3.2. *Let $w_\circ \in W = S_{n+1}$ be the longest element. Then, the following equality holds in $QK_H(Fl_{n+1})$:*

$$[\mathcal{O}^{w_\circ}] = \prod_{k=1}^{n}{}^{\star} \left(\sum_{p=0}^{k} (-1)^p \mathbf{e}^{-p\,\epsilon_{n+1-k}} \mathcal{F}_p^k \right) \in QK_H(Fl_{n+1}). \tag{3.8}$$

3.3. *Proof of the main result for $w = w_\circ$*

We know from [MaNS, Section 6] that there exists an $R(H)[[Q]]$-algebra isomorphism

$$\Psi^Q : (R(H)[[Q]])[x_1, \ldots, x_n, x_{n+1}]/\mathcal{I}^Q \xrightarrow{\cong} QK_H(Fl_{n+1}), \tag{3.9}$$

where $\mathcal{I}^Q$ denotes the ideal of $(R(H)[[Q]])[x_1, \ldots, x_n, x_{n+1}]$ generated by

$$\sum_{\substack{J \subset [n+1] \\ |J|=l}} \prod_{j \in J,\, j+1 \notin J} (1-Q_j) \prod_{j \in J} (1-x_j) - \sum_{\substack{J \subset [n+1] \\ |J|=l}} \mathbf{e}^{\epsilon_J}, \qquad 1 \le l \le n+1, \tag{3.10}$$

with $[n+1] = \{1, 2, \ldots, n+1\}$; here, we understand that $Q_{n+1} = 0$. Also, we know that the isomorphism Ψ^Q maps the residue class $(1-Q_j)(1-x_j) \bmod \mathcal{I}^Q$ to $[\mathcal{O}_{Fl_{n+1}}(-\epsilon_j)]$ for $1 \le j \le n$, and the residue class $(1-x_{n+1}) \bmod \mathcal{I}^Q$ to $[\mathcal{O}_{Fl_{n+1}}(-\epsilon_{n+1})]$; note that $-\epsilon_{n+1} = \epsilon_1 + \cdots + \epsilon_n$.

In the following, by $1 - y_j = \mathbf{e}^{-\epsilon_j}$ for $1 \le j \le n$, we identify

(ID1) $R(H) \cong \mathbb{Z}[\mathbf{e}^{\pm\epsilon_1}, \ldots, \mathbf{e}^{\pm\epsilon_n}]$ with $\mathbb{Z}[(1-y_1)^{\pm 1}, \ldots, (1-y_n)^{\pm 1}]$, and
(ID2) $(R(H)[[Q]])[x_1, \ldots, x_n, x_{n+1}]$ with $(\mathbb{Z}[(1-y_1)^{\pm 1}, \ldots, (1-y_n)^{\pm 1}][[Q]])[x_1, \ldots, x_n, x_{n+1}]$.

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For $0 \le p \le k \le n$, we define

$$F_p^k(x_1,\dots,x_n,x_{n+1}) := \sum_{\substack{J\subset[k]\\ |J|=p}} \prod_{j\in J,\, j+1\notin J} (1-Q_j) \prod_{j\in J}(1-x_j) \in \mathbb{Z}[[Q]][x_1,\dots,x_n,x_{n+1}]. \tag{3.11}$$

Then, we can easily verify that for $0 \le p \le k \le n$,

$$\Psi^Q(F_p^k(x_1,\dots,x_n,x_{n+1}) \bmod \mathcal{I}^Q) = \mathcal{F}_p^k \in QK_H(Fl_{n+1}).$$

Therefore, we deduce from Proposition 3.2 that the residue class

$$\prod_{k=1}^{n}\left(\sum_{p=0}^{k}(-1)^p(1-y_{n+1-k})^p F_p^k(x_1,\dots,x_n,x_{n+1})\right) \bmod \mathcal{I}^Q$$

is mapped under the $R(H)[[Q]]$-algebra isomorphism Ψ^Q to

$$\prod_{k=1}^{n}{}^{\star}\left(\sum_{p=0}^{k}(-1)^p \mathbf{e}^{-p\epsilon_{n+1-k}}\mathcal{F}_p^k\right) = [\mathcal{O}^{w_\circ}] \in QK_H(Fl_{n+1});$$

recall the identification $1-y_j = \mathbf{e}^{-\epsilon_j}$ for $1 \le j \le n$.

Now, let us set

$$\begin{aligned}\mathfrak{G}_{w_\circ}^Q(x,y) := \prod_{k=1}^{n}&\left(\sum_{l=0}^{k}(-1)^l(1-y_{n+1-k})^l F_l^k(x_1,\dots,x_n,x_{n+1})\right)\\ &\in \mathbb{Z}[[Q]][x_1,\dots,x_n,x_{n+1}][(1-y_1)^{\pm1},\dots,(1-y_n)^{\pm1}],\end{aligned} \tag{3.12}$$

which can be thought of as a subring of $(\mathbb{Z}[(1-y_1)^{\pm1},\dots,(1-y_n)^{\pm1}][[Q]])[x_1,\dots,x_n,x_{n+1}]$; see Section 4.1 below. Then, from the above, we obtain the following.

Theorem 3.3. *Let $w_\circ \in W = S_{n+1}$ be the longest element. Then, under the identification (ID2), the following equality holds in $QK_H(Fl_{n+1})$:*

$$\Psi^Q(\mathfrak{G}_{w_\circ}^Q(x,y) \bmod \mathcal{I}^Q) = [\mathcal{O}^{w_\circ}] \in QK_H(Fl_{n+1}), \tag{3.13}$$

where Ψ^Q is the $R(H)[[Q]]$-algebra isomorphism in (3.9).

4. Proof of the main result for an arbitrary *w*

4.1. Quantum double Grothendieck polynomials

First we recall from [LM, Section 8] the Demazure operators $\pi_i^{(y)}$ acting on the y variables; note that we use Demazure operators acting on the y variables, while in [LM], they are acting on the x variables. Let $i \in [n] = \{1,2,\dots,n\}$. We define the Demazure operator $\pi_i^{(y)}$ (with respect to the y variables) on $\mathbb{Z}[(1-y_1)^{\pm1},\dots,(1-y_n)^{\pm1}]$ by

$$\pi_i^{(y)} = \mathrm{id} + \frac{1-y_i}{y_i - y_{i+1}}(\mathrm{id} - s_i),$$

where we identify the Laurent polynomial ring $\mathbb{Z}[(1-y_1)^{\pm1},\dots,(1-y_n)^{\pm1}]$ with the quotient of the polynomial ring $\mathbb{Z}[(1-y_1),\dots,(1-y_n),(1-y_{n+1})]$ by the relation $(1-y_1)\cdots(1-y_n)(1-y_{n+1}) = 1$; we extend it to $\mathbb{Z}[[Q]][x_1,\dots,x_n,x_{n+1}][(1-y_1)^{\pm1},\dots,(1-y_n)^{\pm1}]$ by $\mathbb{Z}[[Q]][x_1,\dots,x_n,x_{n+1}]$-linearity,

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and also to $(\mathbb{Z}[(1-y_1)^{\pm 1},\dots,(1-y_n)^{\pm 1}][[Q]])[x_1,\dots,x_n,x_{n+1}]$ by letting it act trivially on the elements of $\mathbb{Z}[[Q]][x_1.\dots,x_n,x_{n+1}]$. Also, for $w \in W = S_{n+1}$ with reduced expression $w = s_{i_1}\cdots s_{i_l}$, we define $\pi_w^{(y)} := \pi_{i_1}^{(y)}\cdots\pi_{i_l}^{(y)}$; it is well-known that this definition does not depend on the choice of a reduced expression for w.

Now, let us recall from [LM, Section 8] the definition of quantum double Grothendieck polynomials. For the longest element $w_\circ \in W = S_{n+1}$, we define $\mathfrak{G}_{w_\circ}^Q(x,y)$ as in (3.12). Note that the polynomial $\mathfrak{G}_{w_\circ}^Q(x,y)$ is the image under the quantization map $\mathcal{Q}$ (with respect to the x variables) of the ordinary double Grothendieck polynomial

$$\mathfrak{G}_{w_\circ}(x,y) := \prod_{k=1}^{n}\left(\sum_{l=0}^{k}(-1)^l(1-y_{n+1-k})^l f_l^k(x_1,\dots,x_n,x_{n+1})\right),$$

where

$$f_l^k(x_1,\dots,x_n,x_{n+1}) := \sum_{\substack{J\subset[k]\\ |J|=l}}\prod_{j\in J}(1-x_j),$$

with $[k] = \{1,2,\dots,k\}$; for more details on the quantization map $\mathcal{Q}$, we refer the reader to [LM, Sections 3 and 5]. For an arbitrary $w \in W = S_{n+1}$, the quantum double Grothendieck polynomial $\mathfrak{G}_w^Q(x,y)$ is defined by

$$\mathfrak{G}_w^Q(x,y) := \pi_{ww_\circ}^{(y)}\mathfrak{G}_{w_\circ}^Q(x,y) \in \mathbb{Z}[[Q]][x_1,\dots,x_n,x_{n+1}][(1-y_1)^{\pm 1},\dots,(1-y_n)^{\pm 1}],$$

which is a subring of $(\mathbb{Z}[(1-y_1)^{\pm 1},\dots,(1-y_n)^{\pm 1}][[Q]])[x_1,\dots,x_n,x_{n+1}]$.

Remark 4.1. In our definition of quantum double Grothendieck polynomials $\mathfrak{G}_w^Q(x,y)$, the roles of the variables x and y are interchanged from those in [LM, Section 8]. In addition, we use the Demazure operator $\pi_{ww_\circ}^{(y)}$ instead of $\pi_{w^{-1}w_\circ}^{(y)}$. Hence, our polynomial $\mathfrak{G}_w^Q(x,y)$ coincides with $\mathfrak{G}_{w^{-1}}^q(y,x)$ in the notation of [LM, Section 8].

Remark 4.2. Our polynomial $\mathfrak{G}_w^Q(x,y)$ coincides with the one in [MaNS, Appendix B]; this is because the quantization map $\mathcal{Q}$ (with respect to the x variables) commutes with the Demazure operators $\pi_i^{(y)}$, $1 \le i \le n$, and we have $\mathfrak{G}_w(y,x) = \mathfrak{G}_{w^{-1}}(x,y)$ for ordinary double Grothendieck polynomials.

4.2. Quantum left Demazure operators

Let $K_H(Fl_{n+1})$ be the H-equivariant (ordinary) K-theory ring of the (full) flag manifold Fl_{n+1} of type A_n. The ring $K_H(Fl_{n+1})$ admits a left action of the Weyl group $W = S_{n+1}$, given by the left multiplication on the flag manifold $Fl_{n+1} = SL_{n+1}(\mathbb{C})/B$, where B is the Borel subgroup of $G = SL_{n+1}(\mathbb{C})$ consisting of the upper triangular matrices in $G = SL_{n+1}(\mathbb{C})$; for each $w \in W = S_{n+1}$, let w^L denote the corresponding ring automorphism of $K_H(Fl_{n+1})$. By using the ring automorphisms s_i^L, $i \in [n]$, of $K_H(Fl_{n+1})$, one can define the (dual) left Demazure operators $\delta_i^\vee$, $i \in [n]$, on $K_H(Fl_{n+1})$ by

$$\delta_i^\vee := \frac{1}{1-\mathbf{e}^{-(\epsilon_i-\epsilon_{i+1})}}(\mathrm{id} - \mathbf{e}^{-(\epsilon_i-\epsilon_{i+1})}s_i^L). \tag{4.1}$$

Here, we recall the well-known presentation

$$K_H(Fl_{n+1}) = (\mathbb{Z}[\mathbf{e}^{\pm\epsilon_1},\dots,\mathbf{e}^{\pm\epsilon_n}]\otimes\mathbb{Z}[x_1^{\pm 1},\dots,x_n^{\pm 1}])/\mathcal{I},$$

where $\mathcal{I}$ is the ideal of $\mathbb{Z}[\mathbf{e}^{\pm\epsilon_1},\dots,\mathbf{e}^{\pm\epsilon_n}]\otimes\mathbb{Z}[x_1^{\pm 1},\dots,x_n^{\pm 1}]$ generated by $\iota(f)\otimes 1 - 1\otimes f$, $f \in \mathbb{Z}[x_1^{\pm 1},\dots,x_n^{\pm 1}]^{S_{n+1}}$, with $\iota : \mathbb{Z}[x_1^{\pm 1},\dots,x_n^{\pm 1}] \to \mathbb{Z}[\mathbf{e}^{\pm\epsilon_1},\dots,\mathbf{e}^{\pm\epsilon_n}]$ the natural isomorphism given

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by $\iota(x_i^{\pm 1}) = \mathbf{e}^{\pm\epsilon_i}$, $1 \le i \le n$; we identify the Laurent polynomial ring $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ with the quotient of $\mathbb{Z}[x_1, \dots, x_n, x_{n+1}]$ by the relation $x_1 \cdots x_n x_{n+1} = 1$, and the Laurent polynomial ring $\mathbb{Z}[\mathbf{e}^{\pm\epsilon_1}, \dots, \mathbf{e}^{\pm\epsilon_n}]$ with the quotient of $\mathbb{Z}[\mathbf{e}^{\epsilon_1}, \dots, \mathbf{e}^{\epsilon_n}, \mathbf{e}^{\epsilon_{n+1}}]$ by the relation $\mathbf{e}^{\epsilon_1} \cdots \mathbf{e}^{\epsilon_n} \mathbf{e}^{\epsilon_{n+1}} = 1$. We know from (the dual version of) [MNS, Proposition 9.5] that under the presentation above of $K_H(Fl_{n+1})$, the action of the (dual) left Demazure operator $\delta_i^\vee$ on $K_H(Fl_{n+1})$ coincides with that on the quotient ring $(\mathbb{Z}[\mathbf{e}^{\pm\epsilon_1}, \dots, \mathbf{e}^{\pm\epsilon_n}] \otimes \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}])/\mathcal{I}$ induced by the (dual) Demazure operator on the left factor $\mathbb{Z}[\mathbf{e}^{\pm\epsilon_1}, \dots, \mathbf{e}^{\pm\epsilon_n}]$, given by

$$\frac{1}{1 - \mathbf{e}^{-(\epsilon_i - \epsilon_{i+1})}} (\mathrm{id} - \mathbf{e}^{-(\epsilon_i - \epsilon_{i+1})} s_i), \tag{4.2}$$

where s_i denotes the natural action of the simple reflection $s_i \in W = S_{n+1}$ on $R(H) \cong \mathbb{Z}[\mathbf{e}^{\pm\epsilon_1}, \dots, \mathbf{e}^{\pm\epsilon_n}]$.

Remark 4.3. It is easy to see that under the identification (ID1) in Section 3.3, the action of the operator (4.2) on $\mathbb{Z}[\mathbf{e}^{\pm\epsilon_1}, \dots, \mathbf{e}^{\pm\epsilon_n}]$ coincides with that of the Demazure operator $\pi_i^{(y)}$ on $\mathbb{Z}[(1-y_1)^{\pm 1}, \dots, (1-y_n)^{\pm 1}]$.

Let $QK_H(Fl_{n+1}) = K_H(Fl_{n+1}) \otimes_{R(H)} R(H)[[Q]]$ be the H-equivariant (small) quantum K-theory ring of Fl_{n+1}. We know from [MNS, Section 8] that for each $w \in W = S_{n+1}$, the ring automorphism w^L of $K_H(Fl_{n+1})$ extends by $\mathbb{Z}[[Q]]$-linearity to the ring automorphism of $QK_H(Fl_{n+1})$, and hence, the (dual) left Demazure operators $\delta_i^\vee$, $1 \le i \le n$, on $K_H(Fl_{n+1})$ extend by $\mathbb{Z}[[Q]]$-linearity to $QK_H(Fl_{n+1})$. Moreover, we know from [MNS, Proposition 8.3] that the following equalities hold in $QK_H(Fl_{n+1})$ for $1 \le i \le n$:

$$\delta_i^\vee(\bullet \star \mathcal{O}_{Fl_{n+1}}(\nu)) = \delta_i^\vee(\bullet) \star \mathcal{O}_{Fl_{n+1}}(\nu) \tag{4.3}$$

for $\nu \in P$ and $\bullet \in QK_H(Fl_{n+1})$, and

$$\delta_i^\vee([\mathcal{O}^w]) = \begin{cases} [\mathcal{O}^{s_i w}] & \text{if } s_i w < w, \\ [\mathcal{O}^w] & \text{if } s_i w > w \end{cases} \tag{4.4}$$

for $w \in W = S_{n+1}$.

4.3. *Proof of the main result for an arbitrary w*

Recall the $R(H)[[Q]]$-algebra isomorphism Ψ^Q from (3.9). Also, recall the identification (ID1) and (ID2) from Section 3.3. Let $i \in [n] = \{1, \dots, n\}$. If we extend the Demazure operator $\pi_i^{(y)}$ on $\mathbb{Z}[(1-y_1)^{\pm 1}, \dots, (1-y_n)^{\pm 1}]$ by $\mathbb{Z}[[Q]]$-linearly to $(\mathbb{Z}[(1-y_1)^{\pm 1}, \dots, (1-y_n)^{\pm 1}][[Q]])[x_1, \dots, x_n, x_{n+1}]$ by letting it act trivially on the elements of $\mathbb{Z}[x_1, \dots, x_n, x_{n+1}]$, then the (resulting) Demazure operator $\pi_i^{(y)}$ stabilizes the ideal $\mathcal{I}^Q$ generated by the elements in (3.10), since the elementary symmetric functions $\sum_{J \subset [n+1],\, |J|=l} \mathbf{e}^{\epsilon_J}$, $1 \le l \le n+1$, are invariant under the natural action of s_i and the Demazure operator $\pi_i^{(y)}$ satisfies the (twisted) Leibniz rule. Hence, it induces a $\mathbb{Z}[[Q]]$-linear operator, also denoted by $\pi_i^{(y)}$, acting trivially on the (residue classes of the) variables x_i for $1 \le i \le n+1$ on the quotient ring $(\mathbb{Z}[(1-y_1)^{\pm 1}, \dots, (1-y_n)^{\pm 1}][[Q]])[x_1, \dots, x_n, x_{n+1}]/\mathcal{I}^Q$. Also, by (4.3) and the fact that Ψ^Q maps the residue class of $(1-Q_j)(1-x_j)$ modulo $\mathcal{I}^Q$ to $[\mathcal{O}_{Fl_{n+1}}(-\epsilon_j)]$ for $1 \le j \le n+1$ (recall that $Q_{n+1} = 0$ by convention), it follows that the quantum (dual) left Demazure operator $\delta_i^\vee$ on $QK_H(Fl_{n+1})$ is a $\mathbb{Z}[[Q]]$-linear operator acting trivially on the variables x_i for $1 \le i \le n+1$. Since $QK_H(Fl_{n+1})$ and the quotient ring $(\mathbb{Z}[(1-y_1)^{\pm 1}, \dots, (1-y_n)^{\pm 1}][[Q]])[x_1, \dots, x_n, x_{n+1}]/\mathcal{I}^Q$ are both generated by the (image under the quotient map of the) elements of $\mathbb{Z}[x_1, \dots, x_n, x_{n+1}]$ over $\mathbb{Z}[\mathbf{e}^{\pm\epsilon_1}, \dots, \mathbf{e}^{\pm\epsilon_n}][[Q]] \cong \mathbb{Z}[(1-y_1)^{\pm 1}, \dots, (1-y_n)^{\pm 1}][[Q]]$, and since the (dual) left Demazure operator $\delta_i^\vee$ and the Demazure operator $\pi_i^{(y)}$ coincide on $\mathbb{Z}[\mathbf{e}^{\pm\epsilon_1}, \dots, \mathbf{e}^{\pm\epsilon_n}] \cong \mathbb{Z}[(1-y_1)^{\pm 1}, \dots, (1-y_n)^{\pm 1}]$ by Remark 4.3, together with the comment on $\delta_i^\vee$ preceding it, we deduce that the operators $\delta_i^\vee$ and $\pi_i^{(y)}$ coincide on

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$$QK_H(Fl_{n+1}) \cong (\mathbb{Z}[(1-y_1)^{\pm 1}, \dots, (1-y_n)^{\pm 1}][[Q]])[x_1, \dots, x_n, x_{n+1}]/\mathcal{I}^Q;$$

recall the identification (ID2). Namely, we obtain the following commutative diagram for all $1 \le i \le n$:

$$\begin{tikzcd} (\mathbb{Z}[(1-y_1)^{\pm 1}, \dots, (1-y_n)^{\pm 1}][[Q]])[x_1, \dots, x_n, x_{n+1}]/\mathcal{I}^Q \arrow[r, "\Psi^Q", "\cong"'] \arrow[d, "\pi_i^{(y)}"'] & QK_H(Fl_{n+1}) \arrow[d, "\delta_i^\vee"] \\ (\mathbb{Z}[(1-y_1)^{\pm 1}, \dots, (1-y_n)^{\pm 1}][[Q]])[x_1, \dots, x_n, x_{n+1}]/\mathcal{I}^Q \arrow[r, "\Psi^Q", "\cong"'] & QK_H(Fl_{n+1}). \end{tikzcd} \tag{4.5}$$

Now, we are ready to state and prove the main result of this paper.

Theorem 4.4. *Let w be an arbitrary element of $W = S_{n+1}$. Then, under the identification (ID2), the following equality holds in $QK_H(Fl_{n+1})$:*

$$\Psi^Q(\mathfrak{G}_w^Q(x, y) \bmod \mathcal{I}^Q) = [\mathcal{O}^w],$$

where Ψ^Q is the $R(H)[[Q]]$-algebra isomorphism in (3.9).

Proof. The assertion of the theorem is already proved for $w = w_\circ \in W = S_{n+1}$ by Theorem 3.3:

$$\Psi^Q(\mathfrak{G}_{w_\circ}^Q(x, y) \bmod \mathcal{I}^Q) = [\mathcal{O}^{w_\circ}] \in QK_H(Fl_{n+1}). \tag{4.6}$$

For an arbitrary $w \in W = S_{n+1}$, let $ww_\circ = s_{i_1} \cdots s_{i_l}$ be a reduced expression; notice that $l = \ell(w_\circ) - \ell(w)$. Then, by the definition of quantum double Grothendieck polynomials, we have

$$\mathfrak{G}_w^Q(x, y) = \pi_{i_1}^{(y)} \cdots \pi_{i_l}^{(y)} \mathfrak{G}_{w_\circ}^Q(x, y). \tag{4.7}$$

Here, note that $w = s_{i_1} \cdots s_{i_l} w_\circ$, and that $w < s_{i_2} \cdots s_{i_l} w_\circ < \cdots < s_{i_l} w_\circ < w_\circ$ in the Bruhat order $<$ on $W = S_{n+1}$, since $\ell(w) = \ell(w_\circ) - l$. Therefore, by the property (4.4) of quantum (dual) left Demazure operators $\delta_i^\vee$, we deduce that

$$(\delta_{i_1}^\vee \cdots \delta_{i_l}^\vee)[\mathcal{O}^{w_\circ}] = [\mathcal{O}^w] \in QK_H(Fl_{n+1}). \tag{4.8}$$

From the equalities (4.6), (4.7) and (4.8), using the commutative diagram (4.5), we conclude that $\Psi^Q(\mathfrak{G}_w^Q(x, y) \bmod \mathcal{I}^Q) = [\mathcal{O}^w] \in QK_H(Fl_{n+1})$. This proves the theorem. □

Remark 4.5. Let $R(H)[Q]_{\mathrm{loc}}$ denote the localization of the polynomial ring $R(H)[Q] := R(H)[Q_1, \dots, Q_n]$ with respect to the multiplicative set $1 + (Q_1, \dots, Q_n)$, where $(Q_1, \dots, Q_n)$ is the ideal of $R(H)[Q]$ generated by the variables $Q_1, \dots, Q_n$. Also, let $\mathcal{I}_{\mathrm{loc}}^Q$ denote the ideal of $(R(H)[Q]_{\mathrm{loc}})[x_1, \dots, x_n, x_{n+1}]$ generated by the same elements as for the ideal $\mathcal{I}^Q$. In [MaNS, Remark 6.3], we stated that the quotient ring $(R(H)[Q]_{\mathrm{loc}})[x_1, \dots, x_n, x_{n+1}]/\mathcal{I}_{\mathrm{loc}}^Q$ is isomorphic to the subring $QK_H(Fl_{n+1})_{\mathrm{loc}} := K_H(Fl_{n+1}) \otimes_{R(H)} R(H)[Q]_{\mathrm{loc}}$ of $QK_H(Fl_{n+1}) = K_H(Fl_{n+1}) \otimes_{R(H)} R(H)[[Q]]$. Based on this result, by the same line of argument as for the proof of Theorem 4.4, we can prove that the assertion of Theorem 4.4 still holds in this setting (i.e., in the setting where we use $R(H)[Q]_{\mathrm{loc}}$ in place of $R(H)[[Q]]$).

5. Proof of Proposition 2.4

5.1. General Chevalley formula

We briefly review the general Chevalley formula for $K_H(\mathbf{Q}_G)$, following [LNS] (cf. [KoLN, Theorem 5.16]); this formula plays an important role in the proof of Proposition 2.4. We use the notation and

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setting of Section 2.4. Recall that $\mathfrak{g}$ is an (arbitrary) finite-dimensional simple Lie algebra over $\mathbb{C}$. Let $\mathfrak{h}^*_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} P$ be a real form of $\mathfrak{h}^*$, and set

$$H_{\beta,l} := \left\{\mu \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle \mu, \beta^\vee \rangle = l\right\} \quad \text{for } \beta \in \Delta \text{ and } l \in \mathbb{Z}.$$

We denote by $s_{\beta,l}$ the affine reflection in the affine hyperplane $H_{\beta,l}$. The affine hyperplanes $H_{\beta,l}$, $\beta \in \Delta$, $l \in \mathbb{Z}$, divide the real vector space $\mathfrak{h}^*_{\mathbb{R}}$ into open regions, called alcoves; the fundamental alcove is defined as

$$A_\circ := \left\{\mu \in \mathfrak{h}^*_{\mathbb{R}} \mid 0 < \langle \mu, \alpha^\vee \rangle < 1 \quad \text{for all } \alpha \in \Delta^+\right\}.$$

We say that two alcoves are adjacent if they are distinct and have a common wall. Given a pair of adjacent alcoves A and B, we write $A \xrightarrow{\beta} B$ for $\beta \in \Delta$ if the common wall is orthogonal to β and β points in the direction from A to B.

Definition 5.1 [LP]. An alcove path is a sequence of alcoves $(A_0, A_1, \ldots, A_m)$ such that A_{j-1} and A_j are adjacent for $j = 1, \ldots, m$. We say that $(A_0, A_1, \ldots, A_m)$ is reduced if it has minimal length among all alcove paths from A_0 to A_m.

Let $\lambda \in P$, and let $A_\lambda = A_\circ + \lambda$ be the translation of the fundamental alcove $A_\circ$ by the weight λ.

Definition 5.2 [LP]. Let $\lambda \in P$. A sequence $(\beta_1, \beta_2, \ldots, \beta_m)$ of roots is called a reduced λ-chain (of roots) if

$$A_\circ = A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \cdots \xrightarrow{-\beta_m} A_m = A_{-\lambda} \tag{5.1}$$

is a reduced alcove path.

A reduced alcove path $(A_0 = A_\circ, A_1, \ldots, A_m = A_{-\lambda})$ can be identified with the corresponding total order on the hyperplanes, to be called λ-hyperplanes, which separate $A_\circ$ from $A_{-\lambda}$. This total order is given by the sequence $H_{\beta_i,-l_i}$ for $i = 1, \ldots, m$, where $H_{\beta_i,-l_i}$ contains the common wall of A_{i-1} and A_i. Note that $\langle \lambda, \beta_i^\vee \rangle \geq 0$, and that the integers l_i, called heights, have the following ranges:

$$\begin{aligned} &0 \leq l_i \leq \langle \lambda, \beta_i^\vee \rangle - 1 \quad \text{if} \quad \beta_i \in \Delta^+, \\ &1 \leq l_i \leq \langle \lambda, \beta_i^\vee \rangle \quad \text{if} \quad \beta_i \in \Delta^- = -\Delta^+. \end{aligned}$$

Note also that a reduced λ-chain $(\beta_1, \ldots, \beta_m)$ determines the corresponding reduced alcove path, and hence we can identify them.

Let $\lambda \in P$, and fix a reduced λ-chain, which we denote by $\Gamma(\lambda) = (\beta_1, \ldots, \beta_m)$; later in Section 5.2, for $\lambda = -\epsilon_J \in P$ with $J \subset [k]$, we make a specific choice of reduced λ-chain and denote it by Γ_J. Let $w \in W$.

Definition 5.3 [LL]. A (possibly empty) subset $A = \{j_1 < j_2 < \cdots < j_s\}$ of $[m] = \{1, \ldots, m\}$ is called a w-admissible subset if we have the following directed path in the quantum Bruhat graph $\mathrm{QBG}(W)$:

$$\begin{aligned} \Pi(w, A) : w &\xrightarrow{|\beta_{j_1}|} w s_{\beta_{j_1}} \xrightarrow{|\beta_{j_2}|} w s_{\beta_{j_1}} s_{\beta_{j_2}} \xrightarrow{|\beta_{j_3}|} \cdots \\ &\cdots \xrightarrow{|\beta_{j_s}|} w s_{\beta_{j_1}} s_{\beta_{j_2}} \cdots s_{\beta_{j_s}} =: \mathrm{end}(A). \end{aligned} \tag{5.2}$$

Let $\mathcal{A}(w, \Gamma(\lambda))$ denote the collection of all w-admissible subsets of $[m] = \{1, \ldots, m\}$.

Let $A = \{j_1 < \cdots < j_s\} \in \mathcal{A}(w, \Gamma(\lambda))$. The weight of A is defined by

$$\mathrm{wt}(A) := -w s_{\beta_{j_1}, -l_{j_1}} \cdots s_{\beta_{j_s}, -l_{j_s}}(-\lambda). \tag{5.3}$$

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Also, we set

$$n(A) := \#\{\beta_{j_1}, \dots, \beta_{j_s}\} \cap \Delta^-, \tag{5.4}$$

$$A^- := \left\{ j_i \in A \;\middle|\; \begin{array}{l} w s_{\beta_{j_1}} \cdots s_{\beta_{j_{i-1}}} \xrightarrow{|\beta_i|} w s_{\beta_{j_1}} \cdots s_{\beta_{j_{i-1}}} s_{\beta_{j_i}} \\ \text{is a quantum edge} \end{array} \right\}, \tag{5.5}$$

$$\mathrm{down}(A) := \sum_{j \in A^-} |\beta_j|^\vee \in Q^{\vee,+}. \tag{5.6}$$

We write $\lambda \in P$ as $\lambda = \sum_{i \in I} \lambda_i \varpi_i \in P$, with $\lambda_i \in \mathbb{Z}$ for $i \in I$. Following [LNS, Section 4.1], let $\overline{\mathrm{Par}(\lambda)}$ denote the set of I-tuples of partitions $\boldsymbol{\chi} = (\chi^{(i)})_{i \in I}$ such that $\chi^{(i)}$ is a partition of length at most $\max(\lambda_i, 0)$; in [NS1, Section 2.5], for $\lambda \in P^+$, we introduced the set $\mathrm{Par}(\lambda)$ of I-tuples of partitions $\boldsymbol{\chi} = (\chi^{(i)})_{i \in I}$ such that $\chi^{(i)}$ is a partition of length less than λ_i (which we do not use in this paper). For $\boldsymbol{\chi} = (\chi^{(i)})_{i \in I} \in \overline{\mathrm{Par}(\lambda)}$, we set $\iota(\boldsymbol{\chi}) := \sum_{i \in I} \chi_1^{(i)} \alpha_i^\vee \in Q^{\vee,+}$, with $\chi_1^{(i)}$ the first part of the partition $\chi^{(i)}$ for each $i \in I$.

By specializing at $q = 1$ in [LNS, Theorem 33] (cf. [KoLN, Theorem 5.16]), we obtain the following general Chevalley formula for $K_H(\mathbf{Q}_G)$.

Theorem 5.4. *Let $\lambda = \sum_{i \in I} \lambda_i \varpi_i \in P$ be an arbitrary weight, $\Gamma(\lambda)$ an arbitrary reduced λ-chain, and $x = w t_\xi \in W_{\mathrm{af}}^{\ge 0} \simeq W \times Q^{\vee,+}$. Then, the following equality holds in $K_H(\mathbf{Q}_G)$:*

$$\begin{aligned}&[\mathcal{O}_{\mathbf{Q}_G}(-w_\circ \lambda)] \otimes [\mathcal{O}_{\mathbf{Q}_G(x)}] \\ &\quad = \sum_{A \in \mathcal{A}(w, \Gamma(\lambda))} \sum_{\boldsymbol{\chi} \in \overline{\mathrm{Par}(\lambda)}} (-1)^{n(A)} \mathbf{e}^{\mathrm{wt}(A)} [\mathcal{O}_{\mathbf{Q}_G(\mathrm{end}(A) t_{\xi + \mathrm{down}(A) + \iota(\boldsymbol{\chi})})}].\end{aligned} \tag{5.7}$$

In order to prove equation (2.6), we compute

$$\left(\left(\prod_{j \in J^-} (1 - \mathrm{t}_j) \right) [\mathcal{O}_{\mathbf{Q}_G}(w_\circ \epsilon_J)] \right) \otimes [\mathcal{O}_{\mathbf{Q}_G(w_{k+1})}] \in K_H(\mathbf{Q}_G) \tag{5.8}$$

for $J = \{j_1, j_2, \dots, j_p\} \subset [k] = \{1, 2, \dots, k\}$, with $1 \le j_1 < j_2 < \cdots < j_p \le k$; recall Remark 2.3 and the definition of J^- from (2.4).

Lemma 5.5. *Keep the setting above. The following equality holds in $K_H(\mathbf{Q}_G)$:*

$$\begin{aligned}&\left(\left(\prod_{j \in J^-} (1 - \mathrm{t}_j) \right) [\mathcal{O}_{\mathbf{Q}_G}(w_\circ \epsilon_J)] \right) \otimes [\mathcal{O}_{\mathbf{Q}_G(w_{k+1})}] \\ &\qquad = \sum_{A \in \mathcal{A}(w_{k+1}, \Gamma(-\epsilon_J))} (-1)^{n(A)} \mathbf{e}^{\mathrm{wt}(A)} [\mathcal{O}_{\mathbf{Q}_G(\mathrm{end}(A) t_{\mathrm{down}(A)})}].\end{aligned} \tag{5.9}$$

Proof. Applying Theorem 5.4 to the case that $\lambda = -\epsilon_J$ and $x = w_{k+1}$, we obtain

$$[\mathcal{O}_{\mathbf{Q}_G}(w_\circ \epsilon_J)] \otimes [\mathcal{O}_{\mathbf{Q}_G(w_{k+1})}] = \sum_{A \in \mathcal{A}(w_{k+1}, \Gamma(-\epsilon_J))} \sum_{\boldsymbol{\chi} \in \overline{\mathrm{Par}(-\epsilon_J)}} (-1)^{n(A)} \mathbf{e}^{\mathrm{wt}(A)} [\mathcal{O}_{\mathbf{Q}_G(\mathrm{end}(A) t_{\mathrm{down}(A) + \iota(\boldsymbol{\chi})})}].$$

Here, observe that for $1 \le j \le n$, we have $\langle -\epsilon_J, \alpha_j^\vee \rangle \in \{-1, 0, 1\}$, and that the equality $\langle -\epsilon_J, \alpha_j^\vee \rangle = 1$ holds if and only if $j \in J^-$. This implies that $-\epsilon_J = \sum_{i \in J^-} \varpi_i$. Hence, $\boldsymbol{\chi} = (\chi^{(i)})_{i \in I} \in \overline{\mathrm{Par}(-\epsilon_J)}$ if and only if $\chi^{(i)}$ is a partition of length at most one for all $i \in J^-$ and the empty partition for all $i \notin J^-$.

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By the definition of $\iota(\chi)$, we deduce that the map $\chi \mapsto \iota(\chi)$ gives a bijection from $\overline{\mathrm{Par}(-\epsilon_J)}$ onto $\sum_{j\in J^-} \mathbb{Z}_{\geq 0}\alpha_j^\vee$. Therefore, we see that

$$\begin{aligned}
&\sum_{A\in\mathcal{A}(w_{k+1},\Gamma(-\epsilon_J))}\ \sum_{\chi\in\overline{\mathrm{Par}(-\epsilon_J)}} (-1)^{n(A)}\mathbf{e}^{\mathrm{wt}(A)}[\mathcal{O}_{\mathbf{Q}_G(\mathrm{end}(A)t_{\mathrm{down}(A)+\iota(\chi)})}]\\
&= \sum_{A\in\mathcal{A}(w_{k+1},\Gamma(-\epsilon_J))}\ \sum_{\zeta\in\sum_{j\in J^-}\mathbb{Z}_{\geq 0}\alpha_j^\vee} (-1)^{n(A)}\mathbf{e}^{\mathrm{wt}(A)}[\mathcal{O}_{\mathbf{Q}_G(\mathrm{end}(A)t_{\mathrm{down}(A)+\zeta})}]\\
&= \left(\prod_{j\in J^-}\frac{1}{1-\mathrm{t}_j}\right) \sum_{A\in\mathcal{A}(w_{k+1},\Gamma(-\epsilon_J))} (-1)^{n(A)}\mathbf{e}^{\mathrm{wt}(A)}[\mathcal{O}_{\mathbf{Q}_G(\mathrm{end}(A)t_{\mathrm{down}(A)})}].
\end{aligned}$$

From this, we deduce that in $K_H(\mathbf{Q}_G)$,

$$\begin{aligned}
&\left(\left(\prod_{j\in J^-}(1-\mathrm{t}_j)\right)[\mathcal{O}_{\mathbf{Q}_G(w_\circ\epsilon_J)}]\right)\otimes[\mathcal{O}_{\mathbf{Q}_G(w_{k+1})}]\\
&= \left(\prod_{j\in J^-}(1-\mathrm{t}_j)\right)\left([\mathcal{O}_{\mathbf{Q}_G(w_\circ\epsilon_J)}]\otimes[\mathcal{O}_{\mathbf{Q}_G(w_{k+1})}]\right) \quad \text{(see Remark 2.3)}\\
&= \left(\prod_{j\in J^-}(1-\mathrm{t}_j)\right)\left(\left(\prod_{j\in J^-}\frac{1}{1-\mathrm{t}_j}\right)\sum_{A\in\mathcal{A}(w_{k+1},\Gamma(-\epsilon_J))}(-1)^{n(A)}\mathbf{e}^{\mathrm{wt}(A)}[\mathcal{O}_{\mathbf{Q}_G(\mathrm{end}(A)t_{\mathrm{down}(A)})}]\right)\\
&= \sum_{A\in\mathcal{A}(w_{k+1},\Gamma(-\epsilon_J))}(-1)^{n(A)}\mathbf{e}^{\mathrm{wt}(A)}[\mathcal{O}_{\mathbf{Q}_G(\mathrm{end}(A)t_{\mathrm{down}(A)})}],
\end{aligned}$$

as desired. This proves the lemma. □

5.2. A special reduced chain (of roots)

Let us fix $J = \{j_1, j_2, \ldots, j_p\} \subset [k]$, with $1 \leq j_1 < j_2 < \cdots < j_p \leq k$. Following [LNOS, Lemma 4.1], we take a special reduced $(-\epsilon_J)$-chain (of roots) from $A_\circ$ to A_{ϵ_J} (see Definition 5.2) as follows. Notice that $\epsilon_J \in W\varpi_p$. Let x_J be the (unique) minimal-length element in $\{w \in W \mid w\varpi_p = \epsilon_J\}$, and let $x_J = s_{i_a}\cdots s_{i_1}$, with $a := \ell(x_J)$, be the reduced expression of the following form:

$$\begin{aligned}
x_J &= s_{i_a}\cdots s_{i_1}\\
&= \underbrace{(s_{j_1-1}\cdots s_2 s_1)}_{\text{mapping } \epsilon_1 \text{ to } \epsilon_{j_1}.}\ \underbrace{(s_{j_2-1}\cdots s_3 s_2)}_{\text{mapping } \epsilon_2 \text{ to } \epsilon_{j_2}.}\cdots\\
&\qquad \cdots \underbrace{(s_{j_{p-1}-1}\cdots s_p s_{p-1})}_{\text{mapping } \epsilon_{p-1} \text{ to } \epsilon_{j_{p-1}}.}\ \underbrace{(s_{j_p-1}\cdots s_{p+1}s_p)}_{\text{mapping } \epsilon_p \text{ to } \epsilon_{j_p}.}.
\end{aligned} \tag{5.10}$$

Also, let y_J be the (unique) element such that $y_J x_J$ is the (unique) minimal-length element in $\{w \in W \mid w\varpi_p = w_\circ\varpi_p\}$, and let $y_J = s_{k_1}\cdots s_{k_b}$, with $b := \ell(y_J)$, be the reduced expression of the following form:

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$$\begin{aligned} y_J &= s_{k_1} \cdots s_{k_b} \\ &= \underbrace{(s_{n-p+1} \cdots s_{j_1+1} s_{j_1})}_{\text{mapping } \epsilon_{j_1} \text{ to } \epsilon_{n-p+2}.} \underbrace{(s_{n-p+2} \cdots s_{j_2+1} s_{j_2})}_{\text{mapping } \epsilon_{j_2} \text{ to } \epsilon_{n-p+3}.} \cdots \\ &\qquad \cdots \underbrace{(s_{n-1} \cdots s_{j_{p-1}+1} s_{j_{p-1}})}_{\text{mapping } \epsilon_{j_{p-1}} \text{ to } \epsilon_{n}.} \underbrace{(s_{n} \cdots s_{j_p+1} s_{j_p})}_{\text{mapping } \epsilon_{j_p} \text{ to } \epsilon_{n+1}.}. \end{aligned} \tag{5.11}$$

We set

$$\beta_c := s_{i_a} \cdots s_{i_{c+1}} \alpha_{i_c} \in \Delta^+ \quad \text{for } 1 \le c \le a, \tag{5.12}$$

$$\gamma_d := s_{k_b} \cdots s_{k_{d+1}} \alpha_{k_d} \in \Delta^+ \quad \text{for } 1 \le d \le b. \tag{5.13}$$

We call the sequences $\boldsymbol{\beta} := (\beta_a, \dots, \beta_1)$ and $\boldsymbol{\gamma} := (\gamma_1, \dots, \gamma_b)$ the $\boldsymbol{\beta}$-sequence and $\boldsymbol{\gamma}$-sequence for J, respectively.

Proposition 5.6 [LNOS, Lemma 4.1]. *If we set*

$$\Gamma_J = (\zeta_1, \dots, \zeta_{a+b}) := (-\beta_a, \dots, -\beta_1, \gamma_1, \dots, \gamma_b), \tag{5.14}$$

then Γ_J is a reduced $(-\epsilon_J)$-chain from $A_\circ$ to A_{ϵ_J}. Moreover, for $1 \le c \le a$, the affine hyperplane between the $(c-1)$-th alcove and the c-th alcove in Γ_J is $H_{-\beta_{a-c+1},0}$, and for $1 \le d \le b$, the affine hyperplane between the $(a+d-1)$-th alcove and the $(a+d)$-th alcove in Γ_J is $H_{\gamma_d,1}$; remark that $\epsilon_J \in H_{\gamma_d,1}$, and hence, $s_{\gamma_d,1}(\epsilon_J) = \epsilon_J$ for all $1 \le d \le b$ (see also Remark 5.11 below).

Example 5.7. Assume that $n = 5$, $k = 4$ and $J = \{2, 4\}$. In this case, we see that

$$x_J = s_1 s_3 s_2, \qquad y_J = s_4 s_3 s_2 s_5 s_4;$$

in particular, $a = \ell(x_J) = 3$ and $b = \ell(y_J) = 5$. We have

$$\begin{aligned} \beta_3 &= \alpha_1 = (1,2), \\ \beta_2 &= s_1 \alpha_3 = \alpha_3 = (3,4), \\ \beta_1 &= s_1 s_3 \alpha_2 = \alpha_1 + \alpha_2 + \alpha_3 = (1,4), \\ \gamma_1 &= s_4 s_5 s_2 s_3 \alpha_4 = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = (2,6), \\ \gamma_2 &= s_4 s_5 s_2 \alpha_3 = \alpha_2 + \alpha_3 + \alpha_4 = (2,5), \\ \gamma_3 &= s_4 s_5 \alpha_2 = \alpha_2 = (2,3), \\ \gamma_4 &= s_4 \alpha_5 = \alpha_4 + \alpha_5 = (4,6), \\ \gamma_5 &= \alpha_4 = (4,5). \end{aligned}$$

Thus,

$$\begin{aligned} \Gamma_J = (&-\alpha_1, -\alpha_3, -(\alpha_1 + \alpha_2 + \alpha_3), \\ &\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_2 + \alpha_3 + \alpha_4, \alpha_2, \alpha_4 + \alpha_5, \alpha_4) \end{aligned}$$

is a reduced $(-\epsilon_J)$-chain from $A_\circ$ to A_{ϵ_J}.

Let us compute $\beta_c \in \Delta^+$, $1 \le c \le a$, more explicitly. For $1 \le u \le p$ and $u \le t \le j_u - 1$, we set

$$\beta_{u,t} := (s_{j_1-1} \cdots s_2 s_1) \cdots (s_{j_{u-1}-1} \cdots s_u s_{u-1})(s_{j_u-1} \cdots s_{t+1}) \alpha_t;$$

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notice that the sequence $\boldsymbol{\beta} = (\beta_a, \ldots, \beta_1)$ is identical to

$$\begin{aligned}&(\beta_{1,j_1-1}, \ldots, \beta_{1,2}, \beta_{1,1},\ \beta_{2,j_2-1}, \ldots, \beta_{2,3}, \beta_{2,2},\ \ldots\ldots,\\ &\beta_{p-1,j_{p-1}-1}, \ldots, \beta_{p-1,p}, \beta_{p-1,p-1},\ \beta_{p,j_p-1}, \ldots, \beta_{p,p+1}, \beta_{p,p}).\end{aligned} \tag{5.15}$$

We set $\tau(u,t) := |\{1 \le r \le u-1 \mid t-r+1 \le j_{u-r}\}|$.

Lemma 5.8. *The equality* $\beta_{u,t} = \epsilon_{t-\tau(u,t)} - \epsilon_{j_u}$ *holds for* $1 \le u \le p$ *and* $u \le t \le j_u - 1$.

Proof. For $1 \le u \le p$ and $u \le t \le j_u - 1$, we have

$$\begin{aligned}\beta_{u,t} &:= (s_{j_1-1}\cdots s_2 s_1)\cdots(s_{j_{u-1}-1}\cdots s_u s_{u-1})(s_{j_u-1}\cdots s_{t+1})(\epsilon_t - \epsilon_{t+1})\\ &= (s_{j_1-1}\cdots s_2 s_1)\cdots(s_{j_{u-1}-1}\cdots s_u s_{u-1})(\epsilon_t - \epsilon_{j_u})\\ &= \begin{cases}\epsilon_t - \epsilon_{j_u} & \text{if } t > j_{u-1},\\ (s_{j_1-1}\cdots s_2 s_1)\cdots(s_{j_{u-2}-1}\cdots s_{u-1} s_{u-2})(\epsilon_{t-1} - \epsilon_{j_u}) & \text{otherwise.}\end{cases}\end{aligned}$$

In the case $t \le j_{u-1}$, we have

$$\begin{aligned}&(s_{j_1-1}\cdots s_2 s_1)\cdots(s_{j_{u-2}-1}\cdots s_{u-1} s_{u-2})(\epsilon_{t-1} - \epsilon_{j_u})\\ &= \begin{cases}\epsilon_{t-1} - \epsilon_{j_u} & \text{if } t-1 > j_{u-2},\\ (s_{j_1-1}\cdots s_2 s_1)\cdots(s_{j_{u-3}-1}\cdots s_{u-2} s_{u-3})(\epsilon_{t-2} - \epsilon_{j_u}) & \text{otherwise.}\end{cases}\end{aligned}$$

Repeating this computation, we deduce that $\beta_{u,t} = \epsilon_{t-\tau(u,t)} - \epsilon_{j_u}$. This proves the lemma. ◻

Remark 5.9. Let $1 \le u \le p$, and write

$$\{1, 2, \ldots, j_u - 1\} \setminus \{j_1, \ldots, j_{u-1}\} = \{t_{u,1} < t_{u,2} < \cdots < t_{u,m_u}\}, \tag{5.16}$$

where $m_u = (j_u - 1) - (u - 1) = j_u - u$; we set $t_{u,m_u+1} := j_u$ by convention. Then, we have

$$(\beta_{u,j_u-1}, \ldots, \beta_{u,u+1}, \beta_{u,u}) = (\epsilon_{t_{u,m_u}} - \epsilon_{j_u},\ \epsilon_{t_{u,m_u-1}} - \epsilon_{j_u},\ \ldots,\ \epsilon_{t_{u,2}} - \epsilon_{j_u},\ \epsilon_{t_{u,1}} - \epsilon_{j_u}).$$

Hence, the sequence $\boldsymbol{\beta} = (\beta_a, \ldots, \beta_1)$ is of the form

$$\begin{aligned}&(j_1-1, j_1), (j_1-2, j_1), \ldots, (2, j_1), (1, j_1),\\ &(j_2-1, j_2), (j_2-2, j_2), \ldots, \cancel{(j_1, j_2)}, \ldots, (2, j_2), (1, j_2),\\ &(j_3-1, j_3), (j_3-2, j_3), \ldots, \cancel{(j_2, j_3)}, \ldots, \cancel{(j_1, j_3)}, \ldots, (2, j_3), (1, j_3),\\ &\ldots\ldots\\ &(j_p-1, j_p), (j_p-2, j_p), \ldots, \cancel{(j_{p-1}, j_p)}, \ldots, \cancel{(j_2, j_p)}, \ldots, \cancel{(j_1, j_p)}, \ldots, (2, j_p), (1, j_p),\end{aligned}$$

where (j_r, j_s) does not appear in the sequence above for any $1 \le r < s \le p$.

Similarly, let us compute $\gamma_d \in \Delta^+$, $1 \le d \le b$, more explicitly. For $1 \le u \le p$ and $j_u \le t \le n-p+u$, we set

$$\gamma_{u,t} := (s_{j_p} s_{j_p+1}\cdots s_n)\cdots(s_{j_{u+1}} s_{j_{u+1}+1}\cdots s_{n-p+u+1})(s_{j_u} s_{j_u+1}\cdots s_{t-1})\alpha_t;$$

notice that the sequence $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_b)$ is identical to

$$\begin{aligned}&(\gamma_{1,n-p+1}, \ldots, \gamma_{1,j_1+1}, \gamma_{1,j_1},\ \gamma_{2,n-p+2}, \ldots, \gamma_{2,j_2+1}, \gamma_{2,j_2},\ \ldots\ldots,\\ &\gamma_{p-1,n-1}, \ldots, \gamma_{p-1,j_{p-1}+1}, \gamma_{p-1,j_{p-1}},\ \gamma_{p,n}, \ldots, \gamma_{p,j_p+1}, \gamma_{p,j_p}).\end{aligned} \tag{5.17}$$

We set $\sigma(u,t) := |\{u+1 \leq r \leq p \mid t+r-u \geq j_r\}|$. The proof of the following lemma is similar to that of Lemma 5.8.

Lemma 5.10. *The equality* $\gamma_{u,t} = \epsilon_{j_u} - \epsilon_{t+1+\sigma(u,t)}$ *holds for* $1 \leq u \leq p$ *and* $j_u \leq t \leq n-p+u$.

Remark 5.11. Let $1 \leq u \leq p$, and write

$$\{j_u+1, j_u+2, \ldots, n+1\} \setminus \{j_{u+1}, \ldots, j_p\} = \{t_1 < t_2 < \cdots < t_m\},$$

where $m = (n+1-j_u)-(p-u) = n+1-p-j_u+u$. Then, we have

$$\begin{aligned}&(\gamma_{u,n-p+u}, \ldots, \gamma_{u,j_u+1}, \gamma_{u,j_u})\\ &= (\epsilon_{j_u}-\epsilon_{t_m},\ \epsilon_{j_u}-\epsilon_{t_{m-1}},\ \ldots,\ \epsilon_{j_u}-\epsilon_{t_2},\ \epsilon_{j_u}-\epsilon_{t_1}).\end{aligned}$$

Hence, the sequence $\gamma = (\gamma_1, \ldots, \gamma_b)$ is of the form

$$\begin{aligned}&(j_1, n+1), (j_1, n), \ldots, \cancel{(j_1, j_p)}, \ldots, \cancel{(j_1, j_2)}, \ldots, (j_1, j_1+2), (j_1, j_1+1),\\ &(j_2, n+1), (j_2, n), \ldots, \cancel{(j_2, j_p)}, \ldots, \cancel{(j_2, j_3)}, \ldots, (j_2, j_2+2), (j_2, j_2+1),\\ &\cdots\cdots\\ &(j_{p-1}, n+1), (j_{p-1}, n), \ldots, \cancel{(j_{p-1}, j_p)}, \ldots, (j_{p-1}, j_{p-1}+2), (j_{p-1}, j_{p-1}+1),\\ &(j_p, n+1), (j_p, n), \ldots, (j_p, j_p+2), (j_p, j_p+1),\end{aligned}$$

where (j_r, j_s) does not appear in the sequence above for any $1 \leq r < s \leq p$.

Remark 5.12. By Remarks 5.9 and 5.11, the intersection $\{\beta_c \mid 1 \leq c \leq a\} \cap \{\gamma_d \mid 1 \leq d \leq b\}$ is empty.

5.3. Proof of Proposition 2.4

For each $J = \{j_1, j_2, \ldots, j_p\} \subset [k]$, let $\mathbf{D}_J$ be the set of directed paths in $\mathrm{QBG}(W)$ of the form

$$\begin{aligned}&\mathbf{p} : w_{k+1} = \underbrace{x_s \xrightarrow{\beta_{a_s}} \cdots \xrightarrow{\beta_{a_1}} x_0}_{=:\,\mathbf{p}_\beta,\text{ the }\beta\text{-part of }\mathbf{p}} = \underbrace{y_0 \xrightarrow{\gamma_{b_1}} \cdots \xrightarrow{\gamma_{b_t}} y_t}_{=:\,\mathbf{p}_\gamma,\text{ the }\gamma\text{-part of }\mathbf{p}} =: \mathrm{end}(\mathbf{p}),\\ &\text{with } s \geq 0, a \geq a_s > \cdots > a_1 \geq 1,\\ &\text{and } t \geq 0, 1 \leq b_1 < b_2 < \cdots < b_t \leq b.\end{aligned} \tag{5.18}$$

We call the directed subpath $w_{k+1} = x_s \xrightarrow{\beta_{a_s}} \cdots \xrightarrow{\beta_{a_1}} x_0$ of $\mathbf{p}$ the β-part of $\mathbf{p}$ and denote it by $\mathbf{p}_\beta$. Notice that the label sequence $(\beta_{a_s}, \ldots, \beta_{a_1})$ of $\mathbf{p}_\beta$ is a subsequence of $\boldsymbol{\beta} = (\beta_a, \ldots, \beta_1)$, described in Lemma 5.8 and Remark 5.9, with $a = \ell(x_J)$; the subscript $\boldsymbol{\beta}$ of $\mathbf{p}_\beta$ indicates that $\mathbf{p}_\beta$ is obtained from $\mathbf{p}$ by taking the first consecutive edges whose labels are contained in the $\boldsymbol{\beta}$-sequence. Similarly, we call the directed subpath $y_0 \xrightarrow{\gamma_{b_1}} \cdots \xrightarrow{\gamma_{b_t}} y_t = \mathrm{end}(\mathbf{p})$ of $\mathbf{p}$ the γ-part of $\mathbf{p}$ and denote it by $\mathbf{p}_\gamma$. Notice that the label sequence $(\gamma_{b_1}, \ldots, \gamma_{b_t})$ is a subsequence of $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_b)$, described in Lemma 5.15 and Remark 5.11, with $b = \ell(y_J)$; the subscript $\boldsymbol{\gamma}$ of $\mathbf{p}_\gamma$ indicates that $\mathbf{p}_\gamma$ is obtained from $\mathbf{p}$ by taking the second consecutive edges whose labels are contained in the $\boldsymbol{\gamma}$-sequence.

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Proposition 5.13. *The following equality holds in $K_H(\mathbf{Q}_G)$:*

$$\overbrace{\left(\sum_{p=0}^{k}(-1)^p \mathbf{e}^{p\,\epsilon_{n+1-k}}\mathfrak{F}_p^k\right)\otimes[\mathcal{O}_{\mathbf{Q}_G(w_{k+1})}]}^{\text{LHS of (2.6)}}$$
$$= \sum_{J\subset[k]}\sum_{\mathbf{p}\in\mathbf{D}_J}\mathbf{e}^{|J|\epsilon_{n+1-k}-\mathrm{end}(\mathbf{p}_\beta)\epsilon_J}(-1)^{|J|+\ell(\mathbf{p}_\gamma)}[\mathcal{O}_{\mathbf{Q}_G(\mathrm{end}(\mathbf{p})t_{\mathrm{wt}(\mathbf{p})})}]. \tag{5.19}$$

Proof. First recall Lemma 5.5. It is easy to see that there exists a bijection between the sets $\mathbf{D}_J$ and $\mathcal{A}(w_{k+1},\Gamma_J)$. Indeed, for a subset $A=\{p_1,\dots,p_u\}$ of $[a+b]=\{1,2,\dots,a+b\}$, A is an element of $\mathcal{A}(w_{k+1},\Gamma_J)$ if and only if

$$\mathbf{p}(A): w_{k+1}=w_0\xrightarrow{\zeta_{p_1}}w_1\xrightarrow{\zeta_{p_2}}\cdots\xrightarrow{\zeta_{p_u}}w_u \tag{5.20}$$

is an element of $\mathbf{D}_J$. Notice that $\mathrm{end}(A)=\mathrm{end}(\mathbf{p}(A))$ and $\mathrm{down}(A)=\mathrm{wt}(\mathbf{p})$. For $\mathbf{p}\in\mathbf{D}_J$ of the form (5.18), we set $\mathrm{end}(\mathbf{p}_\beta):=x_0$ and $\ell(\mathbf{p}_\gamma):=t$; we see that for $A\in\mathcal{A}(w_{k+1},\Gamma_J)$, $\mathrm{wt}(A)=-\mathrm{end}(\mathbf{p}(A)_\beta)\epsilon_J$ and $n(A)=\ell(\mathbf{p}(A)_\gamma)$. Therefore, we deduce that in $K_H(\mathbf{Q}_G)$,

$$\left(\left(\prod_{j\in J^-}(1-\mathrm{t}_j)\right)[\mathcal{O}_{\mathbf{Q}_G}(w_\circ\epsilon_J)]\right)\otimes[\mathcal{O}_{\mathbf{Q}_G(w_{k+1})}]$$
$$=\sum_{A\in\mathcal{A}(w_{k+1},\Gamma(-\epsilon_J))}(-1)^{n(A)}\mathbf{e}^{\mathrm{wt}(A)}[\mathcal{O}_{\mathbf{Q}_G(\mathrm{end}(A)t_{\mathrm{down}(A)})}] \qquad \text{by (5.9)}$$
$$=\sum_{\mathbf{p}\in\mathbf{D}_J}\mathbf{e}^{-\mathrm{end}(\mathbf{p}_\beta)\epsilon_J}(-1)^{\ell(\mathbf{p}_\gamma)}[\mathcal{O}_{\mathbf{Q}_G(\mathrm{end}(\mathbf{p})t_{\mathrm{wt}(\mathbf{p})})}],$$

and hence that

$$\left(\sum_{p=0}^{k}(-1)^p \mathbf{e}^{p\,\epsilon_{n+1-k}}\mathfrak{F}_p^k\right)\otimes[\mathcal{O}_{\mathbf{Q}_G(w_{k+1})}]$$
$$=\sum_{p=0}^{k}(-1)^p\mathbf{e}^{p\,\epsilon_{n+1-k}}\sum_{\substack{J\subset[k]\\|J|=p}}\left(\left(\prod_{j\in J^-}(1-\mathrm{t}_j)\right)[\mathcal{O}_{\mathbf{Q}_G}(w_\circ\epsilon_J)]\right)\otimes[\mathcal{O}_{\mathbf{Q}_G(w_{k+1})}]$$
$$=\sum_{J\subset[k]}(-1)^{|J|}\mathbf{e}^{|J|\epsilon_{n+1-k}}\left(\left(\prod_{j\in J^-}(1-\mathrm{t}_j)\right)[\mathcal{O}_{\mathbf{Q}_G}(w_\circ\epsilon_J)]\right)\otimes[\mathcal{O}_{\mathbf{Q}_G(w_{k+1})}]$$
$$=\sum_{J\subset[k]}\sum_{\mathbf{p}\in\mathbf{D}_J}\mathbf{e}^{|J|\epsilon_{n+1-k}-\mathrm{end}(\mathbf{p}_\beta)\epsilon_J}(-1)^{|J|+\ell(\mathbf{p}_\gamma)}[\mathcal{O}_{\mathbf{Q}_G(\mathrm{end}(\mathbf{p})t_{\mathrm{wt}(\mathbf{p})})}].$$

This proves the proposition. □

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Recall that

$$\begin{aligned} w_{k+1} &:= (s_1 s_2 \cdots s_n)(s_1 s_2 \cdots s_{n-1}) \cdots (s_1 s_2 \cdots s_{k+1}) \\ &= (1, 2, \ldots, n+1)(1, 2, \ldots, n) \cdots (1, 2, \ldots, k+2) \\ &= \begin{pmatrix} 1 & 2 & \cdots & k & k+1 & k+2 & \cdots & n & n+1 \\ n-k+1 & n-k+2 & \cdots & n & n+1 & n-k & \cdots & 2 & 1 \end{pmatrix}. \end{aligned}$$

Also, recall that $J = \{j_1, j_2, \ldots, j_p\} \subset [k]$, with $1 \le j_1 < j_2 < \cdots < j_p \le k$, and recall from (5.14) that $\Gamma_J = (-\beta_a, \ldots, -\beta_1, \gamma_1, \ldots, \gamma_b)$. Let $\mathbf{p} \in \mathbf{D}_J$. We see from Remark 5.9 that the (label sequence of the) β-part $\mathbf{p}_\beta$ of $\mathbf{p}$ is of the following form:

$$\begin{aligned} & w_{k+1} \xrightarrow{(t_{1,m_1}, j_1)} \bullet \xrightarrow{(t_{1,m_1-1}, j_1)} \cdots \xrightarrow{(t_{1,m_1-r_1+1}, j_1)} \bullet \\ & \xrightarrow{(t_{2,m_2}, j_2)} \bullet \xrightarrow{(t_{2,m_2-1}, j_2)} \cdots \xrightarrow{(t_{2,m_2-r_2+1}, j_2)} \bullet \\ & \cdots\cdots \\ & \xrightarrow{(t_{p,m_p}, j_p)} \bullet \xrightarrow{(t_{p,m_p-1}, j_p)} \cdots \xrightarrow{(t_{p,m_p-r_p+1}, j_p)} \bullet, \end{aligned} \tag{5.21}$$

where $m_q \ge r_q \ge 0$ for $1 \le q \le p$; for each $1 \le q \le p$ such that $r_q \ge 1$, we have

$$\begin{aligned} & j_q > t_{q,m_q} > t_{q,m_q-1} > \cdots > t_{q,m_q-r_q+1} \ge 1, \\ & \qquad\qquad \text{with } t_{q,m_q-r+1} \ne j_1, j_2, \ldots, j_{q-1} \text{ for } 1 \le r \le r_q. \end{aligned} \tag{5.22}$$

By convention, we set $t_{q,m_q+1} := j_q$ for $1 \le q \le p$.

Lemma 5.14. *Keep the setting above. If (5.21) is the β-part of an element $\mathbf{p} \in \mathbf{D}_J$, then the following are satisfied:*

(i) $t_{p,m_p-r_p+1} \ge \cdots \ge t_{2,m_2-r_2+1} \ge t_{1,m_1-r_1+1}$;
(ii) *for each $1 \le q \le p$ such that $r_q \ge 1$, the sequence $t_{q,m_q} > t_{q,m_q-1} > \cdots > t_{q,m_q-r_q+1}$ is identical to the largest r_q elements in $\{1, 2, \ldots, j_q - 1\} \setminus \{j_1, j_2, \ldots, j_{q-1}\}$.*

Moreover, all the edges in (5.21) are Bruhat edges. Conversely, let $0 \le r_q \le m_q$ for $1 \le q \le p$, and assume that $\{t_{q,m_q-r+1} \mid 1 \le q \le p,\ 1 \le r \le r_q\}$ satisfy (5.22), and (i), (ii) above. Then, (5.21) is a directed path in QBG(W) *whose edges are all Bruhat edges; in particular, there exists an element* $\mathbf{p}$ *whose β-part $\mathbf{p}_\beta$ is of the form (5.21).*

Proof. It follows from Lemma 2.2 that

$$w_{k+1} = x_0 \xrightarrow{(k_1, j_1)} x_1 \xrightarrow{(k_2, j_1)} \cdots \xrightarrow{(k_r, j_1)} x_r \tag{5.23}$$

is a directed path in QBG(W), where $r \ge 0$ and $j_1 - 1 \ge k_1 > k_2 > \cdots > k_r \ge 1$, if and only if $k_1 = j_1 - 1$, $k_2 = j_1 - 2, \ldots, k_r = j_1 - r$; in this case, all the edges in (5.23) are Bruhat edges.

Also, it follows from Lemma 2.2 that

$$\begin{aligned} w_{k+1} = x_0 & \xrightarrow{(j_1-1, j_1)} x_1 \xrightarrow{(j_1-2, j_1)} \cdots \xrightarrow{(j_1-r_1, j_1)} x_{r_1} \\ & \xrightarrow{(k_1, j_2)} x_{r_1+1} \xrightarrow{(k_2, j_2)} \cdots \xrightarrow{(k_r, j_2)} x_{r_1+r} \end{aligned} \tag{5.24}$$

is a directed path in QBG(W), where $r_1, r \ge 0$ and $j_2 - 1 \ge k_1 > k_2 > \cdots > k_r \ge 1$ with $k_{r'} \ne j_1$ for any $1 \le r' \le r$, if and only if $\{k_1, k_2, \ldots . k_r\}$ is identical to the largest r elements in $\{1, 2, \ldots, j_2 - 1\} \setminus \{j_1\}$ and $k_r \ge j_1 - r_1$; in this case, all the edges in (5.24) are Bruhat edges.

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Similarly, it follows from Lemma 2.2 that

$$\begin{aligned} w_{k+1} = x_0 &\xrightarrow{(j_1-1,j_1)} x_1 \xrightarrow{(j_1-2,j_1)} \cdots \xrightarrow{(j_1-r_1,j_1)} x_{r_1} \\ &\xrightarrow{(t_{2,m_2},j_2)} x_{r_1+1} \xrightarrow{(t_{2,m_2-1},j_2)} \cdots \xrightarrow{(t_{2,m_2-r_2+1},j_2)} x_{r_1+r_2} \\ &\xrightarrow{(k_1,j_3)} x_{r_1+r_2+1} \xrightarrow{(k_2,j_3)} \cdots \xrightarrow{(k_r,j_3)} x_{r_1+r_2+r} \end{aligned} \tag{5.25}$$

is a directed path in $\mathrm{QBG}(W)$, where $r_1, r_2, r \geq 0$, $t_{2,m_2-r_2+1} \geq j_1 - r_1$, and $j_3 - 1 \geq k_1 > k_2 > \cdots > k_r \geq 1$ with $k_{r'} \neq j_1, j_2$ for any $1 \leq r' \leq r$, if and only if $\{k_1, k_2, \ldots .k_r\}$ is identical to the largest r elements in $\{1, 2, \ldots, j_3 - 1\} \setminus \{j_1, j_2\}$ and $k_r \geq t_{2,m_2-r_2+1}$; in this case, all the edges in (5.25) are Bruhat edges.

Repeating the argument above, we can verify the assertions of the lemma. □

We now define

$$\mathbf{D} := \bigsqcup_{J \subset [k]} \mathbf{D}_J. \tag{5.26}$$

Let $J = \{j_1, j_2, \ldots, j_p\} \subset [k]$, and let $\mathbf{p} \in \mathbf{D}$ be of the form (5.18); for simplicity of notation, we will henceforth write as

$$\begin{aligned} \mathbf{p} &= [\beta_{a_s}, \ldots, \beta_{a_1} \mid \gamma_{b_1}, \ldots, \gamma_{b_t}], \\ \mathbf{p}_\beta &= [\beta_{a_s}, \ldots, \beta_{a_1}], \quad \mathbf{p}_\gamma = [\gamma_{b_1}, \ldots, \gamma_{b_t}]. \end{aligned}$$

Then we define subsets $\mathbf{D}_J^{\mathrm{A}}$, $\mathbf{D}_J^{\mathrm{B}}$, and $\mathbf{D}_J^{\mathrm{C}}$ of $\mathbf{D}_J$ as follows (recall (5.21), (5.22) and Lemma 5.14):

(A) When $J \neq \emptyset$ and $j_1 = \min J \geq 2$ (and hence $1 \notin J$), we define $\mathbf{D}_J^{\mathrm{A}}$ to be the subset of $\mathbf{D}_J$ consisting of all those elements $\mathbf{p} \in \mathbf{D}_J$ whose β-part $\mathbf{p}_\beta$ is of the form

$$\begin{aligned} \mathbf{p}_\beta = [&(j_1 - 1, j_1), (j_1 - 2, j_1), \ldots, (2, j_1), (1, j_1), \\ &\underbrace{(t_{2,m_2}, j_2), \ldots, (t_{2,m_2-r_2+1}, j_2), \ldots\ldots, (t_{p,m_p}, j_p), \ldots, (t_{p,m_p-r_p+1}, j_p)}_{=:\mathbf{s}}]. \end{aligned} \tag{5.27}$$

(B) When $J \neq \emptyset$ and $j_1 = \min J$ is equal to 1, we set $\mathbf{D}_J^{\mathrm{B}} := \mathbf{D}_J$; note that the β-part $\mathbf{p}_\beta$ of $\mathbf{p} \in \mathbf{D}_J^{\mathrm{B}}$ is of the form

$$\begin{aligned} \mathbf{p}_\beta = [&(t_{2,m_2}, j_2), \ldots, (t_{2,m_2-r_2+1}, j_2), \\ &\ldots\ldots, (t_{p,m_p}, j_p), \ldots, (t_{p,m_p-r_p+1}, j_p)]. \end{aligned} \tag{5.28}$$

(C) When $J = \emptyset$, we set $\mathbf{D}_J^{\mathrm{C}} := \mathbf{D}_J$. Also, when $J \neq \emptyset$ and $j_1 = \min J \geq 2$, we define $\mathbf{D}_J^{\mathrm{C}}$ to be the subset of $\mathbf{D}_J$ consisting of all those elements $\mathbf{p} \in \mathbf{D}_J$ whose β-part $\mathbf{p}_\beta$ is of the form

$$\begin{aligned} \mathbf{p}_\beta = [&(j_1 - 1, j_1), (j_1 - 2, j_1), \ldots, (j_1 - r_1 + 1, j_1), (j_1 - r_1, j_1), \\ &(t_{2,m_2}, j_2), \ldots, (t_{2,m_2-r_2+1}, j_2), \ldots\ldots, (t_{p,m_p}, j_p), \ldots, (t_{p,m_p-r_p+1}, j_p)], \end{aligned} \tag{5.29}$$

where $r_1 = 0$, or $r_1 \geq 1$ and $j_1 - r_1 \geq 2$.

It is easily seen that $\mathbf{D}_J = \mathbf{D}_J^{\mathrm{A}} \sqcup \mathbf{D}_J^{\mathrm{B}} \sqcup \mathbf{D}_J^{\mathrm{C}}$.

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Lemma 5.15. *Let $\mathbf{p} \in \mathbf{D}_J^{\mathrm{A}} \sqcup \mathbf{D}_J^{\mathrm{B}}$, and assume that the γ-part $\mathbf{p}_\gamma$ of $\mathbf{p}$ is of the form*

$$\mathrm{end}(\mathbf{p}_\beta) \xrightarrow{(j_1,k_r)} \bullet \xrightarrow{(j_1,k_{r-1})} \cdots \xrightarrow{(j_1,k_2)} \bullet \xrightarrow{(j_1,k_1)} \bullet \\ \cdots\cdots,$$

where $r \geq 0$, and $n+1 \geq k_r > k_{r-1} > \cdots > k_1 \geq j_1 + 1$ with $k_u \notin \{j_2, \ldots, j_p\}$ for any $1 \leq u \leq r$. Then, either of the following holds: (i) $r = 0$, or (ii) $r = 1$ and $k_1 = j_1 + 1$; note that case (ii) does not occur if $j_1 + 1 \in J$. In case (ii), the edge $\mathrm{end}(\mathbf{p}_\beta) \xrightarrow{(j_1,j_1+1)} \bullet$ is a Bruhat edge.

Proof. Assume that $r > 0$. Note that $\mathrm{end}(\mathbf{p}_\beta)(j_1) = w_{k+1}(1) = n - k + 1$; see (5.27) and (5.28). If $k_r \geq k+2$, then

$$\mathrm{end}(\mathbf{p}_\beta)(k_r) = w_{k+1}(k_r) = n - k_r + 2 < n - k + 1 = \mathrm{end}(\mathbf{p}_\beta)(j_1).$$

However, $j_1 < k+1 < k_r$, and $\mathrm{end}(\mathbf{p}_\beta)(k+1) = n+1$ is greater than both $\mathrm{end}(\mathbf{p}_\beta)(k_r)$ and $\mathrm{end}(\mathbf{p}_\beta)(j_1)$, which is a contradiction (see Lemma 2.2). Thus, we see that $k_r \leq k+1$.

Suppose, for a contradiction, that $j_1 + 1 = j_2 \in J$; note that $j_2 < k_r \leq k+1$ and $k_r \notin \{j_3, \ldots, j_p\}$. In this case, it follows rom Lemma 5.14 and (5.27), (5.28) that $\mathrm{end}(\mathbf{p}_\beta)(j_2) = w_{k+1}(t_1) = n - k + t_1$ for some $1 \leq t_1 \leq j_2$, and hence that

$$\mathrm{end}(\mathbf{p}_\beta)(j_2) = n - k + t_1 > n - k + 1 = \mathrm{end}(\mathbf{p}_\beta)(j_1).$$

Since $j_2 < k_r \leq k+1$ and $k_r \notin \{j_3, \ldots, j_p\}$, we deduce by Lemma 5.14 and (5.27), (5.28) that $\mathrm{end}(\mathbf{p}_\beta)(k_r) = w_{k+1}(t_2) = n - k + t_2$ for some $j_2 < t_2 \leq k+1$. Hence, we obtain

$$\underbrace{\mathrm{end}(\mathbf{p}_\beta)(j_1)}_{=n-k+1} < \underbrace{\mathrm{end}(\mathbf{p}_\beta)(j_2)}_{=n-k+t_1} < \underbrace{\mathrm{end}(\mathbf{p}_\beta)(k_r)}_{=n-k+t_2},$$

which is a contradiction (see Lemma 2.2). Thus, we see that $j_1 + 1 \notin J$.

Suppose, for a contradiction, that $j_1 + 1 \notin J$ and $j_1 + 1 < k_r \leq k+1$; recall that $k_r \notin \{j_2, \ldots, j_p\}$. In this case, we deduce by (5.27) and (5.28) that

$$\mathrm{end}(\mathbf{p}_\beta)(j_1) < \mathrm{end}(\mathbf{p}_\beta)(j_1+1) < \mathrm{end}(\mathbf{p}_\beta)(k_r),$$

which is a contradiction (see Lemma 2.2). Thus, we see that $k_r = j_1 + 1$; since $\mathrm{end}(\mathbf{p}_\beta)(j_1) < \mathrm{end}(\mathbf{p}_\beta)(j_1+1)$, the edge $\mathrm{end}(\mathbf{p}_\beta) \xrightarrow{(j_1,j_1+1)} \bullet$ is a Bruhat edge. This proves the lemma. □

Let $\mathbf{D}_J^{\mathrm{A}_1}$ (resp., $\mathbf{D}_J^{\mathrm{A}_2}$) denote the subset of $\mathbf{D}_J^{\mathrm{A}}$ consisting of all those elements $\mathbf{p} \in \mathbf{D}_J^{\mathrm{A}}$ whose initial label of the γ-part $\mathbf{p}_\gamma$ is (resp., is not) (j_1, j_1+1); see Lemma 5.15. Similarly, let $\mathbf{D}_J^{\mathrm{B}_1}$ (resp., $\mathbf{D}_J^{\mathrm{B}_2}$) denote the subset of $\mathbf{D}_J^{\mathrm{B}}$ consisting of all those elements $\mathbf{p} \in \mathbf{D}_J^{\mathrm{B}}$ whose initial label of the γ-part $\mathbf{p}_\gamma$ is (resp., is not) $(1, 2)$.

We can easily show the following; the element $\mathbf{q}$ in this lemma is a unique element of $\mathbf{D}$ fixed by our 'sijection' (i.e., a bijection between signed sets) defined below.

Lemma 5.16. *We have*

$$\mathbf{q} := [(k-1,k), (k-2,k), \ldots, (1,k) \mid (k,k+1)] \in \mathbf{D}_{\{k\}}^{\mathrm{A}_1}; \tag{5.30}$$

note that $\{k\} \subset [k]$ is a unique subset $J \subset [k]$ such that $\mathbf{q} \in \mathbf{D}_J^{\mathrm{A}_1}$. Also, we have

$$\mathrm{end}(\mathbf{q}_\beta)\epsilon_J = \epsilon_{n+1-k}, \quad \ell(\mathbf{p}_\gamma) = 1, \quad \mathrm{end}(\mathbf{p})t_{\mathrm{wt}(\mathbf{p})} = w_k. \tag{5.31}$$

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Let us complete the proof of Proposition 2.4. For this purpose, we define a 'sijection' (i.e., a bijection between signed sets) $\Phi : \mathbf{D} \to \mathbf{D}$ as follows. We set $\Phi(\mathbf{q}) := \mathbf{q}$ (see Lemma 5.16). Let $\mathbf{p} \in \mathbf{D}_J^{\mathrm{A}_2}$; note that $j_1 = \min J \geq 2$ and $j_1 - 1 \notin J$. We set $\Phi(J) := (J \setminus \{j_1\}) \sqcup \{j_1 - 1\}$ and then define $\Phi(\mathbf{p}) \in \mathbf{D}_{\Phi(J)}^{\mathrm{A}_1} \sqcup \mathbf{D}_{\Phi(J)}^{\mathrm{B}_1}$ as follows. Recall that the β-part $\mathbf{p}_\beta$ of $\mathbf{p}$ is of the form (5.27), and that if the γ-part $\mathbf{p}_\gamma$ of $\mathbf{p}$ is nontrivial, then it starts at the edge whose label is (j_u, t) for some $2 \leq u \leq p$ and $j_u \leq t \leq n+1$, with $t \notin \{j_{u+1}, \dots, j_p\}$ (see Lemma 5.15). We define $\Phi(\mathbf{p})$ by

$$\Phi(\mathbf{p}) = [(j_1 - 2, j_1 - 1), \dots, (2, j_1 - 1), (1, j_1 - 1), \mathbf{s}' \mid (j_1 - 1, j_1), \mathbf{p}_\gamma], \tag{5.32}$$

where $\mathbf{s}'$ is obtained from $\mathbf{s}$ by replacing $(j_1 - 1, j_u)$ appearing in $\mathbf{p}_\beta$ with (j_1, j_u) for each $2 \leq u \leq p$; by [NS2, Lemma 2.3 (2)] and Lemma 5.14, we deduce that $\Phi(\mathbf{p}) \in \mathbf{D}_{\Phi(J)}^{\mathrm{A}_1} \sqcup \mathbf{D}_{\Phi(J)}^{\mathrm{B}_1}$ (note that $\Phi(\mathbf{p}) \in \mathbf{D}_{\Phi(J)}^{\mathrm{B}_1}$ if and only if $j_1 = 2$). Also, it follows that

$$\begin{aligned}
&|\Phi(J)| = |J|, \\
&|\Phi(J)|\epsilon_{n+1-k} - \mathrm{end}(\Phi(\mathbf{p})_\beta)\epsilon_{\Phi(J)} = |J|\epsilon_{n+1-k} - \mathrm{end}(\mathbf{p}_\beta)\epsilon_J, \\
&\ell(\Phi(\mathbf{p})_\gamma) = \ell(\mathbf{p}_\gamma) + 1, \quad \mathrm{end}(\Phi(\mathbf{p}))t_{\mathrm{wt}(\Phi(\mathbf{p}))} = \mathrm{end}(\mathbf{p})t_{\mathrm{wt}(\mathbf{p})}.
\end{aligned} \tag{5.33}$$

Let $\mathbf{p} \in \mathbf{D}_J^{\mathrm{A}_1} \sqcup \mathbf{D}_J^{\mathrm{B}_1}$, with $J \neq \{k\}$; recall that $\mathbf{D}_{\{k\}}^{\mathrm{A}_1} \sqcup \mathbf{D}_{\{k\}}^{\mathrm{B}_1} = \{\mathbf{q}\}$, and note that $j_1 + 1 \notin [k] \setminus J$. We set $\Phi(J) := (J \setminus \{j_1\}) \sqcup \{j_1 + 1\}$ and then define $\Phi(\mathbf{p}) \in \mathbf{D}_{\Phi(J)}^{\mathrm{A}_1}$ as follows. Recall that the β-part $\mathbf{p}_\beta$ of $\mathbf{p}$ is of the form (5.27) or (5.28), and that the γ-part $\mathbf{p}_\gamma$ of $\mathbf{p}$ starts at the edge whose label is $(j_1, j_1 + 1)$ (see Lemma 5.15). We define $\Phi(\mathbf{p})$ by

$$\Phi(\mathbf{p}) = [(j_1, j_1 + 1), (j_1 - 1, j_1 + 1), \dots, (2, j_1 + 1), (1, j_1 + 1), \mathbf{s}'' \mid \mathbf{p}_\gamma \setminus (j_1, j_1 + 1)], \tag{5.34}$$

where $\mathbf{s}''$ is obtained from $\mathbf{s}$ by replacing $(j_1 + 1, j_u)$ appearing in $\mathbf{p}_\beta$ with (j_1, j_u) for each $2 \leq u \leq p$, and $\mathbf{p}_\gamma \setminus (j_1, j_1 + 1)$ is the sequence obtained form $\mathbf{p}_\gamma$ by removing the initial label $(j_1, j_1 + 1)$; by [NS2, Lemma 2.3 (4)] and Lemma 5.14, we deduce that $\Phi(\mathbf{p}) \in \mathbf{D}_{\Phi(J)}^{\mathrm{A}_2}$. Also, it follows that

$$\begin{aligned}
&|\Phi(J)| = |J|, \\
&|\Phi(J)|\epsilon_{n+1-k} - \mathrm{end}(\Phi(\mathbf{p})_\beta)\epsilon_{\Phi(J)} = |J|\epsilon_{n+1-k} - \mathrm{end}(\mathbf{p}_\beta)\epsilon_J, \\
&\ell(\Phi(\mathbf{p})_\gamma) = \ell(\mathbf{p}_\gamma) - 1, \quad \mathrm{end}(\Phi(\mathbf{p}))t_{\mathrm{wt}(\Phi(\mathbf{p}))} = \mathrm{end}(\mathbf{p})t_{\mathrm{wt}(\mathbf{p})}.
\end{aligned} \tag{5.35}$$

If $\mathbf{p} \in \mathbf{D}_J^{\mathrm{B}_2}$, then $\mathbf{p} \in \mathbf{D}_{\Phi(J)}^{\mathrm{C}}$ with $\Phi(J) := J \setminus \{1\}$. We define $\Phi(\mathbf{p}) := \mathbf{p} \in \mathbf{D}_{\Phi(J)}^{\mathrm{C}}$ for $\mathbf{p} \in \mathbf{D}_J^{\mathrm{B}_2}$. It follows that

$$\begin{aligned}
&|\Phi(J)| = |J| - 1, \\
&|\Phi(J)|\epsilon_{n+1-k} - \mathrm{end}(\Phi(\mathbf{p})_\beta)\epsilon_{\Phi(J)} = |J|\epsilon_{n+1-k} - \mathrm{end}(\mathbf{p}_\beta)\epsilon_J, \\
&\ell(\Phi(\mathbf{p})_\gamma) = \ell(\mathbf{p}_\gamma), \qquad \mathrm{end}(\Phi(\mathbf{p}))t_{\mathrm{wt}(\Phi(\mathbf{p}))} = \mathrm{end}(\mathbf{p})t_{\mathrm{wt}(\mathbf{p})}.
\end{aligned} \tag{5.36}$$

Similarly, if $\mathbf{p} \in \mathbf{D}_J^{\mathrm{C}}$, then $\mathbf{p} \in \mathbf{D}_{\Phi(J)}^{\mathrm{B}_2}$ with $\Phi(J) := J \sqcup \{1\}$. We define $\Phi(\mathbf{p}) := \mathbf{p} \in \mathbf{D}_{\Phi(J)}^{\mathrm{B}_2}$ for $\mathbf{p} \in \mathbf{D}_J^{\mathrm{C}}$. It follows that

$$\begin{aligned}
&|\Phi(J)| = |J| + 1, \\
&|\Phi(J)|\epsilon_{n+1-k} - \mathrm{end}(\Phi(\mathbf{p})_\beta)\epsilon_{\Phi(J)} = |J|\epsilon_{n+1-k} - \mathrm{end}(\mathbf{p}_\beta)\epsilon_J, \\
&\ell(\Phi(\mathbf{p})_\gamma) = \ell(\mathbf{p}_\gamma), \qquad \mathrm{end}(\Phi(\mathbf{p}))t_{\mathrm{wt}(\Phi(\mathbf{p}))} = \mathrm{end}(\mathbf{p})t_{\mathrm{wt}(\mathbf{p})}.
\end{aligned} \tag{5.37}$$

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Thus, we have obtained a bijection $\Phi : \mathbf{D} \to \mathbf{D}$ such that (i) $\Phi(\mathbf{q}) = \mathbf{q}$, (ii) $\Phi(\mathbf{p}) \neq \mathbf{p}$, $\Phi(\Phi(\mathbf{p})) = \mathbf{p}$, and such that

$$\begin{aligned}
&\mathbf{e}^{|\Phi(J)|\epsilon_{n+1-k}-\mathrm{end}(\Phi(\mathbf{p})_\beta)\epsilon_{\Phi(J)}}(-1)^{|\Phi(J)|+\ell(\Phi(\mathbf{p})_\gamma)}[\mathcal{O}_{\mathbf{Q}_G(\mathrm{end}(\Phi(\mathbf{p}))t_{\mathrm{wt}(\Phi(\mathbf{p}))})}]\\
&= -\mathbf{e}^{|J|\epsilon_{n+1-k}-\mathrm{end}(\mathbf{p}_\beta)\epsilon_J}(-1)^{|J|+\ell(\mathbf{p}_\gamma)}[\mathcal{O}_{\mathbf{Q}_G(\mathrm{end}(\mathbf{p})t_{\mathrm{wt}(\mathbf{p})})}]
\end{aligned}$$

for all $\mathbf{p} \in \mathbf{D}_J \setminus \{\mathbf{q}\}$ with $J \subset [k]$. Therefore, the right-hand side of equation (5.19) becomes

$$\mathbf{e}^{|\{k\}|\epsilon_{n+1-k}-\mathrm{end}(\mathbf{q}_\beta)\epsilon_{\{k\}}}(-1)^{|\{k\}|+\ell(\mathbf{q}_\gamma)}[\mathcal{O}_{\mathbf{Q}_G(\mathrm{end}(\mathbf{q})t_{\mathrm{wt}(\mathbf{q})})}] = [\mathcal{O}_{\mathbf{Q}_G(w_k)}] \qquad \text{by Lemma 5.16,}$$

as desired. This completes the proof of Proposition 2.4.

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