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RESEARCH ARTICLE

Borel-type presentation of the torus-equivariant quantum K-ring of flag manifolds of type C

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Abstract
We give a presentation of the torus-equivariant (small) quantum K-ring of flag manifolds of type C as an explicit quotient of a Laurent polynomial ring; our presentation can be thought of as a quantization of the classical Borel presentation of the ordinary K-ring of flag manifolds. Also, we give an explicit Laurent polynomial representative for each special Schubert class in our Borel-type presentation of the quantum K-ring.

Contents

1. Introduction

Let G be a connected, simply connected, simple algebraic group over $\mathbb{C}$, B its Borel subgroup, $T \subset B$ a maximal torus, and G/B the associated flag manifold. The T-equivariant (small) quantum K-ring $QK_T(G/B)$ of the flag manifold G/B, defined by Givental [4] and Lee [15], is, as a module,

$$QK_T(G/B) := K_T(G/B) \otimes_{R(T)} R(T)[[Q_1, \dots, Q_n]],$$

where $K_T(G/B)$ is the (ordinary) T-equivariant K-ring of G/B, $R(T)$ is the representation ring of T, and Q_k, $k = 1, \dots, n$, with $n = \operatorname{rank} G$, are the (Novikov) variables. The quantum K-ring $QK_T(G/B)$ is also equipped with the quantum product $\star$, defined in terms of the two-point and three-point (genus 0, equivariant) K-theoretic Gromov-Witten invariants, and becomes an associative, commutative algebra (for details, see [4] and [15]). The quantum K-ring $QK_T(G/B)$ has a good basis as an $R(T)[[Q_1, \dots, Q_n]]$-module, called the *Schubert basis*. Let W be the Weyl group of G, and B^- the opposite Borel subgroup of G such that $B \cap B^- = T$. Then, for $w \in W$, the (opposite) *Schubert variety* X^w is defined as $X^w := \overline{B^- wB/B}$, where $\overline{\cdot}$ denotes the Zariski closure; we denote by $\mathcal{O}^w$, $w \in W$, the structure sheaf of X^w. Then the classes $[\mathcal{O}^w] \in K_T(G/B)$, $w \in W$, form an $R(T)$-basis of $K_T(G/B)$, and hence they form an $R(T)[[Q_1, \dots, Q_n]]$-basis of $QK_T(G/B)$, called the *Schubert basis*.

In the ordinary K-theory, we know the isomorphism:

$$K_T(G/B) \simeq R(T) \otimes_{R(T)^W} R(T) \tag{1.1}$$

of $R(T)$-algebras (see [24]); here, $R(T)^W$ denotes the subring of $R(T)$ consisting of all W-invariant elements. This isomorphism is the K-theory version of the *Borel presentation* of the cohomology ring $H(G/B; \mathbb{C})$ of G/B. The purpose of this paper is to extend the K-theoretic Borel presentation to the quantum case, that is, to give a presentation of the quantum K-ring $QK_T(G/B)$ analogous to the above one. In type A_n, that is, in the case that $G = SL_{n+1}(\mathbb{C})$, Maeno–Naito–Sagaki [21] (see also [1]) has given a presentation of the quantum K-ring $QK_T(G/B)$ as the quotient of a polynomial ring by an explicit ideal. More precisely, Maeno–Naito–Sagaki gave an explicit ideal $\mathcal{I}^Q$ of a polynomial ring $(R(T)[[Q_1, \dots, Q_n]])[x_1, \dots, x_n, x_{n+1}]$ such that

$$(R(T)[[Q_1, \dots, Q_n]])[x_1, \dots, x_n, x_{n+1}]/\mathcal{I}^Q \simeq QK_T(Fl_{n+1}) \tag{1.2}$$

as $R(T)[[Q_1, \dots, Q_n]]$-algebras; this isomorphism is called a *Borel-type presentation*. We can think of this Borel-type presentation as a quantization of the classical K-theoretic Borel presentation as follows. Let P be the weight lattice of G given as:

$$P = \left(\bigoplus_{1 \le i \le n+1} \mathbb{Z}\varepsilon_i \right) \Big/ \mathbb{Z}(\varepsilon_1 + \cdots + \varepsilon_n + \varepsilon_{n+1}).$$

Let $\mathcal{I}$ denote the ideal of $R(T)[x_1, \dots, x_{n+1}]$ generated by

$$\{e_l(x_1, \dots, x_{n+1}) - e_l(\mathbf{e}^{\varepsilon_1}, \dots, \mathbf{e}^{\varepsilon_{n+1}}) \mid 1 \le l \le n+1\},$$

where e_l is the l-th elementary symmetric polynomial for $1 \le l \le n+1$ and $\mathbf{e}^\lambda$, $\lambda \in P$, are the canonical $\mathbb{Z}$-basis of the group algebra $\mathbb{Z}[P]$ of P, identified with $R(T)$. Then, (1.1) implies the isomorphism:

$$R(T)[x_1, \dots, x_{n+1}]/\mathcal{I} \xrightarrow{\sim} K_T(G/B) \tag{1.3}$$

of $R(T)$-algebras. From the definition of the ideal $\mathcal{I}^Q$ in [21], we see that (1.3) is obtained from (1.2) by the specialization $Q_1 = \cdots = Q_n = 0$. Note that Maeno–Naito–Sagaki [22] also proved that *quantum double Grothendieck polynomials*, introduced by Lenart–Maeno [18], represent Schubert classes under the Borel-type presentation of $QK_T(G/B)$.

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At present, the Borel-type presentation of $QK_T(G/B)$ is known only for G/B of type A; even an explicit conjecture does not exist other than in type A. Hence, giving a Borel-type presentation beyond type A is an important problem to consider. In this paper, we deal with the flag manifold G/B of type C. Let $G = \mathrm{Sp}_{2n}(\mathbb{C})$ be the symplectic group of rank n, and $\{\varepsilon_1, \dots, \varepsilon_n\}$ the standard basis for the weight lattice P of G. For $\lambda \in P$, we denote by $\mathcal{O}_{G/B}(\lambda)$ the line bundle associated to λ. Now, we give an explicit ideal $\mathcal{I}^Q$ of $(R(T)[[Q_1, \dots, Q_n]])[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ such that the quotient $(R(T)[[Q_1, \dots, Q_n]])[z_1^{\pm 1}, \dots, z_n^{\pm 1}]/\mathcal{I}^Q$ is isomorphic to $QK_T(G/B)$ as $R(T)[[Q_1, \dots, Q_n]]$-algebras, and such that the residue class of the variable z_j corresponds to (a scalar multiple of) the class $[\mathcal{O}_{G/B}(-\varepsilon_j)]$ of the line bundle $\mathcal{O}_{G/B}(-\varepsilon_j)$ for $1 \le j \le n$. Let $[1, \overline{1}] := \{1 < 2 < \cdots < n < \overline{n} < \cdots < \overline{2} < \overline{1}\}$ be a totally ordered set. We define $F_l \in (R(T)[[Q_1, \dots, Q_n]])[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$, $1 \le l \le n$, by

$$F_l := \sum_{\substack{J \subset [1,\overline{1}] \\ |J| = l}} \left(\prod_{1 \le j \le \overline{1}} \zeta_J(j) \right) \left(\prod_{j \in J} z_j \right);$$

here, $\zeta_J(j)$ for $J \subset [1, \overline{1}]$ and $1 \le j \le \overline{1}$ are elements of $\mathbb{Z}[[Q_1, \dots, Q_n]]$ given in Definition 3.1. Then we define an ideal $\mathcal{I}^Q$ of $(R(T)[[Q_1, \dots, Q_n]])[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ to be the one generated by

$$F_l - \underbrace{e_l(\mathbf{e}^{\varepsilon_1}, \dots, \mathbf{e}^{\varepsilon_n}, \mathbf{e}^{-\varepsilon_n}, \dots, \mathbf{e}^{-\varepsilon_1})}_{=: E_l}, \quad 1 \le l \le n,$$

where $e_l(x_1, \dots, x_{2n})$ is the l-th elementary symmetric polynomial in the $2n$ variables $x_1, \dots, x_{2n}$ for $1 \le l \le n$, and $\mathbf{e}^\lambda$, $\lambda \in P$, are the canonical $\mathbb{Z}$-basis of the group algebra $\mathbb{Z}[P]$ of P, identified with $R(T)$. It follows from the definition that the elements $F_1, \dots, F_n$ satisfy $F_l|_{Q_1 = \cdots = Q_n = 0} = e_l(z_1, \dots, z_n, z_n^{-1}, \dots, z_1^{-1})$ for $1 \le l \le n$.

The following is the main result of this paper.

Theorem A (= Theorem 3.6). *There exists an $R(T)[[Q]]$-algebra isomorphism*

$$\begin{array}{rcl} \Psi^Q : (R(T)[[Q]])[z_1^{\pm 1}, \dots, z_n^{\pm 1}]/\mathcal{I}^Q & \xrightarrow{\sim} & QK_T(G/B) \\ z_j + \mathcal{I}^Q & \mapsto & \dfrac{1}{1 - Q_j}[\mathcal{O}_{G/B}(-\varepsilon_j)], \quad 1 \le j \le n, \end{array} \tag{1.4}$$

where $R(T)[[Q]] := R(T)[[Q_1 \dots, Q_n]]$.

If we set $Q_1 = \cdots = Q_n = 0$, then the isomorphism Ψ^Q above specializes to the $R(T)$-algebra isomorphism

$$R(T)[z_1^{\pm 1}, \dots, z_n^{\pm 1}]/\mathcal{I} \simeq K_T(G/B), \tag{1.5}$$

where $\mathcal{I}$ is the ideal of $R(T)[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ generated by

$$e_l(z_1, \dots, z_n, z_n^{-1}, \dots, z_1^{-1}) - E_l, \quad 1 \le l \le n;$$

this isomorphism is just the classical K-theoretic Borel presentation (1.1) of the flag manifold G/B of type C_n.

The strategy for the proof of Theorem A is basically the same as that of [21]. The explicit form of the elements $F_1, \dots, F_n$ comes from the *inverse Chevalley formula* (see Section 4.2), in the T-equivariant K-group $K_T(\mathbf{Q}_G)$ of the semi-infinite flag manifold $\mathbf{Q}_G$ associated to $G = \mathrm{Sp}_{2n}(\mathbb{C})$, which is a (reduced) scheme of infinite type (see Section 4), through the $R(T)$-module isomorphism $QK_T(G/B) \simeq K_T(\mathbf{Q}_G)$ established in [9]; for this isomorphism, see Theorem 4.1. In types A, D, E, the inverse Chevalley formula is obtained in the general case in [14] and [19], and in type C, it is obtained in some special cases in [13]. From the inverse Chevalley formula, we obtain key relations for the elements $\mathfrak{F}_l \in K_T(\mathbf{Q}_G)$, $0 \le l \le 2n$, defined in Definition 5.5 (3). More precisely, the inverse Chevalley formula in this situation, that is, equations (4.3) and (4.4), can be regarded as recurrence relations for $[\mathcal{O}_{\mathbf{Q}_G(s_1)}]$, $[\mathcal{O}_{\mathbf{Q}_G(s_1 s_2)}], \ldots$, $[\mathcal{O}_{\mathbf{Q}_G(s_1 s_2 \cdots s_n)}]$, $[\mathcal{O}_{\mathbf{Q}_G(s_1 s_2 \cdots s_n s_{n-1})}], \ldots$, $[\mathcal{O}_{\mathbf{Q}_G(s_1 s_2 \cdots s_n s_{n-1} \cdots s_1)}]$, along with 0. By solving these recurrence relations, we obtain a fundamental relation (5.6) for $\mathfrak{F}_l$, $0 \le l \le 2n$, with coefficients in $R(T)$. Then, by applying Demazure operators to the fundamental relation repeatedly, we obtain a system of linear equations (5.10) for $\mathfrak{F}_l$ with coefficients in $R(T)$, from which the $\mathfrak{F}_l \in K_T(\mathbf{Q}_G)$, $0 \le l \le n$, are uniquely determined. Hence, by solving these equations, we conclude that $\mathfrak{F}_l = E_l$ for $0 \le l \le n$ (Corollary 5.15). Although this argument is almost the same as that in [21], we need an additional symmetry $\mathfrak{F}_l = \mathfrak{F}_{2n-l}$ for $1 \le l \le n$ (Proposition 5.7) in type C in order to determine $\mathfrak{F}_l$, $0 \le l \le 2n$, uniquely.

Next, by sending the $\mathfrak{F}_l$, $0 \le l \le n$, to $QK_T(G/B)$ under the isomorphism $QK_T(G/B) \simeq K_T(\mathbf{Q}_G)$, we see that the element $\mathcal{F}_l$ given by Definition 6.2 (3) is identical to E_l for $0 \le l \le n$ (see Corollary 6.4). Thus, we can figure out the explicit form, given in Definition 3.3, of the generators of the ideal $\mathcal{I}^Q$.

Finally, to complete the proof of Theorem A (= Theorem 3.6), we use Nakayama-type arguments. Namely, we make use of a lemma (see Lemma 6.13), due to Gu–Mihalcea–Sharpe–Zou [6], in commutative ring theory. Based on this lemma, we can deduce (1.4) from (1.5).

In the course of the proof of Theorem A, we obtain the description of an explicit Laurent polynomial representative for each special Schubert class in the Borel-type presentation (1.4). In what follows, for $a, b \in [1, \overline{1}]$ with $a < b$, we set $[a, b] := \{j \in [1, \overline{1}] \mid a \le j \le b\}$. Now, for $1 \le k \le n$ and $0 \le l \le k$, we define $F_l^k \in (R(T)[[Q]])[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ as

$$F_l^k := \sum_{\substack{J \subset [1,k] \\ |J|=l}} \left(\prod_{1 \le j \le \overline{1}} \zeta_J(j) \right) \left(\prod_{j \in J} z_j \right).$$

Also, for $0 \le k \le n$ and $0 \le l \le 2n - k$, we define $F_l^{\overline{k}} \in (R(T)[[Q]])[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ as

$$F_l^{\overline{k}} := \sum_{\substack{J \subset [1,\overline{k+1}] \\ |J|=l}} \left(\prod_{1 \le j \le \overline{1}} \zeta_J(j) \right) \left(\prod_{j \in J} z_j \right),$$

where we understand that $\overline{n+1} := n$.

Theorem B (= Corollary 6.12). (1) *For* $1 \le k \le n$*, we have*

$$[\mathcal{O}^{s_1 \cdots s_k}] = \widehat{\Psi}^Q \left(\sum_{l=0}^{k} (-1)^l \mathbf{e}^{-l\varepsilon_1} F_l^k \right).$$

(2) *For* $1 \le k \le n$*, we have*

$$[\mathcal{O}^{s_1 \cdots s_n s_{n-1} \cdots s_k}] = \widehat{\Psi}^Q \left(\sum_{l=0}^{2n-k} (-1)^l \mathbf{e}^{-l\varepsilon_1} F_l^{\overline{k}} \right).$$

This paper is organized as follows. In Section 2, we fix our basic notation for root systems, in particular, for the root system of type C. Also, we briefly recall the definition of the T-equivariant quantum K-ring $QK_T(G/B)$ of the flag manifold G/B. In Section 3, we state a Borel-type presentation (Theorem A) of $QK_T(G/B)$ in type C, which is the main result of this paper. In Section 4, we recall the definition of the T-equivariant K-group $K_T(\mathbf{Q}_G)$ of the semi-infinite flag manifold $\mathbf{Q}_G$, and its relationship with $QK_T(G/B)$. In addition, we review the inverse Chevalley formula for $K_T(\mathbf{Q}_G)$, which is used in the proof of our Borel-type presentation. In Section 5, we deduce key relations in $K_T(\mathbf{Q}_G)$ by using the inverse Chevalley formula. In Section 6, we give a proof of our Borel-type presentation of $QK_T(G/B)$ in type C; we also give a proof of Theorem B.

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2. Basic setting

We fix our basic notation for root systems, in particular, for the root system of type C. Also, we briefly recall the definition of the torus-equivariant (small) quantum K-ring of flag manifolds.

2.1. Notation for root systems

Let G be a connected, simply connected, simple algebraic group over $\mathbb{C}$, B its Borel subgroup, $T \subset B$ a maximal torus, and I the index set for the simple roots of G. Let $\mathfrak{h}$ be the Lie algebra of T; we denote by $\langle \cdot, \cdot \rangle$ the canonical pairing of $\mathfrak{h}^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ and $\mathfrak{h}$. Let $P = \sum_{i \in I} \mathbb{Z}\varpi_i$ be the weight lattice of G, with ϖ_i the i-th fundamental weight for $i \in I$, and $Q^\vee = \sum_{i \in I} \mathbb{Z}\alpha_i^\vee$ the coroot lattice of G, with $\alpha_i^\vee$ the coroot such that $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{i,j}$ for $i, j \in I$; we set $Q^{\vee,+} := \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i^\vee$. In addition, let W be the Weyl group of G, which is generated by the simple reflections s_i, $i \in I$. For $\lambda \in P$, we denote by $\mathbf{e}^\lambda$ the character of the one-dimensional representation of T whose weight is λ. Then, the representation ring $R(T)$ of T is written as $R(T) = \bigoplus_{\lambda \in P} \mathbb{Z}\mathbf{e}^\lambda$, with product given by $\mathbf{e}^\lambda \cdot \mathbf{e}^\mu := \mathbf{e}^{\lambda+\mu}$ for $\lambda, \mu \in P$; $R(T)$ is canonically identified with the group algebra $\mathbb{Z}[P]$ of P, on which W acts by $w \cdot \mathbf{e}^\lambda := \mathbf{e}^{w\lambda}$ for $w \in W$ and $\lambda \in P$.

2.2. The quantum K-ring of flag manifolds

Let $K_T(G/B)$ denote the (ordinary) T-equivariant K-ring of the flag manifold G/B. The *T-equivariant (small) quantum K-ring* $QK_T(G/B)$ of the flag manifold G/B, defined by Givental [4] and Lee [15], is, as a module,

$$QK_T(G/B) := K_T(G/B) \otimes_{R(T)} R(T)[[Q_1, \dots, Q_n]],$$

where n is the rank of G; here, $R(T)[[Q_1, \dots, Q_n]]$ denotes the formal power series ring in the (Novikov) variables $Q_1, \dots, Q_n$ with coefficients in $R(T)$. Under the quantum product $\star$, defined in terms of the two-point and three-point (genus 0, equivariant) K-theoretic Gromov-Witten invariants, $QK_T(G/B)$ becomes an associative and commutative algebra (see [4] and [15] for details). For $\xi = \sum_{i \in I} c_i\alpha_i^\vee \in Q^{\vee,+}$, we set $Q^\xi := \prod_{i \in I} Q_i^{c_i}$.

For $w \in W$, we define the (opposite) *Schubert variety* $X^w \subset G/B$ by $X^w := \overline{B^- wB/B}$, where B^- is the Borel subgroup (opposite to B) of G such that $B \cap B^- = T$, and $\overline{\cdot}$ denotes the Zariski closure; let $\mathcal{O}^w$ denote the structure sheaf of X^w. Then it is well known that the classes $[\mathcal{O}^w]$, $w \in W$, of $\mathcal{O}^w$ in $K_T(G/B)$ form a basis of $K_T(G/B)$ as an $R(T)$-module, which are called the (opposite) *Schubert classes*. It follows that the (opposite) Schubert classes $[\mathcal{O}^w]$, $w \in W$, form a basis of $QK_T(G/B)$ as an $R(T)[[Q_1, \dots, Q_n]]$-module.

To each $\lambda \in P$, we can associate the line bundle $G \times_B \mathbb{C}_{-\lambda} \twoheadrightarrow G/B$ over G/B, denoted by $\mathcal{O}_{G/B}(\lambda)$, where $\mathbb{C}_{-\lambda}$ is the one-dimensional representation of B whose weight is $-\lambda$; let $[\mathcal{O}_{G/B}(\lambda)] \in K_T(G/B) \subset QK_T(G/B)$ denote the class of the line bundle $\mathcal{O}_{G/B}(\lambda)$.

2.3. The root system of type C_n

We fix the notation for the root system of type C_n. Let J_n be the $n \times n$ matrix given by

$$J_n := \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

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and set

$$J_{2n}^{-} := \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix};$$

the corresponding symplectic form $(\cdot,\cdot)$ on $\mathbb{C}^{2n}$ is given by: $(u,v) = {}^{t}uJ_{2n}^{-}v$ for $u,v \in \mathbb{C}^{2n}$. Then, the *symplectic group* $\mathrm{Sp}_{2n}(\mathbb{C})$ of rank n over $\mathbb{C}$ is defined as:

$$\mathrm{Sp}_{2n}(\mathbb{C}) = \{A \in \mathrm{GL}_{2n}(\mathbb{C}) \mid {}^{t}AJ_{2n}^{-}A = J_{2n}^{-}\}.$$

Let T be a maximal torus of $\mathrm{Sp}_{2n}(\mathbb{C})$ given by:

$$T = \{\mathrm{diag}(x_1,\dots,x_n,x_n^{-1},\dots,x_1^{-1}) \mid x_1,\dots,x_n \in \mathbb{C}^*\},$$

where $\mathrm{diag}(a_1,\dots,a_{2n})$ denotes the $2n \times 2n$ diagonal matrix with entries $a_1,\dots,a_{2n}$; its Lie algebra $\mathfrak{h} = \mathrm{Lie}(T)$ can be written as:

$$\mathfrak{h} = \{\mathrm{diag}(a_1,\dots,a_n,-a_n,\dots,-a_1) \mid a_1,\dots,a_n \in \mathbb{C}\}.$$

We define $\varepsilon_k \in \mathfrak{h}^*$, $1 \le k \le n$, to be

$$\varepsilon_k : \mathfrak{h} \to \mathbb{C},\ \mathrm{diag}(a_1,\dots,a_n,-a_n,\dots,-a_1) \mapsto a_k.$$

Then, the set Δ^+ of positive roots of G can be written as:

$$\Delta^+ = \{\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le n\} \cup \{2\varepsilon_j \mid 1 \le j \le n\};$$

the simple roots $\alpha_1,\dots,\alpha_n$ are $\alpha_j = \varepsilon_j - \varepsilon_{j+1}$, $1 \le j \le n-1$, and $\alpha_n = 2\varepsilon_n$. We set

$$\begin{aligned} (i,j) &:= \varepsilon_i - \varepsilon_j \quad \text{for } 1 \le i < j \le n, \\ (i,\overline{j}) &:= \varepsilon_i + \varepsilon_j \quad \text{for } 1 \le i < j \le n, \text{ and} \\ (i,\overline{i}) &:= 2\varepsilon_i \quad \text{for } 1 \le i \le n. \end{aligned}$$

The fundamental weight ϖ_j for $1 \le j \le n$ is of the form $\varpi_j = \varepsilon_1 + \cdots + \varepsilon_j$. It follows that $R(T) = \mathbb{Z}[P] = \mathbb{Z}[\mathbf{e}^{\pm\varepsilon_1},\dots,\mathbf{e}^{\pm\varepsilon_n}]$.

The Weyl group $W \simeq S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ of $\mathrm{Sp}_{2n}(\mathbb{C})$ can be regarded as the group of signed permutations on $[1,\overline{1}] := \{1 < 2 < \cdots < n < \overline{n} < \cdots < \overline{2} < \overline{1}\}$. For $1 \le i < j \le n$, the reflection $s_{(i,j)}$ (resp., $s_{(i,\overline{j})}$) corresponding to (i,j) (resp., $(i,\overline{j})$) is viewed as the transposition of i and j (resp., i and $\overline{j}$). In addition, for $1 \le i \le n$, the reflection $s_{(i,\overline{i})}$ corresponding to $(i,\overline{i})$ can be regarded as the transposition of i and $\overline{i}$. The action of W on P is a natural extension of that on $[1,\overline{1}]$, given by: $w \cdot \varepsilon_k = \varepsilon_{w(k)}$ for $w \in W$ and $k = 1,\dots,n$, where we set $\varepsilon_{\overline{j}} := -\varepsilon_j$ for $1 \le j \le n$. We sometimes write $w \in W$ as $w = [w(1), w(2),\dots,w(n)]$ by regarding it as a signed permutation as above; this notation is called the *window notation*.

3. Borel-type presentation of $QK_T(G/B)$

Let $G = \mathrm{Sp}_{2n}(\mathbb{C})$ be the symplectic group of rank n. We give a presentation of $QK_T(G/B)$ as an explicit quotient of a certain Laurent polynomial ring.

For a commutative ring A, we denote by $A[[Q]] := A[[Q_1,\dots,Q_n]]$ the formal power series ring in the (Novikov) variables $Q_1,\dots,Q_n$ with coefficients in A. For $1 \le k \le 2n$, let $e_k(x_1,\dots,x_{2n})$ be the

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Table 1. *The list of $\zeta_J(j)$ for $1 \le j \le \overline{1}$.*

j	1	2	3	4	$\overline{4}$	$\overline{3}$	$\overline{2}$	$\overline{1}$
$\zeta_J(j)$	1	1	$1-Q_3$	1	$1+\frac{Q_3Q_4}{1-Q_3}$	$1-Q_2$	1	1

k-th elementary symmetric polynomial in the $2n$ variables $x_1, \ldots, x_{2n}$, that is,

$$e_k(x_1, \ldots, x_{2n}) := \sum_{1 \le i_1 < \cdots < i_k \le 2n} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

For $0 \le l \le 2n$, we define an element $E_l \in R(T)$ by

$$E_l := e_l(\mathbf{e}^{\varepsilon_1}, \ldots, \mathbf{e}^{\varepsilon_n}, \mathbf{e}^{-\varepsilon_n}, \ldots, \mathbf{e}^{-\varepsilon_1}). \tag{3.1}$$

Observe that $E_{n+l} = E_{n-l}$ for $1 \le l \le n$ since

$$e_{n+l}(x_1, \ldots, x_n, x_n^{-1}, \ldots, x_1^{-1}) = e_{n-l}(x_1, \ldots, x_n, x_n^{-1}, \ldots, x_1^{-1}) \tag{3.2}$$

for $1 \le l \le n$.

Also, we consider the Laurent polynomial ring $(R(T)[[Q]])[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ with coefficients in the formal power series ring $R(T)[[Q]] = R(T)[[Q_1, \ldots, Q_n]]$.

Definition 3.1. Let $J \subset [1, \overline{1}]$.

(1) For $1 \le j \le n$, we define $\zeta_J(j) \in \mathbb{Z}[Q] \subset \mathbb{Z}[[Q]]$ by

$$\zeta_J(j) := \begin{cases} 1 - Q_j & \text{if } j \in J \text{ and } j+1 \notin J, \\ 1 & \text{otherwise,} \end{cases}$$

where $n+1$ is understood to be $\overline{n}$.

(2) For $2 \le j \le n$, we define $\zeta_J(\overline{j}) \in \mathbb{Z}[[Q]]$ by

$$\zeta_J(\overline{j}) := \begin{cases} 1 + \dfrac{Q_{j-1}Q_j \cdots Q_n}{1 - Q_{j-1}} & \text{if } J \text{ is of the form } \{\cdots < j-1 < \overline{j-1} < \cdots\}, \\ 1 - Q_{j-1} & \text{if } \overline{j} \in J \text{ and } \overline{j-1} \notin J, \\ 1 & \text{otherwise.} \end{cases}$$

(3) We set $\zeta_J(\overline{1}) := 1 \in \mathbb{Z}[[Q]]$.

Example 3.2. Let $n = 4$ and $J = \{2, 3, \overline{3}, \overline{1}\}$. Then, the elements $\zeta_J(j)$ for $1 \le j \le \overline{1}$ are as in Table 1.

We set $z_{\overline{j}} := z_j^{-1}$ for $1 \le j \le n$.

Definition 3.3. For $0 \le l \le 2n$, we define $F_l \in (R(T)[[Q]])[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ by

$$F_l := \sum_{\substack{J \subset [1, \overline{1}] \\ |J| = l}} \left(\prod_{1 \le j \le \overline{1}} \zeta_J(j) \right) \left(\prod_{j \in J} z_j \right).$$

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Remark 3.4. Under the specialization $Q_1 = \cdots = Q_n = 0$, the F_l becomes:

$$F_l|_{Q_1=\cdots=Q_n=0} = \sum_{\substack{J\subset[1,\overline{1}] \\ |J|=l}} \left(\prod_{j\in J} z_j\right) = e_l(z_1, \dots, z_n, z_n^{-1}, \dots, z_1^{-1})$$

for $0 \le l \le 2n$.

Definition 3.5. We define an ideal $\mathcal{I}^Q$ of $(R(T)[[Q]])[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ to be the one generated by the elements $F_l - E_l$, $1 \le l \le n$.

The main result of this paper is the following.

Theorem 3.6. *There exists an $R(T)[[Q]]$-algebra isomorphism*

$$\begin{aligned} \Psi^Q : (R(T)[[Q]])[z_1^{\pm 1}, \dots, z_n^{\pm 1}]/\mathcal{I}^Q &\xrightarrow{\sim} \quad QK_T(G/B) \\ z_j + \mathcal{I}^Q &\mapsto \frac{1}{1-Q_j}[\mathcal{O}_{G/B}(-\varepsilon_j)], \quad 1 \le j \le n. \end{aligned} \tag{3.3}$$

Remark 3.7. The flag manifold G/B of type C_n is the manifold parametrizing all flags

$$0 \subset V_1 \subset \cdots \subset V_n \subset \mathbb{C}^{2n}$$

of isotropic subspaces with respect to the symplectic form $(\cdot,\cdot)$ on $\mathbb{C}^{2n}$ such that $\dim_{\mathbb{C}} V_j = j$ for $1 \le j \le n$. Let $0 = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \mathcal{S}_2 \subset \cdots \subset \mathcal{S}_n \subset \mathbb{C}^{2n}$ be the tautological filtration by subbundles of the trivial bundle over G/B, with $\mathrm{rk}(\mathcal{S}_j) = j$ for $1 \le j \le n$; we have $\mathcal{S}_j/\mathcal{S}_{j-1} \simeq \mathcal{O}_{G/B}(-\varepsilon_j)$ for $1 \le j \le n$.

4. Equivariant K-groups of semi-infinite flag manifolds

In this section, we recall the definition of the torus-equivariant K-group of semi-infinite flag manifolds, and its relationship with the torus-equivariant quantum K-ring. Also, we briefly review the inverse Chevalley formula for the torus-equivariant K-group of semi-infinite flag manifolds.

4.1. Semi-infinite flag manifolds and the quantum K-ring

Following [21, §3.1] (which is based on [9, §1.4 and §1.5]) and [23, §2.1 and §2.3], we recall semi-infinite flag manifolds and their equivariant K-groups (see also [10, §4.2]).

Let G be of an arbitrary type, and N the unipotent radical of B. The *semi-infinite flag manifold* $\mathbf{Q}_G^{\mathrm{rat}}$ is a (reduced) ind-scheme of ind-infinite type whose set of $\mathbb{C}$-valued points is $G(\mathbb{C}((z)))/(T(\mathbb{C})\cdot N(\mathbb{C}((z))))$. The semi-infinite flag manifold $\mathbf{Q}_G^{\mathrm{rat}}$ can be thought of as an inductive limit of copies of the scheme $\mathbf{Q}_G$ of infinite type, introduced in [3, §4.1]. The scheme $\mathbf{Q}_G$ can be described as follows. In what follows, for a $\mathbb{C}$-vector space V, let $\mathbb{P}(V)$ denote the projective space of lines in V. For $\lambda \in P^+$, let $V(\lambda)$ be the irreducible highest weight G-module of highest weight λ. The scheme $\mathbf{Q}_G$ parametrizes tuples $(\ell_\lambda)_{\lambda\in P^+}$ of $\mathbb{C}$-lines in $\prod_{\lambda\in P^+} \mathbb{P}(V(\lambda) \otimes_{\mathbb{C}} \mathbb{C}[[z]])$ such that for $\lambda, \mu \in P^+$, the line $\ell_{\lambda+\mu}$ is mapped to $\ell_\lambda \otimes \ell_\mu$ under the embedding

$$V(\lambda+\mu) \otimes_{\mathbb{C}} \mathbb{C}[[z]] \hookrightarrow (V(\lambda) \otimes_{\mathbb{C}} \mathbb{C}[[z]]) \otimes_{\mathbb{C}[[z]]} (V(\mu) \otimes_{\mathbb{C}} \mathbb{C}[[z]])$$

of $\mathbb{C}$-vector spaces induced by the embedding

$$V(\lambda+\mu) \hookrightarrow V(\lambda) \otimes_{\mathbb{C}} V(\mu)$$

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of G-modules (unique up to scalars). Such a tuple $(\ell_\lambda)_{\lambda \in P^+}$ is uniquely determined by ℓ_{ϖ_i} for $i \in I$, and so we have the closed embedding

$$\mathbf{Q}_G \hookrightarrow \mathbb{P} := \prod_{i \in I} \mathbb{P}(L(\varpi_i) \otimes_{\mathbb{C}} \mathbb{C}[[z]]) \tag{4.1}$$

(see [10, Lemma 4.4]). Also, we define the $(G(\mathbb{C}[[z]]) \rtimes \mathbb{C}^*)$-equivariant line bundle $\mathcal{O}_{\mathbf{Q}_G}(\lambda)$ over $\mathbf{Q}_G$ for each $\lambda = \sum_{i \in I} m_i \varpi_i \in P$ as the pullback of the line bundle $\boxtimes_{i \in I} \mathcal{O}(m_i)$ over $\mathbb{P}$ under the embedding (4.1), where the i-th $\mathcal{O}(m_i)$ is over $\mathbb{P}(L(\varpi_i) \otimes_{\mathbb{C}} \mathbb{C}[[z]])$. Note that for $\lambda, \mu \in P$, we have $\mathcal{O}_{\mathbf{Q}_G}(\lambda) \otimes \mathcal{O}_{\mathbf{Q}_G}(\mu) = \mathcal{O}_{\mathbf{Q}_G}(\lambda + \mu)$.

Let $\mathbf{I}$ be the Iwahori subgroup of $G(\mathbb{C}[[z]])$, which is the preimage of B under the evaluation map $G(\mathbb{C}[[z]]) \to G$, $z \mapsto 0$. Also, let $W_{\mathrm{af}} = \{w t_\xi \mid w \in W,\ \xi \in Q^\vee\} \simeq W \ltimes Q^\vee$ be the affine Weyl group of G, where t_ξ for $\xi \in Q^\vee$ denotes the translation element corresponding to ξ. Then, the $(T \times \mathbb{C}^*)$-fixed points of $\mathbf{Q}_G$ are labeled by the subset $W_{\mathrm{af}}^{\geq 0} := \{w t_\xi \mid w \in W,\ \xi \in Q^{\vee,+}\}$ of W_{af} ([10, §4.2], [23, §2.3]); more precisely, the $(T \times \mathbb{C}^*)$-fixed point of $\mathbf{Q}_G$ labeled by $x = w t_\xi \in W_{\mathrm{af}}^{\geq 0}$, with $w \in W$ and $\xi \in Q^{\vee,+}$, is the tuple of $\mathbb{C}$-lines $p_x := (z^{-\langle \lambda, w_\circ \xi \rangle} V(\lambda)_{w w_\circ \lambda})_{\lambda \in P^+}$, where $V(\lambda)_\mu$ is the μ-weight space of $V(\lambda)$ for $\lambda \in P^+$, $\mu \in P$. We denote by $\mathbf{Q}_G(x)$ the closure of the $\mathbf{I}$-orbit of p_x, called the *semi-infinite Schubert variety* associated to $x \in W_{\mathrm{af}}^{\geq 0}$; it is a (reduced) closed subscheme of $\mathbf{Q}_G(e) = \mathbf{Q}_G$, where $e \in W_{\mathrm{af}}$ is the identity element. For $x \in W_{\mathrm{af}}^{\geq 0}$, we denote by $\mathcal{O}_{\mathbf{Q}_G(x)}$ the structure sheaf of $\mathbf{Q}_G(x)$.

For $a = \sum_{\lambda \in P} \sum_{k \in \mathbb{Z}} c_{\lambda,k} q^k \mathbf{e}^\lambda \in \mathbb{Z}[q, q^{-1}][P]$ with $c_{\lambda,k} \in \mathbb{Z}$, we set $|a| := \sum_{\lambda \in P} \sum_{k \in \mathbb{Z}} |c_{\lambda,k}| q^k \mathbf{e}^\lambda$. Let $\widetilde{K}'(\mathbf{Q}_G)$ denote the $\mathbb{Z}[q, q^{-1}][P]$-module consisting of all formal sums $\sum_{x \in W_{\mathrm{af}}^{\geq 0}} a_x [\mathcal{O}_{\mathbf{Q}_G(x)}]$ with coefficients $a_x \in \mathbb{Z}[q, q^{-1}][P]$ such that

$$\sum_{x \in W_{\mathrm{af}}^{\geq 0}} |a_x| \operatorname{gch} H^0(\mathbf{Q}_G, \mathcal{O}_{\mathbf{Q}_G(x)}(\lambda)) \in \mathbb{Z}[P]((q^{-1}))$$

for $\lambda \in P^{++} := \sum_{i \in I} \mathbb{Z}_{>0} \varpi_i$, where gch denotes the character of a weight module over $T \times \mathbb{C}^*$ and q denotes the character of the loop rotation action $\mathbb{C}^* \curvearrowright \mathbb{C}[[z]]$; the classes $[\mathcal{O}_{\mathbf{Q}_G(x)}]$, $x \in W_{\mathrm{af}}^{\geq 0}$, are called *semi-infinite Schubert classes*. By [9, Theorem 1.25], we see that the sheaf $\mathcal{O}_{\mathbf{Q}_G(x)}(\lambda) := \mathcal{O}_{\mathbf{Q}_G(x)} \otimes \mathcal{O}_{\mathbf{Q}_G}(\lambda)$ for $x \in W_{\mathrm{af}}^{\geq 0}$ and $\lambda \in P$ defines the class $[\mathcal{O}_{\mathbf{Q}_G(x)}(\lambda)] \in \widetilde{K}'(\mathbf{Q}_G)$, called the *twisted semi-infinite Schubert class*; in particular, for $\lambda \in P$, $\widetilde{K}'(\mathbf{Q}_G)$ contains the class $[\mathcal{O}_{\mathbf{Q}_G}(\lambda)] = [\mathcal{O}_{\mathbf{Q}_G(e)}(\lambda)]$ of the line bundle $\mathcal{O}_{\mathbf{Q}_G}(\lambda)$. Also, $\widetilde{K}'(\mathbf{Q}_G)$ is stable under the tensor product $\bullet \mapsto \bullet \otimes [\mathcal{O}_{\mathbf{Q}_G}(\lambda)]$ for $\lambda \in P$.

Let $K_{T \times \mathbb{C}^*}(\mathbf{Q}_G)$ denote the $\mathbb{Z}[q, q^{-1}][P]$-submodule of $\widetilde{K}'(\mathbf{Q}_G)$ consisting of all "convergent" (in the sense described in [10, Proposition 5.11]) sums; that is, $K_{T \times \mathbb{C}^*}(\mathbf{Q}_G)$ consists of all formal infinite linear combinations $\sum_{x \in W_{\mathrm{af}}^{\geq 0}} a_x [\mathcal{O}_{\mathbf{Q}_G(x)}]$ of the semi-infinite Schubert classes $[\mathcal{O}_{\mathbf{Q}_G(x)}]$, $x \in W_{\mathrm{af}}^{\geq 0}$, with coefficients $a_x \in \mathbb{Z}[q, q^{-1}][P]$ such that

$$\sum_{x \in W_{\mathrm{af}}^{\geq 0}} |a_x| \in \mathbb{Z}[P]((q^{-1}));$$

we see from [10, Corollary 4.31] that $K_{T \times \mathbb{C}^*}(\mathbf{Q}_G)$ is indeed a $\mathbb{Z}[q, q^{-1}][P]$-submodule of $\widetilde{K}'(\mathbf{Q}_G)$. Also, it follows from [12, Theorem 5.16] and (the proof of) [10, Corollary 5.12] that $[\mathcal{O}_{\mathbf{Q}_G(x)}(\lambda)] \in K_{T \times \mathbb{C}^*}(\mathbf{Q}_G)$ for $x \in W_{\mathrm{af}}^{\geq 0}$ and $\lambda \in P$; in particular, we have $[\mathcal{O}_{\mathbf{Q}_G}(\lambda)] = [\mathcal{O}_{\mathbf{Q}_G(e)}(\lambda)] \in K_{T \times \mathbb{C}^*}(\mathbf{Q}_G)$.

By taking the specialization of $K_{T \times \mathbb{C}^*}(\mathbf{Q}_G)$ at $q = 1$, we obtain the *T-equivariant K-group* $K_T(\mathbf{Q}_G)$ of $\mathbf{Q}_G$, which turns out to be the $R(T)(= \mathbb{Z}[P])$-module consisting of all infinite linear combinations of the semi-infinite Schubert classes $[\mathcal{O}_{\mathbf{Q}_G(x)}]$, $x \in W_{\mathrm{af}}^{\geq 0}$, with coefficients in $R(T)$; namely, we can write each element of $K_T(\mathbf{Q}_G)$ uniquely as an infinite linear combination of the semi-infinite Schubert classes $[\mathcal{O}_{\mathbf{Q}_G(x)}]$, $x \in W_{\mathrm{af}}^{\geq 0}$, with coefficients in $R(T)$ ([9, Lemma 1.22]). It follows that $[\mathcal{O}_{\mathbf{Q}_G(x)}(\lambda)] \in K_T(\mathbf{Q}_G)$ for $\lambda \in P$ and $x \in W_{\mathrm{af}}^{\geq 0}$ (see also [9, Theorem 1.26]); in particular, for $\lambda \in P$, we have the class $[\mathcal{O}_{\mathbf{Q}_G}(\lambda)] = [\mathcal{O}_{\mathbf{Q}_G(e)}(\lambda)] \in K_T(\mathbf{Q}_G)$.

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Theorem 4.1 [9, Corollary 3.13 and Theorem 4.17]. *There exists an isomorphism*

$$\Phi : QK_T(G/B) \xrightarrow{\sim} K_T(\mathbf{Q}_G)$$

of $R(T)$-modules such that

(i) $\Phi(\mathbf{e}^{\mu} Q^{\xi}[\mathcal{O}^w]) = \mathbf{e}^{-\mu}[\mathcal{O}_{\mathbf{Q}_G(wt_\xi)}]$ *for $\mu \in P$, $\xi \in Q^{\vee,+}$, and $w \in W$;*
(ii) *for $\mathcal{Z} \in QK_T(G/B)$, we have*

$$\Phi(\mathcal{Z} \star [\mathcal{O}_{G/B}(-\varpi_i)]) = \Phi(\mathcal{Z}) \otimes [\mathcal{O}_{\mathbf{Q}_G}(w_\circ \varpi_i)], \quad i \in I, \tag{4.2}$$

where $w_\circ$ denotes the longest element of W.

We warn the reader that the convention for line bundles in this paper is different from that in [9] by the twist coming from the involution $-w_\circ$, and so we need $w_\circ$ on the right-hand side of (4.2). If G is of type C, then $w_\circ\lambda = -\lambda$ for all $\lambda \in P$, and hence the right-hand side of (4.2) is $\Phi(\mathcal{Z}) \otimes [\mathcal{O}_{\mathbf{Q}_G}(-\varpi_i)]$.

Note that for a general $\lambda \in P$, the $R(T)$-module isomorphism Φ is not necessarily compatible with the quantum product $\star$ with $\mathcal{O}_{G/B}(\lambda)$ in $QK_T(G/B)$ and the tensor product $\otimes$ with $\mathcal{O}_{\mathbf{Q}_G}(-w_\circ\lambda)$ in $K_T(\mathbf{Q}_G)$. However, if G is of type C, that is, if $G = \mathrm{Sp}_{2n}(\mathbb{C})$, then we have the following, which plays an important role in this paper.

Proposition 4.2 [21, Proposition 5.3]. *Let $G = \mathrm{Sp}_{2n}(\mathbb{C})$ be the symplectic group of rank n. For $\mathcal{Z} \in QK_T(G/B)$, we have*

$$\Phi\left(\mathcal{Z} \star \left(\frac{1}{1-Q_j}[\mathcal{O}_{G/B}(\varepsilon_j)]\right)\right) = \Phi(\mathcal{Z}) \otimes [\mathcal{O}_{\mathbf{Q}_G}(\varepsilon_j)],$$
$$\Phi\left(\mathcal{Z} \star \left(\frac{1}{1-Q_{j-1}}[\mathcal{O}_{G/B}(-\varepsilon_j)]\right)\right) = \Phi(\mathcal{Z}) \otimes [\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_j)]$$

for $1 \le j \le n$; here we set $Q_0 := 0$.

For each $i \in I$, we define an $R(T)$-linear endomorphism t_i of $K_T(\mathbf{Q}_G)$ by

$$\mathsf{t}_i([\mathcal{O}_{\mathbf{Q}_G(x)}]) := \left[\mathcal{O}_{\mathbf{Q}_G\left(xt_{\alpha_i^\vee}\right)}\right]$$

for $x \in W_{\mathrm{af}}^{\ge 0}$. Then, for $i \in I$, $\mathfrak{Z} \in K_T(\mathbf{Q}_G)$, and $\lambda \in P$, we have

$$\mathsf{t}_i(\mathfrak{Z}) \otimes [\mathcal{O}_{\mathbf{Q}_G}(\lambda)] = \mathsf{t}_i(\mathfrak{Z} \otimes [\mathcal{O}_{\mathbf{Q}_G}(\lambda)])$$

(see [21, (3.2)]).

4.2. Inverse Chevalley formula in type C

The *inverse Chevalley formula* is a formula expressing the expansion of the product $\mathbf{e}^{\mu}[\mathcal{O}_{\mathbf{Q}_G(x)}]$ for $\mu \in P$ and $x \in W_{\mathrm{af}}^{\ge 0}$ as an explicit linear combination of the elements $[\mathcal{O}_{\mathbf{Q}_G(y)}] \otimes [\mathcal{O}_{\mathbf{Q}_G}(\lambda)]$, $y \in W_{\mathrm{af}}^{\ge 0}$, $\lambda \in P$, with coefficients in $\mathbb{Z}[q, q^{-1}]$, which is of the following form:

$$\mathbf{e}^{\mu}[\mathcal{O}_{\mathbf{Q}_G(x)}] = \sum_{y \in W_{\mathrm{af}}^{\ge 0},\, \lambda \in P} d_{x,\mu}^{y,\mu}[\mathcal{O}_{\mathbf{Q}_G(y)}] \otimes [\mathcal{O}_{\mathbf{Q}_G}(\lambda)],$$

with $d_{x,\mu}^{y,\mu} \in \mathbb{Z}[q, q^{-1}]$. In types A, D, E, the inverse Chevalley formula in the general case is given in [14] and [19], and in type C, it is given in [13] in some special cases. In this paper, we use the inverse Chevalley formula in type C.

Assume that G is of type C_n, that is, $G = \mathrm{Sp}_{2n}(\mathbb{C})$. From [13, Theorem 4.5], we obtain the following.

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Proposition 4.3. *Let* $1 \le k \le n-1$. *The following equality holds in* $K_{T\times\mathbb{C}^*}(\mathbf{Q}_G)$*:*

$$\begin{aligned}\mathbf{e}^{\varepsilon_1}[\mathcal{O}_{\mathbf{Q}_G(s_1s_2\cdots s_k)}] &= [\mathcal{O}_{\mathbf{Q}_G(s_1s_2\cdots s_k)}(\varepsilon_{k+1})] - [\mathcal{O}_{\mathbf{Q}_G(s_1s_2\cdots s_{k+1})}(\varepsilon_{k+1})] \\ &\quad + q\sum_{j=1}^{k}\left[\mathcal{O}_{\mathbf{Q}_G\left(s_1s_2\cdots s_{j-1}t_{\alpha_j^\vee+\alpha_{j+1}^\vee+\cdots+\alpha_k^\vee}\right)}(\varepsilon_j)\right] \\ &\quad - q\sum_{j=1}^{k}\left[\mathcal{O}_{\mathbf{Q}_G\left(s_1s_2\cdots s_j t_{\alpha_j^\vee+\alpha_{j+1}^\vee+\cdots+\alpha_k^\vee}\right)}(\varepsilon_j)\right].\end{aligned} \tag{4.3}$$

Also, from [13, Theorem 4.3], we have the following.

Proposition 4.4. *Let* $1 \le k \le n$. *The following equality holds in* $K_{T\times\mathbb{C}^*}(\mathbf{Q}_G)$*:*

$$\begin{aligned}&\mathbf{e}^{\varepsilon_1}[\mathcal{O}_{\mathbf{Q}_G(s_1\cdots s_{n-1}s_ns_{n-1}\cdots s_k)}] \\ &= [\mathcal{O}_{\mathbf{Q}_G(s_1\cdots s_{n-1}s_ns_{n-1}\cdots s_k)}(-\varepsilon_k)] - [\mathcal{O}_{\mathbf{Q}_G(s_1\cdots s_{n-1}s_ns_{n-1}\cdots s_{k-1})}(-\varepsilon_k)] \\ &\quad + q\sum_{j=k+1}^{n}\left[\mathcal{O}_{\mathbf{Q}_G\left(s_1\cdots s_{n-1}s_ns_{n-1}\cdots s_j t_{\alpha_k^\vee+\alpha_{k+1}^\vee+\cdots+\alpha_{j-1}^\vee}\right)}(-\varepsilon_j)\right] \\ &\quad - q\sum_{j=k+1}^{n}\left[\mathcal{O}_{\mathbf{Q}_G\left(s_1\cdots s_{n-1}s_ns_{n-1}\cdots s_{j-1} t_{\alpha_k^\vee+\alpha_{k+1}^\vee+\cdots+\alpha_{j-1}^\vee}\right)}(-\varepsilon_j)\right] \\ &\quad + q\sum_{j=1}^{k}\left[\mathcal{O}_{\mathbf{Q}_G\left(s_1s_2\cdots s_{j-1}t_{\alpha_j^\vee+\alpha_{j+1}^\vee+\cdots+\alpha_n^\vee}\right)}(\varepsilon_j)\right] \\ &\quad - q\sum_{j=1}^{k}\left[\mathcal{O}_{\mathbf{Q}_G\left(s_1s_2\cdots s_j t_{\alpha_j^\vee+\alpha_{j+1}^\vee+\cdots+\alpha_n^\vee}\right)}(\varepsilon_j)\right];\end{aligned} \tag{4.4}$$

when $k=1$, *the second term* $[\mathcal{O}_{\mathbf{Q}_G(s_1\cdots s_{n-1}s_ns_{n-1}\cdots s_{k-1})}(-\varepsilon_k)]$ *on the right-hand side of the formula above is understood to be* 0.

We defer the proofs of (4.3) and (4.4) to Appendix A.

4.3. Demazure operators

Following [21, §A], we review Demazure operators acting on $K_{T\times\mathbb{C}^*}(\mathbf{Q}_G)$ and $K_T(\mathbf{Q}_G)$. Let G be of an arbitrary type. The *nil-DAHA* $\mathbb{H}_0$ is defined to be the $\mathbb{Z}[\mathsf{q}^{\pm 1}]$-algebra generated by T_i, $i \in I \sqcup \{0\}$, and X^ν, $\nu \in P$, subject to certain defining relations (including the braid relations); see, for example, [21, (A.2)–(A.6)]). For $i \in I \sqcup \{0\}$, we set $\mathsf{D}_i := 1 + \mathsf{T}_i$ for $i \in I \sqcup \{0\}$.

By [10, Theorem 6.5] (see also [23, §2.6] and [14, §3.1.2]), we have a left $\mathbb{H}_0$-action on $K_{T\times\mathbb{C}^*}(\mathbf{Q}_G^{\mathrm{rat}})$. In particular, for $i \in I$, we have an action of D_i on $K_{T\times\mathbb{C}^*}(\mathbf{Q}_G)$, which is called a Demazure operator; note that these Demazure operators satisfy the braid relations. Let D_i (not to be confused with D_i) for $i \in I$ be an operator on $\mathbb{Z}[P]$ given by

$$D_i(\mathbf{e}^\nu) := \frac{\mathbf{e}^\nu - \mathbf{e}^{\alpha_i}\mathbf{e}^{s_i\nu}}{1-\mathbf{e}^{\alpha_i}}$$

for $\nu \in P$; in other words, we have

$$D_i(\mathbf{e}^\nu) = \begin{cases} \mathbf{e}^\nu(1 + \mathbf{e}^{\alpha_i} + \mathbf{e}^{2\alpha_i} + \cdots + \mathbf{e}^{-\langle \nu, \alpha_i^\vee \rangle \alpha_i}) & \text{if } \langle \nu, \alpha_i^\vee \rangle \leq 0, \\ 0 & \text{if } \langle \nu, \alpha_i^\vee \rangle = 1, \\ -\mathbf{e}^\nu(\mathbf{e}^{-\alpha_i} + \mathbf{e}^{-2\alpha_i^\vee} + \cdots + \mathbf{e}^{-(\langle \nu, \alpha_i^\vee \rangle - 1)\alpha_i}) & \text{if } \langle \nu, \alpha_i^\vee \rangle \geq 2. \end{cases}$$

Lemma 4.5 [21, Lemma A.2]; [23, §2.6] and [14, §3.1.2]. *Let $i \in I$, and let $\xi \in Q^{\vee,+}$, $\nu, \lambda \in P$. In $K_{T \times \mathbb{C}^*}(\mathbf{Q}_G)$, we have*

$$\mathsf{D}_i(\mathbf{e}^\nu[\mathcal{O}_{\mathbf{Q}_G(t_\xi)}(\lambda)]) = D_i(\mathbf{e}^\nu)[\mathcal{O}_{\mathbf{Q}_G(t_\xi)}(\lambda)].$$

5. Key relations in $K_T(\mathbf{Q}_G)$

In order to prove Theorem 3.6, we need to obtain sufficiently many relations in $K_T(\mathbf{Q}_G)$. First, by making use of the inverse Chevalley formula (see (4.3) and (4.4)), we obtain a "fundamental" relation. Then, by applying Demazure operators to the "fundamental" relation, we obtain the other relations. These relations can be thought of as recurrence relations for the elements $\mathfrak{F}_l$, $0 \leq l \leq n$, of $K_T(\mathbf{Q}_G)$ (defined in Definition 5.5) with coefficients in $R(T)$, which uniquely determine the elements $\mathfrak{F}_l$, $0 \leq l \leq n$. Hence, by solving these recurrence relations, we obtain a description (see Corollary 5.15) of the elements $\mathfrak{F}_l$, $0 \leq l \leq n$, as explicit elements of $R(T)$. Although this strategy for obtaining the desired description of $\mathfrak{F}_l$, $0 \leq l \leq n$, is almost the same as that in [21] in type A, we need to overcome some technical difficulties peculiar to the case of type C; in particular, we need Proposition 5.7.

In this section, we assume that $G = \mathrm{Sp}_{2n}(\mathbb{C})$, the symplectic group of rank n.

5.1. "Fundamental" relation

From equations (4.3) and (4.4), which are special cases of the inverse Chevalley formula, we will obtain a "fundamental" relation in $K_T(\mathbf{Q}_G)$. First, by multiplying both sides of (4.3) by $[\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_{k+1})]$ and specializing q to 1, we obtain the following.

Lemma 5.1. *Let $1 \leq k \leq n-1$. The following equality holds in $K_T(\mathbf{Q}_G)$:*

$$\begin{aligned} [\mathcal{O}_{\mathbf{Q}_G(s_1 s_2 \cdots s_{k+1})}] &= -\mathbf{e}^{\varepsilon_1}[\mathcal{O}_{\mathbf{Q}_G(s_1 s_2 \cdots s_k)}(-\varepsilon_{k+1})] + [\mathcal{O}_{\mathbf{Q}_G(s_1 s_2 \cdots s_k)}] \\ &\quad + \sum_{j=1}^{k} \left[\mathcal{O}_{\mathbf{Q}_G\left(s_1 s_2 \cdots s_{j-1} t_{\alpha_j^\vee + \alpha_{j+1}^\vee + \cdots + \alpha_k^\vee}\right)}(\varepsilon_j - \varepsilon_{k+1}) \right] \\ &\quad - \sum_{j=1}^{k} \left[\mathcal{O}_{\mathbf{Q}_G\left(s_1 s_2 \cdots s_j t_{\alpha_j^\vee + \alpha_{j+1}^\vee + \cdots + \alpha_k^\vee}\right)}(\varepsilon_j - \varepsilon_{k+1}) \right]. \end{aligned} \tag{5.1}$$

Also, by multiplying both sides of (4.4) by $[\mathcal{O}_{\mathbf{Q}_G}(\varepsilon_k)]$ and specializing q to 1, we obtain the following.

Lemma 5.2. *Let $1 \leq k \leq n$. The following equality holds in $K_T(\mathbf{Q}_G)$:*

$$\begin{aligned} &[\mathcal{O}_{\mathbf{Q}_G(s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_{k-1})}] \\ &= -\mathbf{e}^{\varepsilon_1}[\mathcal{O}_{\mathbf{Q}_G(s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_k)}(\varepsilon_k)] + [\mathcal{O}_{\mathbf{Q}_G(s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_k)}] \\ &\quad + \sum_{j=k+1}^{n} \left[\mathcal{O}_{\mathbf{Q}_G\left(s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_j t_{\alpha_k^\vee + \alpha_{k+1}^\vee + \cdots + \alpha_{j-1}^\vee}\right)}(\varepsilon_k - \varepsilon_j) \right] \\ &\quad - \sum_{j=k+1}^{n} \left[\mathcal{O}_{\mathbf{Q}_G\left(s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_{j-1} t_{\alpha_k^\vee + \alpha_{k+1}^\vee + \cdots + \alpha_{j-1}^\vee}\right)}(\varepsilon_k - \varepsilon_j) \right] \end{aligned}$$

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$$
\begin{aligned}
&+\sum_{j=1}^{k}\left[\mathcal{O}_{\mathbf{Q}_G\left(s_1s_2\cdots s_{j-1}t_{\alpha_j^\vee+\alpha_{j+1}^\vee+\cdots+\alpha_n^\vee}\right)}(\varepsilon_j+\varepsilon_k)\right]\\
&-\sum_{j=1}^{k}\left[\mathcal{O}_{\mathbf{Q}_G\left(s_1s_2\cdots s_{j}t_{\alpha_j^\vee+\alpha_{j+1}^\vee+\cdots+\alpha_n^\vee}\right)}(\varepsilon_j+\varepsilon_k)\right];
\end{aligned}
\tag{5.2}
$$

when $k=1$, the left-hand side of the formula above is understood to be 0*.*

For $1 \le k \le n$, we set

$$
\begin{aligned}
\mathfrak{P}_k &:= [\mathcal{O}_{\mathbf{Q}_G(s_1s_2\cdots s_k)}],\\
\mathfrak{Q}_k &:= [\mathcal{O}_{\mathbf{Q}_G(s_1\cdots s_{n-1}s_ns_{n-1}\cdots s_k)}].
\end{aligned}
$$

Also, we set $\mathfrak{P}_0 := 1$ and $\mathfrak{Q}_0 := 0$. Then, (5.1) and (5.2) can be rewritten as:

$$
\begin{aligned}
\mathfrak{P}_{k+1} = -\mathbf{e}^{\varepsilon_1}\mathfrak{P}_k \otimes [\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_{k+1})] + \mathfrak{P}_k\\
+\sum_{j=1}^{k}\mathsf{t}_j\mathsf{t}_{j+1}\cdots\mathsf{t}_k\mathfrak{P}_{j-1}\otimes[\mathcal{O}_{\mathbf{Q}_G}(\varepsilon_j-\varepsilon_{k+1})]\\
-\sum_{j=1}^{k}\mathsf{t}_j\mathsf{t}_{j+1}\cdots\mathsf{t}_k\mathfrak{P}_{j}\otimes[\mathcal{O}_{\mathbf{Q}_G}(\varepsilon_j-\varepsilon_{k+1})],
\end{aligned}
\tag{5.3}
$$

$$
\begin{aligned}
\mathfrak{Q}_{k-1} = -\mathbf{e}^{\varepsilon_1}\mathfrak{Q}_k \otimes [\mathcal{O}_{\mathbf{Q}_G}(\varepsilon_{k})] + \mathfrak{Q}_k\\
+\sum_{j=k+1}^{n}\mathsf{t}_k\mathsf{t}_{k+1}\cdots\mathsf{t}_{j-1}\mathfrak{Q}_{j}\otimes[\mathcal{O}_{\mathbf{Q}_G}(\varepsilon_k-\varepsilon_{j})]\\
-\sum_{j=k+1}^{n}\mathsf{t}_k\mathsf{t}_{k+1}\cdots\mathsf{t}_{j-1}\mathfrak{Q}_{j-1}\otimes[\mathcal{O}_{\mathbf{Q}_G}(\varepsilon_k-\varepsilon_{j})]\\
+\sum_{j=1}^{k}\mathsf{t}_j\mathsf{t}_{j+1}\cdots\mathsf{t}_n\mathfrak{P}_{j-1}\otimes[\mathcal{O}_{\mathbf{Q}_G}(\varepsilon_j+\varepsilon_{k})]\\
-\sum_{j=1}^{k}\mathsf{t}_j\mathsf{t}_{j+1}\cdots\mathsf{t}_n\mathfrak{P}_{j}\otimes[\mathcal{O}_{\mathbf{Q}_G}(\varepsilon_j+\varepsilon_{k})].
\end{aligned}
\tag{5.4}
$$

Equations (5.3) and (5.4) can be thought of as recurrence relations for $\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n$ and $\mathfrak{Q}_n, \mathfrak{Q}_{n-1}, \ldots, \mathfrak{Q}_1, \mathfrak{Q}_0$. By solving these recurrence relations, we can write these elements as explicit linear combinations of line bundle classes with coefficients in $\mathrm{End}_{\mathbb{C}}(K_T(\mathbf{Q}_G))$.

Definition 5.3. Let $J \subset [1, \overline{1}]$.

(1) For $1 \le j \le n$, we define $\psi_J(j) \in \mathrm{End}_{\mathbb{C}}(K_T(\mathbf{Q}_G))$ by

$$
\psi_J(j) := \begin{cases} 1-\mathsf{t}_j & \text{if } j \notin J \text{ and } j+1 \in J,\\ 1 & \text{otherwise,}\end{cases}
$$

where we understand that $n+1 := \overline{n}$.

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Table 2. *The list of* $\psi_J(j)$ *for* $1 \le j \le \overline{1}$.

j	1	2	3	4	$\overline{4}$	$\overline{3}$	$\overline{2}$	$\overline{1}$
$\psi_J(j)$	$1-\mathrm{t}_1$	1	1	1	$1-\mathrm{t}_3+\mathrm{t}_3\mathrm{t}_4$	1	$1-\mathrm{t}_1$	1

(2) For $2 \le j \le n$, we define $\psi_J(\overline{j}) \in \mathrm{End}_{\mathbb{C}}(K_T(\mathbf{Q}_G))$ by

$$\psi_J(\overline{j}) := \begin{cases} 1-\mathrm{t}_{j-1}+\mathrm{t}_{j-1}\mathrm{t}_j\cdots\mathrm{t}_n & \text{if } J \text{ is of the form } \{\cdots < j-1 < \overline{j-1} < \cdots\}, \\ 1-\mathrm{t}_{j-1} & \text{if } J \text{ is not of the above form} \\ & \text{ and } \overline{j} \notin J, \overline{j-1} \in J, \\ 1 & \text{otherwise.} \end{cases}$$

(3) We set $\psi_J(\overline{1}) := 1 \in \mathrm{End}_{\mathbb{C}}(K_T(\mathbf{Q}_G))$.

Example 5.4. Let $n = 4$ and $J = \{2, 3, \overline{3}, \overline{1}\}$. Then, the elements $\psi_J(j)$ for $1 \le j \le \overline{1}$ are as in Table 2.

For $J \subset [1, \overline{1}]$, we set

$$\varepsilon_J := \sum_{j \in J} \varepsilon_j.$$

Definition 5.5.

(1) For $1 \le k \le n$ and $0 \le l \le k$, we define $\mathfrak{F}_l^k \in K_T(\mathbf{Q}_G)$ as

$$\mathfrak{F}_l^k := \sum_{\substack{J \subset [1,k] \\ |J|=l}} \left(\prod_{1 \le j \le \overline{1}} \psi_J(j) \right) [\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_J)].$$

(2) For $1 \le k \le n$ and $0 \le l \le 2n-k$, we define $\mathfrak{F}_l^{\overline{k}} \in K_T(\mathbf{Q}_G)$ as

$$\mathfrak{F}_l^{\overline{k}} := \sum_{\substack{J \subset [1,\overline{k+1}] \\ |J|=l}} \left(\prod_{1 \le j \le \overline{1}} \psi_J(j) \right) [\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_J)],$$

where we understand that $\overline{n+1} := n$.

(3) For $0 \le l \le 2n$, we define $\mathfrak{F}_l \in K_T(\mathbf{Q}_G)$ as

$$\mathfrak{F}_l := \sum_{\substack{J \subset [1,\overline{1}] \\ |J|=l}} \left(\prod_{1 \le j \le \overline{1}} \psi_J(j) \right) [\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_J)].$$

Proposition 5.6. (1) *For* $1 \le k \le n$, *we have*

$$\mathfrak{P}_k = \sum_{l=0}^{k} (-1)^l \mathbf{e}^{l\varepsilon_1} \mathfrak{F}_l^k.$$

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(2) *For* $1 \le k \le n$, *we have*

$$\mathfrak{Q}_k = \sum_{l=0}^{2n-k} (-1)^l \mathbf{e}^{l\varepsilon_1} \mathfrak{F}_l^{\overline{k}}. \tag{5.5}$$

(3) *We have*

$$\sum_{l=0}^{2n} (-1)^l \mathbf{e}^{l\varepsilon_1} \mathfrak{F}_l = 0. \tag{5.6}$$

Proof. Part (1) can be proved by the same argument as for [21, Proposition 4.7]. If we set $\mathfrak{F}_l^{\overline{0}} := \mathfrak{F}_l$ for $0 \le l \le 2n$, then part (3) can be regarded as the special case $k = 0$ of (5.5). Hence, to prove parts (2) and (3), it suffices to prove (5.5) for $0 \le k \le n$. We will prove (5.5) by downward induction on $k = n, n-1, \ldots, 1, 0$. Since $\mathfrak{P}_n = \mathfrak{Q}_n$, the case $k = n$ is already proved by part (1). Now assume that (5.5) holds for all k, with $0 < k \le n$. By (5.4), we see that

$$\begin{aligned}
\mathfrak{Q}_{k-1} = &-\mathbf{e}^{\varepsilon_1} \sum_{l=0}^{2n-k} (-1)^l \mathbf{e}^{l\varepsilon_1} \mathfrak{F}_l^{\overline{k}} \otimes [\mathcal{O}_{\mathbf{Q}_G}(\varepsilon_k)] + \sum_{l=0}^{2n-k} (-1)^l \mathbf{e}^{l\varepsilon_1} \mathfrak{F}_l^{\overline{k}} \\
&+ \sum_{i=k+1}^{n} \mathsf{t}_k \mathsf{t}_{k+1} \cdots \mathsf{t}_{i-1} \sum_{l=0}^{2n-i} (-1)^l \mathbf{e}^{l\varepsilon_1} \mathfrak{F}_l^{\overline{i}} \otimes [\mathcal{O}_{\mathbf{Q}_G}(\varepsilon_k - \varepsilon_i)] \\
&- \sum_{i=k+1}^{n} \mathsf{t}_k \mathsf{t}_{k+1} \cdots \mathsf{t}_{i-1} \sum_{l=0}^{2n-i+1} (-1)^l \mathbf{e}^{l\varepsilon_1} \mathfrak{F}_l^{\overline{i-1}} \otimes [\mathcal{O}_{\mathbf{Q}_G}(\varepsilon_k - \varepsilon_i)] \\
&+ \sum_{i=1}^{k} \mathsf{t}_i \mathsf{t}_{i+1} \cdots \mathsf{t}_n \sum_{l=0}^{i-1} (-1)^l \mathbf{e}^{l\varepsilon_1} \mathfrak{F}_l^{i-1} \otimes [\mathcal{O}_{\mathbf{Q}_G}(\varepsilon_i + \varepsilon_k)] \\
&- \sum_{i=1}^{k} \mathsf{t}_i \mathsf{t}_{i+1} \cdots \mathsf{t}_n \sum_{l=0}^{i} (-1)^l \mathbf{e}^{l\varepsilon_1} \mathfrak{F}_l^{i} \otimes [\mathcal{O}_{\mathbf{Q}_G}(\varepsilon_i + \varepsilon_k)] \\
= &\sum_{l=0}^{2n-k} (-1)^{l+1} \mathbf{e}^{(l+1)\varepsilon_1} \sum_{\substack{J \subset [1,\overline{k+1}] \\ |J| = l}} \left(\prod_{1 \le j \le \overline{1}} \psi_J(j) \right) [\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_J + \varepsilon_k)] \\
&+ \sum_{l=0}^{2n-k} (-1)^l \mathbf{e}^{l\varepsilon_1} \sum_{\substack{J \subset [1,\overline{k+1}] \\ |J| = l}} \left(\prod_{1 \le j \le \overline{1}} \psi_J(j) \right) [\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_J)] \\
&+ \sum_{i=k+1}^{n} \mathsf{t}_k \mathsf{t}_{k+1} \cdots \mathsf{t}_{i-1} \sum_{l=0}^{2n-i} (-1)^l \mathbf{e}^{l\varepsilon_1} \sum_{\substack{J \subset [1,\overline{i+1}] \\ |J| = l}} \left(\prod_{1 \le j \le \overline{1}} \psi_J(j) \right) [\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_J + \varepsilon_k - \varepsilon_i)] \\
&- \sum_{i=k+1}^{n} \mathsf{t}_k \mathsf{t}_{k+1} \cdots \mathsf{t}_{i-1} \sum_{l=0}^{2n-i+1} (-1)^l \mathbf{e}^{l\varepsilon_1} \sum_{\substack{J \subset [1,\overline{i}] \\ |J| = l}} \left(\prod_{1 \le j \le \overline{1}} \psi_J(j) \right) [\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_J + \varepsilon_k - \varepsilon_i)] \\
&+ \sum_{i=1}^{k} \mathsf{t}_i \mathsf{t}_{i+1} \cdots \mathsf{t}_n \sum_{l=0}^{i-1} (-1)^l \mathbf{e}^{l\varepsilon_1} \sum_{\substack{J \subset [1,i-1] \\ |J| = l}} \left(\prod_{1 \le j \le \overline{1}} \psi_J(j) \right) [\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_J + \varepsilon_i + \varepsilon_k)]
\end{aligned}$$

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$$
\begin{aligned}
&-\sum_{i=1}^{k} \mathsf{t}_i\mathsf{t}_{i+1}\cdots\mathsf{t}_n \sum_{l=0}^{i}(-1)^l \mathbf{e}^{l\varepsilon_1} \sum_{\substack{J\subset[1,i]\\|J|=l}} \left(\prod_{1\le j\le \overline{1}} \psi_J(j)\right)[\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_J+\varepsilon_i+\varepsilon_k)] \\
=&\sum_{J\subset[1,\overline{k+1}]} (-1)^{|J|+1}\mathbf{e}^{(|J|+1)\varepsilon_1}\left(\prod_{1\le j\le \overline{1}} \psi_J(j)\right)[\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_J+\varepsilon_k)] \\
&+\sum_{J\subset[1,\overline{k+1}]} (-1)^{|J|}\mathbf{e}^{|J|\varepsilon_1}\left(\prod_{1\le j\le \overline{1}} \psi_J(j)\right)[\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_J)] \\
&-\sum_{i=k+1}^{n} \mathsf{t}_k\mathsf{t}_{k+1}\cdots\mathsf{t}_{i-1} \sum_{\substack{J\subset[1,\overline{i}]\\ \overline{i}\in J}} (-1)^{|J|}\mathbf{e}^{|J|\varepsilon_1}\left(\prod_{1\le j\le \overline{1}} \psi_J(j)\right)[\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_J+\varepsilon_k-\varepsilon_i)] \\
&-\sum_{i=1}^{k} \mathsf{t}_i\mathsf{t}_{i+1}\cdots\mathsf{t}_n \sum_{\substack{J\subset[1,i]\\ i\in J}} (-1)^{|J|}\mathbf{e}^{|J|\varepsilon_1}\left(\prod_{1\le j\le \overline{1}} \psi_J(j)\right)[\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_J+\varepsilon_i+\varepsilon_k)]. \qquad (5.7)
\end{aligned}
$$

Let us compute the term corresponding to each $J \subset [1,\overline{k}]$ on the RHS of (5.7).

Let $J \subset [1,\overline{k}]$ be such that $\overline{k} \notin J$; we set $l := |J|$. Then we find that the second sum on the RHS of (5.7) contains the term

$$
\begin{aligned}
&(-1)^l \mathbf{e}^{l\varepsilon_1}\left(\prod_{1\le j\le \overline{1}} \psi_J(j)\right)[\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_J)] \\
&= (-1)^l \mathbf{e}^{l\varepsilon_1}\left(\prod_{1\le j\le \overline{1}} \psi_J(j)\right)[\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_J)],
\end{aligned}
$$

which is just the term in $\mathfrak{Q}_{k-1}$ corresponding to J.

Next, let $J \subset [1,\overline{k}]$ be such that $\overline{k} \in J$; we set $M := \max(J \setminus \{\overline{k}\}) \in [1,\overline{1}]$. Assume that $M < n$. By summing up

- the term in the first sum on the RHS of (5.7) corresponding to $J \setminus \{\overline{k}\}$,
- the terms in the third sum on the RHS of (5.7) corresponding to $(J \setminus \{\overline{k}\}) \sqcup \{\overline{i}\}$ for $k+1 \le i \le n$ and $j \in J$,
- the terms in the fourth sum on the RHS of (5.7) corresponding to $(J \setminus \{\overline{k}\}) \sqcup \{i\}$ for $M+1 \le i \le k$ in the case $M < k$,

we deduce that

$$
\begin{aligned}
&(-1)^{(|J|-1)+1}\mathbf{e}^{((|J|-1)+1)\varepsilon_1}\left(\prod_{1\le j\le \overline{1}} \psi_{J\setminus\{\overline{k}\}}(j)\right)[\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_{J\setminus\{\overline{k}\}}+\varepsilon_k)] \\
&-\sum_{i=k+1}^{n} \mathsf{t}_k\cdots\mathsf{t}_{i-1}(-1)^{|J|}\mathbf{e}^{|J|\varepsilon_1}\left(\prod_{1\le j\le \overline{1}} \psi_{(J\setminus\{\overline{k}\})\sqcup\{\overline{i}\}}(j)\right)[\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_{(J\setminus\{\overline{k}\})\sqcup\{\overline{i}\}}+\varepsilon_k-\varepsilon_i)] \\
&-\sum_{i=M+1}^{k} \mathsf{t}_i\cdots\mathsf{t}_n(-1)^{|J|}\mathbf{e}^{|J|\varepsilon_1}\left(\prod_{1\le j\le \overline{1}} \psi_{(J\setminus\{\overline{k}\})\sqcup\{i\}}(j)\right)[\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_{(J\setminus\{\overline{k}\})\sqcup\{i\}}+\varepsilon_i+\varepsilon_k)]
\end{aligned}
$$

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$$= (-1)^{|J|}\mathbf{e}^{|J|\varepsilon_1}\Bigg(\Bigg(\prod_{1\le j\le \overline{1}}\psi_{J\setminus\{\overline{k}\}}(j)\Bigg) - \sum_{i=k+1}^{n}\mathrm{t}_k\cdots\mathrm{t}_{i-1}\Bigg(\prod_{1\le j\le\overline{1}}\psi_{(J\setminus\{\overline{k}\})\sqcup\{\overline{i}\}}(j)\Bigg)$$
$$- \sum_{i=M+1}^{k}\mathrm{t}_i\cdots\mathrm{t}_n\Bigg(\prod_{1\le j\le\overline{1}}\psi_{(J\setminus\{\overline{k}\})\sqcup\{i\}}(j)\Bigg)\Bigg)[\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_J)].$$
$$= (-1)^{|J|}\mathbf{e}^{|J|\varepsilon_1}\Bigg(\prod_{1\le j\le M}\psi_{J\setminus\{\overline{k}\}}(j)\Bigg)$$
$$\times\Bigg(1 - \sum_{i=k+1}^{n}\mathrm{t}_k\cdots\mathrm{t}_{i-1}\psi_{(J\setminus\{\overline{k}\})\sqcup\{\overline{i}\}}(\overline{i+1})$$
$$- \mathrm{t}_{M+1}\cdots\mathrm{t}_n - \sum_{i=M+2}^{k}\mathrm{t}_i\cdots\mathrm{t}_n\psi_{(J\setminus\{\overline{k}\})\sqcup\{i\}}(i-1)\Bigg)[\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_J)]. \tag{5.8}$$

Here, we set

$$\varphi := 1 - \sum_{i=k+1}^{n}\mathrm{t}_k\cdots\mathrm{t}_{i-1}\psi_{(J\setminus\{\overline{k}\})\sqcup\{\overline{i}\}}(\overline{i+1}) - \mathrm{t}_{M+1}\cdots\mathrm{t}_n - \sum_{i=M+2}^{k}\mathrm{t}_i\cdots\mathrm{t}_n\psi_{(J\setminus\{\overline{k}\})\sqcup\{i\}}(i-1).$$

If $M < k$, then

$$\begin{aligned}\varphi &= 1 - \sum_{i=k+1}^{n}\mathrm{t}_k\cdots\mathrm{t}_{i-1}\psi_{(J\setminus\{\overline{k}\})\sqcup\{\overline{i}\}}(\overline{i+1})\\ &\qquad - \mathrm{t}_{M+1}\cdots\mathrm{t}_n - \sum_{i=M+2}^{k}\mathrm{t}_i\cdots\mathrm{t}_n\psi_{(J\setminus\{\overline{k}\})\sqcup\{i\}}(i-1)\\ &= 1 - \sum_{i=k+1}^{n}\mathrm{t}_k\cdots\mathrm{t}_{i-1}(1-\mathrm{t}_i) - \mathrm{t}_{M+1}\cdots\mathrm{t}_n - \sum_{i=M+2}^{k}\mathrm{t}_i\cdots\mathrm{t}_n(1-\mathrm{t}_{i-1})\\ &= 1-\mathrm{t}_k\\ &= \psi_J(\overline{k+1}).\end{aligned}$$

If $M = k$, then

$$\begin{aligned}\varphi &= 1 - \sum_{i=k+1}^{n}\mathrm{t}_k\cdots\mathrm{t}_{i-1}\underbrace{\psi_{(J\setminus\{\overline{k}\})\sqcup\{\overline{i}\}}(\overline{i+1})}_{=1-\mathrm{t}_j}\\ &= 1-\mathrm{t}_k+\mathrm{t}_k\cdots\mathrm{t}_n\\ &= \psi_J(\overline{k+1}).\end{aligned}$$

If $M > k$, then

$$\begin{aligned}\varphi &= 1 - \sum_{i=k+1}^{n}\mathrm{t}_k\cdots\mathrm{t}_{i-1}\underbrace{\psi_{(J\setminus\{\overline{k}\})\sqcup\{\overline{i}\}}(\overline{i+1})}_{=1-\mathrm{t}_j+\mathrm{t}_j\cdots\mathrm{t}_n\ (\text{if } j=M),\ 1-\mathrm{t}_j\ (\text{if } j\neq M)}\\ &= 1 - \sum_{i=k+1}^{M-1}(1-\mathrm{t}_i) - \mathrm{t}_k\cdots\mathrm{t}_{M-1}(1-\mathrm{t}_M+\mathrm{t}_M\cdots\mathrm{t}_n) - \sum_{i=M+1}^{n}\mathrm{t}_k\cdots\mathrm{t}_{j-1}(1-\mathrm{t}_i)\\ &= 1-\mathrm{t}_k\\ &= \psi_J(\overline{k+1}).\end{aligned}$$

From these, we conclude that the RHS of (5.8) equals

$$(-1)^{|J|}\mathbf{e}^{|J|\varepsilon_1}\left(\prod_{1\le j\le \overline{1}}\psi_J(j)\right)[\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_J)],$$

which is just the term in $\mathfrak{Q}_{k-1}$ corresponding to J.

Assume that $M \ge n$. If $M > n$, then take $1 \le m \le n$ such that $M = \overline{m}$. If $M = n$, then set $m := \overline{n}$. By summing up

- the term in the first sum on the RHS of (5.7) corresponding to $J \setminus \{\overline{k}\}$,
- the terms in the third sum on the RHS of (5.7) corresponding to $(J \setminus \{\overline{k}\}) \sqcup \{\overline{i}\}$ for $k+1 \le i \le m-1$ (if $m = \overline{n}$, then regard $\overline{n}-1$ as n),

we deduce that (if $m = \overline{n}$, then regard $\overline{n}-2$ as $n-1$)

$$\begin{aligned}
&(-1)^{(|J|-1)+1}\mathbf{e}^{((|J|-1)+1)\varepsilon_1}\left(\prod_{1\le j\le \overline{1}}\psi_{J\setminus\{\overline{k}\}}(j)\right)[\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_{J\setminus\{\overline{k}\}}+\varepsilon_k)]\\
&-\sum_{i=k+1}^{m-1}\mathrm{t}_k\cdots\mathrm{t}_{i-1}(-1)^{|J|}\mathbf{e}^{|J|\varepsilon_1}\left(\prod_{1\le j\le \overline{1}}\psi_{(J\setminus\{\overline{k}\})\sqcup\{\overline{i}\}}(j)\right)[\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_{(J\setminus\{\overline{k}\})\sqcup\{\overline{i}\}}+\varepsilon_k-\varepsilon_i)]\\
&=(-1)^{|J|}\mathbf{e}^{|J|\varepsilon_1}\left(\left(\prod_{1\le j\le M}\psi_{J\setminus\{\overline{k}\}}(j)\right)\left(1-\mathrm{t}_k\cdots\mathrm{t}_{m-2}\right.\right.\\
&\qquad\left.\left.-\sum_{i=k+1}^{m-2}\mathrm{t}_k\cdots\mathrm{t}_{i-1}\underbrace{\psi_{(J\setminus\{\overline{k}\})\sqcup\{\overline{i}\}}(\overline{i+1})}_{=1-\mathrm{t}_i}\right)\right)[\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_J)]\\
&=(-1)^{|J|}\mathbf{e}^{|J|\varepsilon_1}\left(\prod_{1\le j\le M}\psi_{J\setminus\{\overline{k}\}}(j)\right)(1-\mathrm{t}_k)[\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_J)]\\
&=(-1)^{|J|}\mathbf{e}^{|J|\varepsilon_1}\left(\prod_{1\le j\le \overline{1}}\psi_{J\setminus\{\overline{k}\}}(j)\right)[\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_J)],
\end{aligned}$$

which is just the term in $\mathfrak{Q}_{k-1}$ corresponding to J. This proves (5.5) for $k-1$. Thus, by downward induction on k, equation (5.5) is proved. This completes the proof of the proposition. □

Also, we can show the following; we defer the proof of this proposition to Appendix B.

Proposition 5.7. *For $0 \le k \le n$, we have $\mathfrak{F}_k = \mathfrak{F}_{2n-k}$.*

From this proposition, by multiplying both sides of (5.6) by $\mathbf{e}^{-n\varepsilon_1}$, we obtain the following “fundamental” relation.

Corollary 5.8. *The following equality holds in $K_T(\mathbf{Q}_G)$:*

$$\sum_{l=0}^{n-1}(-1)^l(\mathbf{e}^{-(n-l)\varepsilon_1}+\mathbf{e}^{(n-l)\varepsilon_1})\mathfrak{F}_l+(-1)^n\mathfrak{F}_n=0. \tag{5.9}$$

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5.2. The other relations

From the fundamental relation (5.9), we will derive sufficiently many relations needed to obtain a Borel-type presentation of $QK_T(G/B)$. For $k \geq 0$, let $h_k(x_1, \dots, x_{2n})$ be the k-th complete symmetric polynomial in the $2n$ variables $x_1, \dots, x_{2n}$, that is,

$$h_k(x_1, \dots, x_{2n}) := \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq 2n} x_{i_1} x_{i_2} \cdots x_{i_{2n}};$$

by convention, we set $h_0(x_1, \dots, x_{2n}) := 0$. For $l \geq 0$ and $k \geq 1$, we set

$$H_l^k := h_l(\mathbf{e}^{\varepsilon_1}, \dots, \mathbf{e}^{\varepsilon_{k-1}}, \mathbf{e}^{\varepsilon_k}, \mathbf{e}^{-\varepsilon_k}, \mathbf{e}^{-\varepsilon_{k-1}}, \dots, \mathbf{e}^{-\varepsilon_1}) \in R(T).$$

The aim of this section is to prove the following.

Proposition 5.9. *The following recurrence relations hold:*

$$\begin{cases} \mathfrak{F}_0 & = 1, \\ \sum_{l=0}^{n-k} (-1)^l (H_{n-l-k}^{k+1} - H_{n-l-k-2}^{k+1}) \mathfrak{F}_l & = 0 \quad \textit{for } 0 \leq k \leq n-1. \end{cases} \tag{5.10}$$

First, by applying the Demazure operator D_1 to (5.9), we obtain the following.

Lemma 5.10. *We have*

$$\sum_{l=0}^{n-1} (-1)^l \mathbf{e}^{-(n-l)\varepsilon_1} \left(\sum_{r=0}^{n-l-1} \mathbf{e}^{r(\varepsilon_1 - \varepsilon_2)} \right) \left(\sum_{s=0}^{n-l-1} \mathbf{e}^{s(\varepsilon_1 + \varepsilon_2)} \right) \mathfrak{F}_l = 0. \tag{5.11}$$

Proof. By multiplying both sides of (5.9) by $\mathbf{e}^{\varepsilon_1}$, we obtain

$$\sum_{l=0}^{n-1} (-1)^l (\mathbf{e}^{-(n-l-1)\varepsilon_1} + \mathbf{e}^{(n-l+1)\varepsilon_1}) \mathfrak{F}_l + (-1)^n \mathbf{e}^{\varepsilon_1} \mathfrak{F}_n = 0.$$

Then, by applying D_1, we see that

$$\begin{aligned} 0 &= \mathsf{D}_1 \left(\sum_{l=0}^{n-1} (-1)^l (\mathbf{e}^{-(n-l-1)\varepsilon_1} + \mathbf{e}^{(n-l+1)\varepsilon_1}) \mathfrak{F}_l + (-1)^n e^{\varepsilon_1} \mathfrak{F}_n \right) \\ &= \sum_{l=0}^{n-1} (-1)^l (D_1(\mathbf{e}^{-(n-l-1)\varepsilon_1}) + D_1(\mathbf{e}^{(n-l+1)\varepsilon_1})) \mathfrak{F}_l + (-1)^n D_1(\mathbf{e}^{\varepsilon_1}) \mathfrak{F}_n \\ &= \sum_{l=0}^{n-1} (-1)^l \left(\frac{\mathbf{e}^{-(n-l-1)\varepsilon_1} - \mathbf{e}^{\alpha_1} \mathbf{e}^{s_1(-(n-l-1)\varepsilon_1)}}{1 - \mathbf{e}^{\alpha_1}} + \frac{\mathbf{e}^{(n-l+1)\varepsilon_1} - \mathbf{e}^{\alpha_1} \mathbf{e}^{s_1((n-l+1)\varepsilon_1)}}{1 - \mathbf{e}^{\alpha_1}} \right) \mathfrak{F}_l \\ &\qquad + (-1)^n \frac{\mathbf{e}^{\varepsilon_1} - \mathbf{e}^{\alpha_1} \mathbf{e}^{s_1 \varepsilon_1}}{1 - \mathbf{e}^{\alpha_1}} \mathfrak{F}_n \\ &= \sum_{l=0}^{n-1} (-1)^l \left(\frac{\mathbf{e}^{-(n-l-1)\varepsilon_1} - \mathbf{e}^{\varepsilon_1 - \varepsilon_2} \mathbf{e}^{-(n-l-1)\varepsilon_2} + \mathbf{e}^{(n-l+1)\varepsilon_1} - \mathbf{e}^{\varepsilon_1 - \varepsilon_2} \mathbf{e}^{(n-l+1)\varepsilon_2}}{1 - \mathbf{e}^{\alpha_1}} \right) \mathfrak{F}_l \\ &\qquad + (-1)^n \frac{\mathbf{e}^{\varepsilon_1} - \mathbf{e}^{\varepsilon_1 - \varepsilon_2} \mathbf{e}^{\varepsilon_2}}{1 - \mathbf{e}^{\alpha_1}} \mathfrak{F}_n \\ &= \mathbf{e}^{\varepsilon_1} \sum_{l=0}^{n-1} (-1)^l \frac{\mathbf{e}^{-(n-l)\varepsilon_1} - \mathbf{e}^{-(n-l)\varepsilon_2} + \mathbf{e}^{(n-l)\varepsilon_1} - \mathbf{e}^{(n-l)\varepsilon_2}}{1 - \mathbf{e}^{\alpha_1}} \mathfrak{F}_l \end{aligned}$$

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$$
\begin{aligned}
&= \mathbf{e}^{\varepsilon_1} \sum_{l=0}^{n-1} (-1)^l \frac{(\mathbf{e}^{-(n-l)\varepsilon_1} - \mathbf{e}^{-(n-l)\varepsilon_2})(1 - \mathbf{e}^{(n-l)(\varepsilon_1+\varepsilon_2)})}{1 - \mathbf{e}^{\alpha_1}} \mathfrak{F}_l \\
&= \mathbf{e}^{\varepsilon_1}(1 - \mathbf{e}^{\varepsilon_1+\varepsilon_2}) \sum_{l=0}^{n-1} (-1)^l \left(\sum_{s=0}^{n-l-1} \mathbf{e}^{s(\varepsilon_1+\varepsilon_2)} \right) \frac{\mathbf{e}^{-(n-l)\varepsilon_1} - \mathbf{e}^{-(n-l)\varepsilon_2}}{1 - \mathbf{e}^{\alpha_1}} \mathfrak{F}_l \\
&= \mathbf{e}^{\varepsilon_1}(1 - \mathbf{e}^{\varepsilon_1+\varepsilon_2}) \sum_{l=0}^{n-1} (-1)^l \left(\sum_{s=0}^{n-l-1} \mathbf{e}^{s(\varepsilon_1+\varepsilon_2)} \right) \frac{\mathbf{e}^{-(n-l)\varepsilon_1}(1 - \mathbf{e}^{(n-l)(\varepsilon_1-\varepsilon_2)})}{1 - \mathbf{e}^{\varepsilon_1-\varepsilon_2}} \mathfrak{F}_l \\
&= \mathbf{e}^{\varepsilon_1}(1 - \mathbf{e}^{\varepsilon_1+\varepsilon_2}) \sum_{l=0}^{n-1} (-1)^l \mathbf{e}^{-(n-l)\varepsilon_1} \left(\sum_{r=0}^{n-l-1} \mathbf{e}^{r(\varepsilon_1-\varepsilon_2)} \right) \left(\sum_{s=0}^{n-l-1} \mathbf{e}^{s(\varepsilon_1+\varepsilon_2)} \right) \mathfrak{F}_l.
\end{aligned}
$$

Finally, by dividing both sides of this equation by $\mathbf{e}^{\varepsilon_1}(1 - \mathbf{e}^{\varepsilon_1+\varepsilon_2})$, we conclude that

$$
\sum_{l=0}^{n-1} (-1)^l \mathbf{e}^{-(n-l)\varepsilon_1} \left(\sum_{r=0}^{n-l-1} \mathbf{e}^{r(\varepsilon_1-\varepsilon_2)} \right) \left(\sum_{s=0}^{n-l-1} \mathbf{e}^{s(\varepsilon_1+\varepsilon_2)} \right) \mathfrak{F}_l = 0,
$$

as desired. This proves the lemma. □

Then, by successively applying Demazure operators to (5.11), we obtain the other relations. For this purpose, we use the following easy lemma.

Lemma 5.11.

(1) *For $k, l \geq 0$ with $k + l \geq 1$ and $1 \leq p < q \leq n$, we have*

$$
\mathbf{e}^{-k\varepsilon_p} \mathbf{e}^{-l\varepsilon_q} - \mathbf{e}^{l\varepsilon_p} \mathbf{e}^{k\varepsilon_q} = \mathbf{e}^{-k\varepsilon_p} \mathbf{e}^{-l\varepsilon_q} (1 - \mathbf{e}^{\varepsilon_p+\varepsilon_q}) \sum_{t=0}^{k+l-1} \mathbf{e}^{t(\varepsilon_p+\varepsilon_q)}.
$$

(2) *For $k \geq 0$, we have*

$$
\begin{aligned}
&\mathbf{e}^{-k\varepsilon_m} \sum_{s=0}^{k-1} \mathbf{e}^{s(\varepsilon_m-\varepsilon_{m+1})} - \mathbf{e}^{k\varepsilon_m} \sum_{s=1}^{k} \mathbf{e}^{s(-\varepsilon_m+\varepsilon_{m+1})} \\
&\quad = \mathbf{e}^{-k\varepsilon_m}(1 - \mathbf{e}^{\varepsilon_m+\varepsilon_{m+1}}) \sum_{t=0}^{k-1} \mathbf{e}^{t(\varepsilon_m-\varepsilon_{m+1})} \sum_{t=0}^{k-1} \mathbf{e}^{t(\varepsilon_m+\varepsilon_{m+1})}.
\end{aligned}
$$

Lemma 5.12. *For $2 \leq k \leq n-1$, we have*

$$
\begin{aligned}
&\sum_{l=0}^{n-k} (-1)^l \Bigg(\sum_{r_1=k-1}^{n-l-1} \sum_{s_1=0}^{n-l-1-r_1} \sum_{r_2=k-2}^{r_1-1} \sum_{s_2=0}^{r_1-1-r_2} \cdots \sum_{r_{k-1}=1}^{r_{k-2}-1} \sum_{s_{k-1}=0}^{r_{k-2}-1-r_{k-1}} \\
&\qquad \mathbf{e}^{(l+r_1+2s_1)\varepsilon_1 + (-r_1+r_2+2s_2)\varepsilon_2 + \cdots + (-r_{k-2}+r_{k-1}+2s_{k-1})\varepsilon_{k-1} + (-r_{k-1})\varepsilon_k} \\
&\qquad \times \sum_{p_k=0}^{r_{k-1}-1} \mathbf{e}^{p_k(\varepsilon_k-\varepsilon_{k+1})} \sum_{q_k=0}^{r_{k-1}-1} \mathbf{e}^{q_k(\varepsilon_k+\varepsilon_{k+1})} \Bigg) \mathfrak{F}_l = 0.
\end{aligned}
\tag{5.12}
$$

Proof. We prove the lemma by induction on k. First, we consider the case $k = 2$; in this case, $n \geq 3$. By multiplying both sides of (5.11) by $\mathbf{e}^{(n-1)\varepsilon_1}$, we obtain

$$
\sum_{l=0}^{n-1} (-1)^l \mathbf{e}^{l\varepsilon_1} \left(\sum_{r=0}^{n-l-1} \sum_{s=0}^{n-l-1} \mathbf{e}^{r(\varepsilon_1-\varepsilon_2)+s(\varepsilon_1+\varepsilon_2)} \right) \mathfrak{F}_l = 0.
\tag{5.13}
$$

Then, by multiplying both sides of (5.13) by $\mathbf{e}^{\varepsilon_2}$, we obtain

$$\sum_{l=0}^{n-1}(-1)^l\left(\sum_{r=0}^{n-l-1}\sum_{s=0}^{n-l-1}\mathbf{e}^{(l+r+s)\varepsilon_1+(-r+s+1)\varepsilon_2}\right)\mathfrak{F}_l=0. \tag{5.14}$$

Here, note that $\langle (l+r+s)\varepsilon_1+(-r+s+1)\varepsilon_2, \alpha_2^\vee\rangle = -r+s+1$. Hence, by applying the Demazure operator D_2 to both sides of (5.14), we deduce that

$$\begin{aligned}\sum_{l=0}^{n-2}(-1)^l\Bigg(&\underbrace{\sum_{r=1}^{n-l-1}\sum_{s=0}^{r-1}\mathbf{e}^{(l+r+s)\varepsilon_1+(-r+s+1)\varepsilon_2}(1+\mathbf{e}^{\alpha_2}+\cdots+\mathbf{e}^{(r-s-1)\alpha_2})}_{(*)}\\ &\underbrace{-\sum_{r=0}^{n-l-2}\sum_{s=r+1}^{n-l-1}\mathbf{e}^{(l+r+s)\varepsilon_1+(-r+s+1)\varepsilon_2}(\mathbf{e}^{-\alpha_2}+\mathbf{e}^{-2\alpha_2}+\cdots+\mathbf{e}^{-(-r+s)\alpha_2})}_{(**)}\Bigg)\mathfrak{F}_l=0.\end{aligned} \tag{5.15}$$

In $(*)$, we put

$$r_1=r-s,\quad s_1=s, \tag{5.16}$$

while in $(**)$, we put

$$r_1=s-r,\quad s_1=r. \tag{5.17}$$

Then, we see that

$$\begin{aligned}&(\text{LHS) of (5.15)}\\ &=\sum_{l=0}^{n-2}(-1)^l\mathbf{e}^{l\varepsilon_1}\Bigg(\sum_{r_1=1}^{n-l-1}\sum_{s_1=0}^{n-l-1-r_1}\mathbf{e}^{(r_1+2s_1)\varepsilon_1+(-r_1+1)\varepsilon_2}(1+\mathbf{e}^{\alpha_2}+\cdots+\mathbf{e}^{(r_1-1)\alpha_2})\\ &\qquad -\sum_{r_1=1}^{n-l-1}\sum_{s_1=0}^{n-l-1-r_1}\mathbf{e}^{(r_1+2s_1)\varepsilon_1+(r_1+1)\varepsilon_2}(\mathbf{e}^{-\alpha_2}+\mathbf{e}^{-2\alpha_2}+\cdots+\mathbf{e}^{-r_1\alpha_1})\Bigg)\mathfrak{F}_l\\ &=\sum_{l=0}^{n-2}(-1)^l\mathbf{e}^{l\varepsilon_1+\varepsilon_2}\Bigg(\sum_{r_1=1}^{n-l-1}\sum_{s_1=0}^{n-l-1-r_1}\mathbf{e}^{(r_1+2s_1)\varepsilon_1}\\ &\qquad\times\Bigg(\mathbf{e}^{-r_1\varepsilon_2}\sum_{t=0}^{r_1-1}\mathbf{e}^{t(\varepsilon_2-\varepsilon_3)}-\mathbf{e}^{r_1\varepsilon_2}\sum_{t=1}^{r_1}\mathbf{e}^{t(-\varepsilon_2+\varepsilon_3)}\Bigg)\Bigg)\mathfrak{F}_l\\ &=\sum_{l=0}^{n-2}(-1)^l\mathbf{e}^{l\varepsilon_1+\varepsilon_2}\Bigg(\sum_{r_1=1}^{n-l-1}\sum_{s_1=0}^{n-l-1-r_1}\mathbf{e}^{(r_1+2s_1)\varepsilon_1}\\ &\qquad\times\Bigg(\mathbf{e}^{-r_1\varepsilon_2}(1-\mathbf{e}^{\varepsilon_2+\varepsilon_3})\sum_{p_2=0}^{r_1-1}\mathbf{e}^{p_2(\varepsilon_2-\varepsilon_3)}\sum_{q_2=0}^{r_1-1}\mathbf{e}^{q_2(\varepsilon_2+\varepsilon_3)}\Bigg)\Bigg)\mathfrak{F}_l,\end{aligned} \tag{5.18}$$

where, for the last equality, we have used Lemma 5.11 (2). Hence, by dividing the rightmost-hand side of (5.18) by $\mathbf{e}^{\varepsilon_2}(1-\mathbf{e}^{\varepsilon_2+\varepsilon_3})$, we obtain

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$$\sum_{l=0}^{n-2}(-1)^l\left(\sum_{r_1=1}^{n-l-1}\sum_{s_1=0}^{n-l-1-r_1}\mathbf{e}^{(l+r_1+2s_1)\varepsilon_1}\left(\mathbf{e}^{-r_1\varepsilon_2}\sum_{p_2=0}^{r_1-1}\mathbf{e}^{p_2(\varepsilon_2-\varepsilon_3)}\sum_{q_2=0}^{r_1-1}\mathbf{e}^{q_2(\varepsilon_2+\varepsilon_3)}\right)\right)\mathfrak{F}_l=0.$$

This proves the lemma for $k=2$.

Now, let $3\le k\le n-1$. We assume that the lemma holds for $k-1$, and prove the assertion of the lemma for k; note that $n\ge k+1$ in this case. By the induction hypothesis, we have

$$\sum_{l=0}^{n-k+1}(-1)^l\left(\sum_{r_1=k-2}^{n-l-1}\sum_{s_1=0}^{n-l-1-r_1}\sum_{r_2=k-3}^{r_1-1}\sum_{s_2=0}^{r_1-1-r_2}\cdots\sum_{r_{k-2}=1}^{r_{k-3}-1}\sum_{s_{k-2}=0}^{r_{k-3}-1-r_{k-2}}\right.$$
$$\mathbf{e}^{(l+r_1+2s_1)\varepsilon_1+(-r_1+r_2+2s_2)\varepsilon_2+\cdots+(-r_{k-3}+r_{k-2}+2s_{k-2})\varepsilon_{k-2}+(-r_{k-2})\varepsilon_{k-1}}$$
$$\left.\times\sum_{p_{k-1}=0}^{r_{k-2}-1}\mathbf{e}^{p_{k-1}(\varepsilon_{k-1}-\varepsilon_k)}\sum_{q_{k-1}=0}^{r_{k-2}-1}\mathbf{e}^{q_{k-1}(\varepsilon_{k-1}+\varepsilon_k)}\right)\mathfrak{F}_l=0.$$

By

(i) multiplying both sides of this equation by $\mathbf{e}^{\varepsilon_k}$,
(ii) applying the Demazure operator D_k to both sides,
(iii) making a change of variables similar to (5.16) and (5.17), and
(iv) dividing both sides by $\mathbf{e}^{\varepsilon_k}(1-\mathbf{e}^{\varepsilon_k+\varepsilon_{k+1}})$,

we deduce that

$$\sum_{l=0}^{n-k+1}(-1)^l\left(\sum_{r_1=k-1}^{n-l-1}\sum_{s_1=0}^{n-l-1-r_1}\sum_{r_2=k-2}^{r_1-1}\sum_{s_2=0}^{r_1-1-r_2}\cdots\sum_{r_{k-1}=1}^{r_{k-2}-1}\sum_{s_{k-1}=0}^{r_{k-2}-1-r_{k-1}}\right.$$
$$\mathbf{e}^{(l+r_1+2s_1)\varepsilon_1+(-r_1+r_2+2s_2)\varepsilon_2+\cdots+(-r_{k-3}+r_{k-2}+2s_{k-2})\varepsilon_{k-2}+(-r_{k-2}+r_{k-1}+2s_{k-1})\varepsilon_{k-1}+(-r_{k-1})\varepsilon_k}$$
$$\left.\times\sum_{p_k=0}^{r_{k-1}-1}\mathbf{e}^{p_k(\varepsilon_k-\varepsilon_{k+1})}\sum_{q_k=0}^{r_{k-1}-1}\mathbf{e}^{q_k(\varepsilon_k+\varepsilon_{k+1})}\right)\mathfrak{F}_l=0,$$

as desired. Thus, by induction on k, the lemma is proved. □

Also, we can prove the following lemma for complete symmetric polynomials. Since the proof of this lemma is elementary, we leave it to the reader.

Lemma 5.13. *In the Laurent polynomial ring $\mathbb{Z}[x_1^{\pm1},\dots,x_n^{\pm1}]$, the following hold.*

(1) *We have $h_0(x_1,\dots,x_n,x_n^{-1},\dots,x_1^{-1})=1$.*
(2) *For $m\ge1$, we have*

$$x_1^m+x_1^{-m}=\begin{cases}h_1(x_1,x_1^{-1}) & \text{if } m=1,\\ h_m(x_1,x_1^{-1})-h_{m-2}(x_1,x_1^{-1}) & \text{if } m\ge2.\end{cases}$$

(3) *For $m\ge1$, we have*

$$x_1^{-m}\left(\sum_{k=0}^{m}(x_1x_2^{-1})^k\right)\left(\sum_{l=0}^{m}(x_1x_2)^l\right)$$
$$=\begin{cases}h_1(x_1,x_2,x_2^{-1},x_1^{-1}) & \text{if } m=1,\\ h_m(x_1,x_2,x_2^{-1},x_1^{-1})-h_{m-2}(x_1,x_2,x_2^{-1},x_1^{-1}) & \text{if } m\ge2.\end{cases}$$

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(4) *If* $n \geq 3$, *then for* $m \geq 1$, *we have*

$$\sum_{r_1=n-2}^{m+n-2}\sum_{s_1=0}^{m+n-2-r_1}\sum_{r_2=n-3}^{r_1-1}\sum_{s_2=0}^{r_1-1-r_2}\cdots\sum_{r_{n-2}=1}^{r_{n-3}-1}\sum_{s_{n-2}=0}^{r_{n-3}-1-r_{n-2}} x_1^{-(m+n-2)+r_1+2s_1}x_2^{-r_1+r_2+2s_2+1}\cdots x_{n-2}^{-r_{n-3}+r_{n-2}+2s_{n-2}+1}x_{n-1}^{-r_{n-2}}$$
$$\times\left(\sum_{k=0}^{r_{n-2}-1}(x_{n-1}x_n^{-1})^k\right)\left(\sum_{l=0}^{r_{n-2}-1}(x_{n-1}x_n)^l\right)$$
$$=\begin{cases} h_1(x_1,\ldots,x_n,x_n^{-1},\ldots,x_1^{-1}) & \text{if } m=1,\\ h_m(x_1,\ldots,x_n,x_n^{-1},\ldots,x_1^{-1})-h_{m-2}(x_1,\ldots,x_n,x_n^{-1},\ldots,x_1^{-1}) & \text{if } m\geq 2.\end{cases}$$

Proof of Proposition 5.9. By multiplying both sides of (5.11) by $\mathbf{e}^{\varepsilon_1}$ and then applying Lemma 5.13 (3) with $x_1=\mathbf{e}^{\varepsilon_1}$, $x_2=\mathbf{e}^{\varepsilon_2}$, and $m=n-l-1$, we obtain

$$\sum_{l=0}^{n-1}(-1)^l(H^2_{n-l-1}-H^2_{n-l-3})\mathfrak{F}_l=0.$$

For $2\leq k\leq n-1$, by multiplying both sides of (5.12) by $\mathbf{e}^{-(2n-k-2)\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_{n-1}}$ and then applying Lemma 5.13 (4) with $x_1=\mathbf{e}^{\varepsilon_1},\ldots,x_{k+1}=\mathbf{e}^{\varepsilon_{k+1}}$, and $m=n-l-k$, we obtain

$$\sum_{l=0}^{n-k}(-1)^l(H^{k+1}_{n-l-k}-H^{k+1}_{n-l-k-2})\mathfrak{F}_l=0.$$

This proves Proposition 5.9. □

5.3. *The solution of the recurrence relations*

In this subsection, we solve the recurrence relations (5.10) for $\mathfrak{F}_l$, $0\leq l\leq n$, with coefficients in $R(T)$. Recall the notation E_l, $0\leq l\leq 2n$, from (3.1).

Lemma 5.14. *We have the following:*

$$E_0=1, \tag{5.19}$$

$$\sum_{l=0}^{n-k}(-1)^l(H^{k+1}_{n-l-k}-H^{k+1}_{n-l-k-2})E_l=0,\quad 0\leq k\leq n-1. \tag{5.20}$$

Proof. First, equation (5.19) is obvious from the definition of elementary symmetric polynomials. We will prove (5.20). Since

$$\sum_{l=0}^{\infty}h_l(x_1,\ldots,x_d)t^l=\prod_{i=1}^{d}\frac{1}{1-x_it}$$

for $1\leq k+1\leq n$, we have

$$\sum_{l=0}^{\infty}H_l^{k+1}t^l=\left(\prod_{i=1}^{k+1}\frac{1}{1-\mathbf{e}^{\varepsilon_i}t}\right)\left(\prod_{j=1}^{k+1}\frac{1}{1-\mathbf{e}^{-\varepsilon_j}t}\right). \tag{5.21}$$

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By multiplying both sides of (5.21) by t^2 and then subtracting the resulting equation from (5.21), we obtain

$$\sum_{l=0}^{\infty}(H_l^{k+1} - H_{l-2}^{k+1})t^l = (1-t^2)\left(\prod_{i=1}^{k+1}\frac{1}{1-\mathbf{e}^{\varepsilon_i}t}\right)\left(\prod_{j=1}^{k+1}\frac{1}{1-\mathbf{e}^{-\varepsilon_j}t}\right). \tag{5.22}$$

Also, since

$$\sum_{l=0}^{d} e_l(x_1,\dots,x_d)t^l = \prod_{i=1}^{d}(1+x_i t),$$

it follows that

$$\sum_{l=0}^{2n} E_l t^d = \left(\prod_{i=1}^{n}(1+\mathbf{e}^{\varepsilon_i}t)\right)\left(\prod_{j=1}^{n}(1+\mathbf{e}^{-\varepsilon_j}t)\right). \tag{5.23}$$

By multiplying (5.23) with the equation obtained from (5.22) by replacing t with $-t$, we deduce that

$$\begin{aligned}&\left(\sum_{l=0}^{\infty}(-1)^l(H_l^{k+1} - H_{l-2}^{k+1})t^l\right)\left(\sum_{m=0}^{2n}E_m t^m\right)\\ &= (1-t^2)\left(\prod_{i=k+2}^{n}(1+\mathbf{e}^{\varepsilon_i}t)\right)\left(\prod_{j=k+2}^{n}(1+\mathbf{e}^{-\varepsilon_j}t)\right).\end{aligned} \tag{5.24}$$

The LHS of (5.24) can be rewritten as:

$$\begin{aligned}&\left(\sum_{l=0}^{\infty}(-1)^l(H_l^{k+1} - H_{l-2}^{k+1})t^l\right)\left(\sum_{m=0}^{2n}E_m t^m\right)\\ &= \sum_{l=0}^{\infty}\left(\sum_{d=0}^{l}(-1)^{l-d}(H_{l-d}^{k+1} - H_{l-d-2}^{k+1})E_d\right)t^l.\end{aligned} \tag{5.25}$$

By setting $e_m(x_1,\dots,x_d) = 0$ if $m < 0$ or $m > d$, the RHS of (5.24) can be rewritten as:

$$\begin{aligned}&(1-t^2)\left(\prod_{i=k+2}^{n}(1+\mathbf{e}^{\varepsilon_i}t)\right)\left(\prod_{j=k+2}^{n}(1+\mathbf{e}^{-\varepsilon_j}t)\right)\\ &= (1-t^2)\sum_{m=0}^{2(n-k-1)} e_m(\mathbf{e}^{\varepsilon_{k+2}},\dots,\mathbf{e}^{\varepsilon_n},\mathbf{e}^{-\varepsilon_n},\dots,\mathbf{e}^{-\varepsilon_{k+2}})t^m\\ &= \sum_{m=0}^{2(n-k-1)}(e_m(\mathbf{e}^{\varepsilon_{k+2}},\dots,\mathbf{e}^{\varepsilon_n},\mathbf{e}^{-\varepsilon_n},\dots,\mathbf{e}^{-\varepsilon_{k+2}}) - e_{m-2}(\mathbf{e}^{\varepsilon_{k+2}},\dots,\mathbf{e}^{\varepsilon_n},\mathbf{e}^{-\varepsilon_n},\dots,\mathbf{e}^{-\varepsilon_{k+2}}))t^m.\end{aligned} \tag{5.26}$$

Therefore, by comparing the coefficients of t^{n-k} in (5.25) and (5.26), we see that

$$\begin{aligned}&\sum_{d=0}^{n-k}(-1)^{n-k-d}(H_{n-k-d}^{k+1} - H_{n-k-d-2}^{k+1})E_d\\ &= e_{n-k}(\mathbf{e}^{\varepsilon_{k+2}},\dots,\mathbf{e}^{\varepsilon_n},\mathbf{e}^{-\varepsilon_n},\dots,\mathbf{e}^{-\varepsilon_{k+2}}) - e_{n-k-2}(\mathbf{e}^{\varepsilon_{k+2}},\dots,\mathbf{e}^{\varepsilon_n},\mathbf{e}^{-\varepsilon_n},\dots,\mathbf{e}^{-\varepsilon_{k+2}})\end{aligned}$$

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$$= e_{(n-k-1)+1}(\underbrace{\mathbf{e}^{\varepsilon_{k+2}}, \ldots, \mathbf{e}^{\varepsilon_n}, \mathbf{e}^{-\varepsilon_n}, \ldots, \mathbf{e}^{-\varepsilon_{k+2}}}_{2(n-k-1) \text{ variables}}) - e_{(n-k-1)-1}(\mathbf{e}^{\varepsilon_{k+2}}, \ldots, \mathbf{e}^{\varepsilon_n}, \mathbf{e}^{-\varepsilon_n}, \ldots, \mathbf{e}^{-\varepsilon_{k+2}})$$

$$= 0 \quad \text{by (3.2)}.$$

Thus, we conclude that

$$\sum_{l=0}^{n-k}(-1)^l (H^{k+1}_{n-l-k} - H^{k+1}_{n-l-k-2})E_l = (-1)^{n-k}\sum_{l=0}^{n-k}(-1)^{n-k-l}(H^{k+1}_{n-l-k} - H^{k+1}_{n-l-k-2})E_l = 0.$$

This proves the Lemma. □

Corollary 5.15. *For* $0 \le l \le 2n$*, we have* $\mathfrak{F}_l = E_l$.

Proof. By Proposition 5.7, it suffices to prove the corollary for $0 \le l \le n$. It follows from Lemma 5.14 that $(E_0, \ldots, E_n)$ is a solution of the recurrence relations (5.10). Since the elements $\mathfrak{F}_l$, $0 \le l \le n$, are uniquely determined by the recurrence relations (5.10), we conclude that $\mathfrak{F}_l = E_l$ for $0 \le l \le n$, as desired. This proves the corollary. □

6. Borel-type presentation

The aim of this section is to give a proof of Theorem 3.6. We assume that $G = \mathrm{Sp}_{2n}(\mathbb{C})$, the symplectic group of rank n. First, we derive some relations in $QK_T(G/B)$ from the corresponding ones in $K_T(\mathbf{Q}_G)$, given in Corollary 5.15. Then, based on these relations, we prove the existence of a homomorphism $(R(T)[[Q]])[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \to QK_T(G/B)$ of $R(T)[[Q]]$-algebras which annihilates the ideal $\mathcal{I}^Q$ of $(R(T)[[Q]])[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$. Finally, we prove that the induced homomorphism $(R(T)[[Q]])[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]/\mathcal{I}^Q \to QK_T(G/B)$ is, in fact, an isomorphism.

6.1. Some relations in $QK_T(G/B)$

We will derive some relations in $QK_T(G/B)$ from the corresponding ones in $K_T(\mathbf{Q}_G)$, given in Corollary 5.15.

Definition 6.1. Let $J \subset [1, \overline{1}]$.

(1) For $1 \le j \le n$, we define $\varphi_J^Q(j) \in \mathbb{Z}[[Q]]$ by

$$\varphi_J^Q(j) := \begin{cases} \dfrac{1}{1 - Q_j} & \text{if } j, j+1 \in J, \\ 1 & \text{otherwise.} \end{cases}$$

(2) For $2 \le j \le n$, we define $\varphi_J^Q(\overline{j}) \in \mathbb{Z}[[Q]]$ by

$$\varphi_J^Q(\overline{j}) := \begin{cases} 1 + \dfrac{Q_{j-1}Q_j \cdots Q_n}{1 - Q_{j-1}} & \text{if } J = \{\cdots < j-1 < \overline{j-1} < \cdots\}, \\ \dfrac{1}{1 - Q_{j-1}} & \text{if } \overline{j}, \overline{j-1} \in J, \\ 1 & \text{otherwise.} \end{cases}$$

(3) We set $\varphi_J^Q(\overline{1}) := 1 \in \mathbb{Z}[[Q]]$.

In the following, $\prod^\star$ denotes the product with respect to the quantum product $\star$.

Definition 6.2. (1) For $1 \leq k \leq n$ and $0 \leq l \leq k$, we define an element $\mathcal{F}_l^k \in QK_T(G/B)$ as

$$\mathcal{F}_l^k := \sum_{\substack{J \subset [1,k] \\ |J|=l}} \left(\prod_{1 \leq j \leq \overline{1}} \varphi_J^Q(j) \right) \left(\prod_{j \in J}{}^{\star} [\mathcal{O}_{G/B}(-\varepsilon_j)] \right).$$

(2) For $0 \leq k \leq n$ and $0 \leq l \leq 2n-k$, we define an element $\mathcal{F}_l^{\overline{k}} \in QK_T(G/B)$ as

$$\mathcal{F}_l^{\overline{k}} := \sum_{\substack{J \subset [1,\overline{k+1}] \\ |J|=l}} \left(\prod_{1 \leq j \leq \overline{1}} \varphi_J^Q(j) \right) \left(\prod_{j \in J}{}^{\star} [\mathcal{O}_{G/B}(-\varepsilon_j)] \right),$$

where we understand that $\overline{n+1} := n$.

(3) We set $\mathcal{F}_l := \mathcal{F}_l^{\overline{0}}$ for $0 \leq l \leq 2n$.

Recall the notation Φ from Theorem 4.1.

Proposition 6.3.

(1) *For* $1 \leq k \leq n$ *and* $0 \leq l \leq k$, *we have* $\Phi(\mathcal{F}_l^k) = \mathfrak{F}_l^k$.

(2) *For* $0 \leq k \leq n$ *and* $0 \leq l \leq 2n-k$, *we have* $\Phi(\mathcal{F}_l^{\overline{k}}) = \mathfrak{F}_l^{\overline{k}}$.

Proof. We prove only (2) since (1) can be proved by the same argument as (2). In this proof, we thought of $1/(1-\mathrm{t}_i)$ for $1 \leq i \leq n$ as the infinite sum $\sum_{d=0}^{\infty} \mathrm{t}_i^d$; this infinite sum is a well-defined operator on $K_T(\mathbf{Q}_G)$. We define $\varphi_J^{\frac{\infty}{2}}(j)$ for $J \subset [1,\overline{1}]$ and $j \in [1,\overline{1}]$ as follows.

(1) For $1 \leq j \leq n$, we set

$$\varphi_J^{\frac{\infty}{2}}(j) := \begin{cases} \dfrac{1}{1-\mathrm{t}_j} & \text{if } j, j+1 \in J, \\ 1 & \text{otherwise.} \end{cases}$$

(2) For $2 \leq j \leq n$, we set

$$\varphi_J^{\frac{\infty}{2}}(\overline{j}) := \begin{cases} 1 + \dfrac{\mathrm{t}_{j-1}\mathrm{t}_j \cdots \mathrm{t}_n}{1-\mathrm{t}_{j-1}} & \text{if } J = \{\cdots < j-1 < \overline{j-1} < \cdots\}, \\ \dfrac{1}{1-\mathrm{t}_{j-1}} & \text{if } \overline{j}, \overline{j-1} \in J, \\ 1 & \text{otherwise.} \end{cases}$$

(3) We set $\varphi_J^{\frac{\infty}{2}}(\overline{1}) := 1$.

Note that $\varphi_J^{\frac{\infty}{2}}$ is obtained from φ_J^Q by replacing Q_i with t_i for $1 \leq i \leq n$.

By Proposition 4.2, we see that

$$\begin{aligned} \Phi(\mathcal{F}_l^{\overline{k}}) &= \sum_{\substack{J \subset [1,\overline{k+1}] \\ |J|=l}} \left(\prod_{1 \leq j \leq \overline{1}} \varphi_J^{\frac{\infty}{2}}(j) \right) \left(\prod_{\substack{1 \leq j \leq n \\ j \in J}} (1-\mathrm{t}_{j-1}) \right) \left(\prod_{\substack{1 \leq j \leq n \\ \overline{j} \in J}} (1-\mathrm{t}_j) \right) [\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_J)] \\ &= \sum_{\substack{J \subset [1,\overline{k+1}] \\ |J|=l}} \left(\prod_{1 \leq j \leq \overline{1}} \varphi_J^{\frac{\infty}{2}}(j) \right) \left(\prod_{\substack{0 \leq j \leq n-1 \\ j+1 \in J}} (1-\mathrm{t}_j) \right) \left(\prod_{\substack{2 \leq j \leq n+1 \\ \overline{j-1} \in J}} (1-\mathrm{t}_{j-1}) \right) [\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_J)]. \end{aligned}$$

Let us take and fix $J \subset [1, \overline{1}]$. For $1 \le j \le n-1$, we set

$$\theta_J(j) := \begin{cases} 1 - \mathsf{t}_j & \text{if } j+1 \in J, \\ 1 & \text{otherwise.} \end{cases}$$

Then, for $1 \le j \le n-1$, it follows that

$$\begin{aligned} \varphi_J^{\frac{\infty}{2}}(j)\theta_J(j) &= \begin{cases} \dfrac{1}{1-\mathsf{t}_j} \cdot (1-\mathsf{t}_j) & \text{if } j \in J \text{ and } j+1 \in J, \\ 1 \cdot 1 & \text{if } j \in J \text{ and } j+1 \notin J, \\ 1 \cdot (1-\mathsf{t}_j) & \text{if } j \notin J \text{ and } j+1 \in J, \\ 1 \cdot 1 & \text{if } j \notin J \text{ and } j+1 \notin J \end{cases} \\ &= \begin{cases} 1-\mathsf{t}_j & \text{if } j \notin J \text{ and } j+1 \in J, \\ 1 & \text{otherwise} \end{cases} \\ &= \psi_J(j). \end{aligned}$$

In addition, we set

$$\theta_J(n) := \begin{cases} 1 - \mathsf{t}_n & \text{if } \overline{n} \in J, \\ 1 & \text{otherwise.} \end{cases}$$

Then it follows that

$$\begin{aligned} \varphi_J^{\frac{\infty}{2}}(n)\theta_J(n) &= \begin{cases} \dfrac{1}{1-\mathsf{t}_n} \cdot (1-\mathsf{t}_n) & \text{if } n \in J \text{ and } \overline{n} \in J, \\ 1 \cdot 1 & \text{if } n \in J \text{ and } \overline{n} \notin J, \\ 1 \cdot (1-\mathsf{t}_n) & \text{if } n \notin J \text{ and } \overline{n} \in J, \\ 1 \cdot 1 & \text{if } n \notin J \text{ and } \overline{n} \notin J \end{cases} \\ &= \begin{cases} 1-\mathsf{t}_n & \text{if } n \notin J \text{ and } \overline{n} \in J, \\ 1 & \text{otherwise} \end{cases} \\ &= \psi_J(n). \end{aligned}$$

Also, for $2 \le j \le n$, we set

$$\theta_J(\overline{j}) := \begin{cases} 1 - \mathsf{t}_{j-1} & \text{if } \overline{j-1} \in J, \\ 1 & \text{otherwise.} \end{cases}$$

Then, for $2 \le j \le n$, it follows that

$$\begin{aligned} \varphi_J^{\frac{\infty}{2}}(\overline{j})\theta_J(\overline{j}) &= \begin{cases} \left(1 + \dfrac{\mathsf{t}_{j-1}\mathsf{t}_j \cdots \mathsf{t}_n}{1-\mathsf{t}_{j-1}}\right) \cdot (1-\mathsf{t}_{j-1}) & \text{if } J = \{\cdots < j-1 < \overline{j-1} < \cdots\}, \\ 1 \cdot (1-\mathsf{t}_{j-1}) & \text{if } \overline{j} \notin J, \overline{j-1} \in J, \text{ and } J \text{ is not of the above form}, \\ 1 \cdot 1 & \text{if } \overline{j} \notin J \text{ and } \overline{j-1} \notin J, \\ \dfrac{1}{1-\mathsf{t}_{j-1}} \cdot (1-\mathsf{t}_{j-1}) & \text{if } \overline{j} \in J \text{ and } \overline{j-1} \in J, \\ 1 \cdot 1 & \text{if } \overline{j} \in J \text{ and } \overline{j-1} \notin J \end{cases} \\ &= \begin{cases} 1 - \mathsf{t}_{j-1} + \mathsf{t}_{j-1}\mathsf{t}_j \cdots \mathsf{t}_n & \text{if } J = \{\cdots < j-1 < \overline{j-1} < \cdots\}, \\ 1 - \mathsf{t}_{j-1} & \text{if } \overline{j} \notin J, \overline{j-1} \in J, \text{ and } J \text{ is not of the above form}, \\ 1 & \text{otherwise} \end{cases} \\ &= \psi_J(\overline{j}). \end{aligned}$$

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Therefore, by setting $\theta_J(\overline{1}) := 1$, we conclude that

$$\begin{aligned}\Phi(\mathcal{F}_l^{\overline{k}}) &= \sum_{\substack{J \subset [1,\overline{k+1}] \\ |J|=l}} \left(\prod_{1 \le j \le \overline{1}} \varphi_J^{\frac{\infty}{2}}(j)\theta_J(j) \right) [\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_J)] \\ &= \sum_{\substack{J \subset [1,\overline{1}] \\ |J|=l}} \left(\prod_{1 \le j \le \overline{1}} \psi_J(j) \right) [\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_J)] \\ &= \mathfrak{F}_l^{\overline{k}},\end{aligned}$$

as desired. This proves the proposition. □

By Proposition 6.3, we obtain the following relations in $QK_T(G/B)$; recall the notation E_l for $0 \le l \le 2n$ from (3.1).

Corollary 6.4. *For $0 \le l \le 2n$, we have $\mathcal{F}_l = E_l$.*

Proof. By Proposition 6.3 and Corollary 5.15, we have

$$\Phi(\mathcal{F}_l - E_l) = \mathfrak{F}_l - E_l = 0;$$

note that $\Phi(E_l) = E_l$, $0 \le l \le 2n$, because we have

$$e_l(\mathbf{e}^{-\varepsilon_1}, \dots, \mathbf{e}^{-\varepsilon_n}, \mathbf{e}^{\varepsilon_n}, \dots, \mathbf{e}^{\varepsilon_1}) = e_l(\mathbf{e}^{\varepsilon_1}, \dots, \mathbf{e}^{\varepsilon_n}, \mathbf{e}^{-\varepsilon_n}, \dots, \mathbf{e}^{-\varepsilon_1}).$$

Since Φ is injective, we conclude that $\mathcal{F}_l = E_l$, as desired. This proves the corollary. □

Also, by combining Proposition 6.3 with Proposition 5.6, we obtain an explicit expression, in terms of line bundle classes, of the Schubert classes $[\mathcal{O}^{s_1 s_2 \cdots s_k}]$ and $[\mathcal{O}^{s_1 s_2 \cdots s_n s_{n-1} \cdots s_k}]$ for $1 \le k \le n$.

Corollary 6.5. (1) *For $1 \le k \le n$, the following equality holds in $QK_T(G/B)$:*

$$[\mathcal{O}^{s_1 \cdots s_k}] = \sum_{l=0}^{k} (-1)^l \mathbf{e}^{-l\varepsilon_1} \mathcal{F}_l^k.$$

(2) *For $1 \le k \le n$, the following equality holds in $QK_T(G/B)$:*

$$[\mathcal{O}^{s_1 \cdots s_n s_{n-1} \cdots s_k}] = \sum_{l=0}^{2n-k} (-1)^l \mathbf{e}^{-l\varepsilon_1} \mathcal{F}_l^{\overline{k}}.$$

6.2. *The existence of a homomorphism*

It follows from Corollary 6.4 that there exists a homomorphism $(R(T)[[Q]])[z_1^{\pm 1}, \dots, z_n^{\pm 1}] \to QK_T(G/B)$ of $R(T)[[Q]]$-algebras, which eventually gives a Borel-type presentation of $QK_T(G/B)$.

Definition 6.6. We define a homomorphism $\widehat{\Psi}^Q : (R(T)[[Q]])[z_1^{\pm 1}, \dots, z_n^{\pm 1}] \to QK_T(G/B)$ of $R(T)[[Q]]$-algebras by:

$$z_j \mapsto \frac{1}{1 - Q_j} [\mathcal{O}_{G/B}(-\varepsilon_j)], \quad 1 \le j \le n.$$

Let us compute the images $\widehat{\Psi}^Q(z_j^{-1})$, $1 \le j \le n$. By convention, we set $Q_0 := 0$.

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Lemma 6.7. *For $1 \le j \le n$, the line bundle class $[\mathcal{O}_{G/B}(-\varepsilon_j)] \in QK_T(G/B)$ is invertible with respect to the quantum product $\star$ in $QK_T(G/B)$, and its inverse is given as:*

$$[\mathcal{O}_{G/B}(-\varepsilon_j)]^{-1} = \frac{1}{(1-Q_j)(1-Q_{j-1})}[\mathcal{O}_{G/B}(\varepsilon_j)].$$

Proof. We set

$$\begin{aligned}\mathcal{L} &:= \frac{1}{(1-Q_j)(1-Q_{j-1})}[\mathcal{O}_{G/B}(\varepsilon_j)] \star [\mathcal{O}_{G/B}(-\varepsilon_j)] \\ &= \left(\frac{1}{1-Q_j}[\mathcal{O}_{G/B}(\varepsilon_j)]\right) \star \left(\frac{1}{1-Q_{j-1}}[\mathcal{O}_{G/B}(-\varepsilon_j)]\right).\end{aligned}$$

It suffices to show that $\mathcal{L} = 1$. By using the map $\Phi : QK_T(G/B) \xrightarrow{\sim} K_T(\mathbf{Q}_G)$ (see Theorem 4.1), we see that

$$\begin{aligned}\Phi(\mathcal{L}) &= \Phi\left(\left(\frac{1}{1-Q_j}[\mathcal{O}_{G/B}(\varepsilon_j)]\right) \star \left(\frac{1}{1-Q_{j-1}}[\mathcal{O}_{G/B}(-\varepsilon_j)]\right)\right) \\ &= [\mathcal{O}_{\mathbf{Q}_G}(\varepsilon_j)] \otimes [\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_j)] \quad \text{(by Proposition 4.2)} \\ &= [\mathcal{O}_{\mathbf{Q}_G}(\varepsilon_j)] \otimes [\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_j)] \\ &= [\mathcal{O}_{\mathbf{Q}_G}].\end{aligned}$$

Since $\Phi(1) = [\mathcal{O}_{\mathbf{Q}_G}]$, we conclude that $\mathcal{L} = 1$, as desired. This proves the lemma. □

Corollary 6.8. *For $1 \le j \le n$, we have*

$$\widehat{\Psi}^Q(z_j^{-1}) = \frac{1}{1-Q_{j-1}}[\mathcal{O}_{G/B}(\varepsilon_j)].$$

Proof. We compute as:

$$\begin{aligned}\widehat{\Psi}^Q(z_j^{-1}) &= \widehat{\Psi}^Q(z_j)^{-1} \\ &= \left(\frac{1}{1-Q_j}[\mathcal{O}_{G/B}(-\varepsilon_j)]\right)^{-1} \\ &= (1-Q_j) \cdot [\mathcal{O}_{G/B}(-\varepsilon_j)]^{-1} \\ &= (1-Q_j) \cdot \frac{1}{(1-Q_j)(1-Q_{j-1})}[\mathcal{O}_{G/B}(\varepsilon_j)] \quad \text{(by Lemma 6.7)} \\ &= \frac{1}{1-Q_{j-1}}[\mathcal{O}_{G/B}(\varepsilon_j)].\end{aligned}$$

This proves the corollary. □

Recall that $z_{\overline{j}} = z_j^{-1}$ for $1 \le j \le n$.

Definition 6.9. (1) For $1 \le k \le n$ and $0 \le l \le k$, we define $F_l^k \in (R(T)[[Q]])[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ by

$$F_l^k := \sum_{\substack{J \subset [1,k] \\ |J| = l}} \left(\prod_{1 \le j \le \overline{1}} \zeta_J(j)\right)\left(\prod_{j \in J} z_j\right).$$

(2) For $0 \le k \le n$ and $0 \le l \le 2n-k$, we define $F_l^{\overline{k}} \in (R(T)[[Q]])[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ by

$$F_l^{\overline{k}} := \sum_{\substack{J \subset [1, \overline{k+1}] \\ |J| = l}} \left(\prod_{1 \le j \le \overline{1}} \zeta_J(j) \right) \left(\prod_{j \in J} z_j \right),$$

where we understand that $\overline{n+1} := n$.

Note that if $k = 0$, then $F_l^{\overline{0}} = F_l$ for $0 \le l \le 2n$.

Lemma 6.10. (1) *For $1 \le k \le n$ and $0 \le l \le k$, we have $\widehat{\Psi}^Q(F_l^k) = \mathcal{F}_l^k$.*
(2) *For $0 \le k \le n$ and $0 \le l \le 2n-k$, we have $\widehat{\Psi}^Q(F_l^{\overline{k}}) = \mathcal{F}_l^{\overline{k}}$.*

Proof. We prove only (2) since (1) can be proved by the same argument as (2). By Corollary 6.8, we see that

$$\begin{aligned}
&\widehat{\Psi}^Q(F_l^{\overline{k}}) \\
&= \sum_{\substack{J \subset [1, \overline{k+1}] \\ |J| = l}} \left(\prod_{1 \le j \le \overline{1}} \zeta_J(j) \right) \left(\prod_{\substack{1 \le j \le n \\ j \in J}}^{\star} \frac{1}{1-Q_j} [\mathcal{O}_{G/B}(-\varepsilon_j)] \right) \left(\prod_{\substack{1 \le j \le n \\ \overline{j} \in J}}^{\star} \frac{1}{1-Q_{j-1}} [\mathcal{O}_{G/B}(\varepsilon_j)] \right) \\
&= \sum_{\substack{J \subset [1, \overline{k+1}] \\ |J| = l}} \left(\prod_{1 \le j \le \overline{1}} \zeta_J(j) \right) \left(\prod_{\substack{1 \le j \le n \\ j \in J}} \frac{1}{1-Q_j} \right) \left(\prod_{\substack{1 \le j \le n \\ \overline{j} \in J}} \frac{1}{1-Q_{j-1}} \right) \left(\prod_{j \in J}^{\star} [\mathcal{O}_{G/B}(-\varepsilon_j)] \right).
\end{aligned}$$

Let us take and fix $J \subset [1, \overline{1}]$. For $1 \le j \le n$, we set

$$\eta_J(j) := \begin{cases} \dfrac{1}{1-Q_j} & \text{if } j \in J, \\ 1 & \text{otherwise.} \end{cases}$$

Then it follows that

$$\begin{aligned}
\zeta_J(j)\eta_J(j) &= \begin{cases} 1 \cdot \dfrac{1}{1-Q_j} & \text{if } j \in J \text{ and } j+1 \in J, \\ (1-Q_j) \cdot \dfrac{1}{1-Q_j} & \text{if } j \in J \text{ and } j+1 \notin J, \\ 1 \cdot 1 & \text{if } j \notin J \end{cases} \\
&= \begin{cases} \dfrac{1}{1-Q_j} & \text{if } j \in J \text{ and } j+1 \in J, \\ 1 & \text{othewise} \end{cases} \\
&= \varphi_J^Q(j).
\end{aligned}$$

Also, for $2 \le j \le n$, we set

$$\eta_J(\overline{j}) := \begin{cases} \dfrac{1}{1-Q_{j-1}} & \text{if } \overline{j} \in J, \\ 1 & \text{otherwise.} \end{cases}$$

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Then it follows that

$$\zeta_J(\overline{j})\eta_J(\overline{j}) = \begin{cases} \left(1 + \dfrac{Q_{j-1}Q_j\cdots Q_n}{1-Q_{j-1}}\right)\cdot 1 & \text{if } J = \{\cdots < j-1 < \overline{j-1} < \cdots\}, \\ 1\cdot 1 & \text{if } \overline{j}\notin J \text{ and } J \text{ is not of the above form}, \\ 1\cdot \dfrac{1}{1-Q_{j-1}} & \text{if } \overline{j}\in J \text{ and } \overline{j-1}\in J, \\ (1-Q_{j-1})\cdot \dfrac{1}{1-Q_{j-1}} & \text{if } \overline{j}\in J \text{ and } \overline{j-1}\notin J \end{cases}$$
$$= \begin{cases} 1 + \dfrac{Q_{j-1}Q_j\cdots Q_n}{1-Q_{j-1}} & \text{if } J = \{\cdots < j-1 < \overline{j-1} < \cdots\}, \\ \dfrac{1}{1-Q_{j-1}} & \text{if } \overline{j}\in J \text{ and } \overline{j-1}\in J, \\ 1 & \text{otherwise} \end{cases}$$
$$= \varphi_J^Q(\overline{j}).$$

From these, we conclude that

$$\begin{aligned} \widehat{\Psi}^Q(F_l^{\overline{k}}) &= \sum_{\substack{J\subset[1,\overline{k+1}]\\ |J|=l}} \left(\prod_{1\le j\le\overline{1}} \zeta_J(j)\eta_J(j)\right)\left(\prod_{j\in J}^{\star}[\mathcal{O}_{G/B}(-\varepsilon_j)]\right) \\ &= \sum_{\substack{J\subset[1,\overline{k+1}]\\ |J|=l}} \left(\prod_{1\le j\le\overline{1}} \varphi_J^Q(j)\right)\left(\prod_{j\in J}^{\star}[\mathcal{O}_{G/B}(-\varepsilon_j)]\right) \\ &= \mathcal{F}_l^{\overline{k}}, \end{aligned}$$

as desired. This proves the lemma. □

By Corollary 6.4, we obtain the following.

Corollary 6.11. *For $0 \le j \le 2n$, we have $\widehat{\Psi}^Q(F_l - E_l) = 0$.*

Hence, the map $\widehat{\Psi}^Q$ induces the $R(T)[[Q]]$-algebra homomorphism (not yet proved to be an isomorphism) Ψ^Q, given by (3.3).

Also, by combining Lemma 6.10 with Corollary 6.5, we obtain an explicit expression, in terms of line bundle classes, of the Schubert classes $[\mathcal{O}^{s_1\cdots s_k}]$ and $[\mathcal{O}^{s_1\cdots s_n s_{n-1}\cdots s_k}]$ for $1 \le k \le n$.

Corollary 6.12. (1) *For $1 \le k \le n$, we have*

$$[\mathcal{O}^{s_1\cdots s_k}] = \widehat{\Psi}^Q\left(\sum_{l=0}^{k}(-1)^l \mathbf{e}^{-l\varepsilon_1}F_l^k\right).$$

(2) *For $1 \le k \le n$, we have*

$$[\mathcal{O}^{s_1\cdots s_n s_{n-1}\cdots s_k}] = \widehat{\Psi}^Q\left(\sum_{l=0}^{2n-k}(-1)^l \mathbf{e}^{-l\varepsilon_1}F_l^{\overline{k}}\right).$$

6.3. Finishing the proof of Theorem 3.6

Let us complete the proof of Theorem 3.6. It remains to prove that the $R(T)[[Q]]$-algebra homomorphism Ψ^Q, given by (3.3), is an isomorphism. For this, we make use of the following lemma, which follows from Nakayama's lemma.

Lemma 6.13 [6, Proposition A.3]. *Let R be a Noetherian domain, $I \subset R$ an ideal. Assume that R is I-adic complete. Let M and N be finitely generated R-modules. In addition, assume that the R-module N and the (R/I)-module N/IN are free of the same finite rank. Then, for a homomorphism $f : M \to N$ of R-modules, if the induced homomorphism $\overline{f} : M/IM \to N/IN$ of (R/I)-modules is an isomorphism, then f is also an isomorphism of R-modules.*

We apply this lemma to the case

$$R = R(T)[[Q]], \quad I = (Q_1, \dots, Q_n),$$
$$M = (R(T)[[Q]])[z_1^{\pm 1}, \dots, z_n^{\pm 1}]/\mathcal{I}^Q, \quad N = QK_T(G/B), \quad \text{and} \quad f = \Psi^Q;$$

note that R is I-adic complete. Recall that N is a free R-module of rank $|W|(< \infty)$ since it admits the Schubert basis $\{[\mathcal{O}_{G/B}^w] \mid w \in W\}$. In addition, we see that $N/IN \simeq K_T(G/B)$ is also a free $R/I \simeq R(T)$-module of rank $|W|$ since it also admits the Schubert basis. Hence, it remains to verify the following:

(i) M is a finitely generated R-module.
(ii) The induced $R(T)$-algebra homomorphism $\overline{\Psi^Q} : M/IM \to N/IN$ is an isomorphism.

Let $\mathcal{I}$ be the ideal of $R(T)[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ generated by:

$$e_l(z_1, \dots, z_n, z_n^{-1}, \dots, z_1^{-1}) - E_l \quad \text{for } 1 \le l \le n.$$

Then, by Remark 3.4, we see that

$$M/IM \simeq R(T)[z_1^{\pm 1}, \dots, z_n^{\pm 1}]/\mathcal{I}. \tag{6.1}$$

It is well-known (see, for example, [24]) that there exists an isomorphism

$$\begin{array}{rcl} \Psi : R(T)[z_1^{\pm 1}, \dots, z_n^{\pm 1}]/\mathcal{I} & \xrightarrow{\sim} & K_T(G/B) \\ z_j + \mathcal{I} & \mapsto & [\mathcal{O}_{G/B}(-\varepsilon_j)], \quad 1 \le j \le n, \end{array} \tag{6.2}$$

of $R(T)$-algebras; this is just the classical *Borel presentation*. Now part (2) follows from (6.2). For part (1), we can apply the following lemma.

Lemma 6.14 [8, Proposition A.5 (1)]. *Let A be a Noetherian ring. Let $R := A[[Q_1, \dots, Q_n]]$, $I := (Q_1, \dots, Q_n) \subset R$, and M an R-module. If M/IM is a finitely generated (R/I)-module, then M is a finitely generated R-module.*

Proposition 6.15. *$(R(T)[[Q]])[z_1^{\pm 1}, \dots, z_n^{\pm 1}]/\mathcal{I}^Q$ is a finitely generated $R(T)[[Q]]$-module.*

Proof. We set $M := (R(T)[[Q]])[z_1^{\pm 1}, \dots, z_n^{\pm 1}]/\mathcal{I}^Q$. By (6.1) and (6.2), M/IM is a free $R(T)$-module of rank $|W|(< \infty)$; in particular, M/IM is a finitely generated $R(T)$-module. Therefore, by applying Lemma 6.14 to the case that $A = R(T)$, we conclude that $M = (R(T)[[Q]])[z_1^{\pm 1}, \dots, z_n^{\pm 1}]/\mathcal{I}^Q$ is a finitely generated $R = R(T)[[Q]]$-module. This proves the proposition. □

Thus, by Lemma 6.13, the proof of Theorem 3.6 is completed.

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Appendices

A. Proofs of Propositions 4.3 and 4.4

In this appendix, we prove Propositions 4.3 and 4.4.

First, let us recall the definition of the quantum Bruhat graph.

Definition A.1 [2, Definition 6.1]. Let W be the Weyl group of G (of an arbitrary type). The *quantum Bruhat graph* $\mathrm{QBG}(W)$ on W is the Δ^+-labeled directed graph with vertex set W and directed edges $x \xrightarrow{\alpha} xs_\alpha$ for $x \in W$ and $\alpha \in \Delta^+$ such that either of the following holds:

(B) $\ell(y) = \ell(x) + 1$, or
(Q) $\ell(y) = \ell(x) - 2\langle \rho, \alpha^\vee \rangle + 1$,

where $\rho := (1/2) \sum_{\alpha \in \Delta^+} \alpha$. If (B) (resp., (Q)) holds, then the edge $x \xrightarrow{\alpha} xs_\alpha$ is called a *Bruhat* (resp., *quantum*) *edge*.

For G of type C, that is, for $G = \mathrm{Sp}_{2n}(\mathbb{C})$, we know the following useful criterion for the directed edges in $\mathrm{QBG}(W)$; for $1 \le j \le n$, we set $\mathrm{sign}(j) := 1$ and $\mathrm{sign}(\overline{j}) := -1$.

Proposition A.2 (cf. [16, Proposition 5.7]). *Let $w \in W$.*

(1) *For $1 \le i < j \le n$, we have the Bruhat edge $w \xrightarrow{(i,j)} ws_{(i,j)}$ in $\mathrm{QBG}(W)$ if and only if $w(i) < w(j)$ and there does not exist any $i < k < j$ such that $w(i) < w(k) < w(j)$.*
(2) *For $1 \le i < j \le n$, we have the quantum edge $w \xrightarrow{(i,j)} ws_{(i,j)}$ in $\mathrm{QBG}(W)$ if and only if $w(i) > w(j)$ and all $i < k < j$ satisfy $w(i) > w(k) > w(j)$.*
(3) *For $1 \le i < j \le n$, we have the edge $w \xrightarrow{(i,\overline{j})} ws_{(i,\overline{j})}$ in $\mathrm{QBG}(W)$ if and only if $w(i) < w(\overline{j})$, $\mathrm{sign}(w(i)) = \mathrm{sign}(w(\overline{j}))$, and there does not exist any $i < k < \overline{j}$ such that $w(i) < w(k) < w(\overline{j})$. In this case, the edge $w \xrightarrow{(i,\overline{j})} ws_{(i,\overline{j})}$ is a Bruhat edge.*
(4) *For $1 \le i \le n$, we have the Bruhat edge $w \xrightarrow{(i,\overline{i})} ws_{(i,\overline{i})}$ in $\mathrm{QBG}(W)$ if and only if $w(i) < w(\overline{i})$ and there does not exist any $i < k < \overline{i}$ such that $w(i) < w(k) < w(\overline{i})$.*
(5) *For $1 \le l \le n$, we have the quantum edge $w \xrightarrow{(i,\overline{i})} ws_{(i,\overline{i})}$ in $\mathrm{QBG}(W)$ if and only if $w(i) > w(\overline{i})$ and all $i < k < \overline{i}$ satisfy $w(i) > w(k) > w(\overline{i})$.*

Let G be of an arbitrary type. Let us briefly review the ("generalized") quantum alcove model, introduced in [17] (see also [20]).

We set

$$|\alpha| := \begin{cases} \alpha & \text{if } \alpha \in \Delta^+, \\ -\alpha & \text{if } \alpha \in -\Delta^+. \end{cases}$$

Definition A.3 [20, Definition 17]. Let $\Gamma = (\gamma_1, \ldots, \gamma_r)$ be a sequence of roots, and $w \in W$. A subset $A = \{i_1, \ldots, i_s\} \subset \{1, \ldots, r\}$ is said to be *w-admissible* if the sequence

$$\Pi(w, A) : w = w_0 \xrightarrow{|\gamma_{i_1}|} w_1 \xrightarrow{|\gamma_{i_2}|} \cdots \xrightarrow{|\gamma_{i_s}|} w_s$$

is a directed path in $\mathrm{QBG}(W)$; we set

$$\mathrm{end}(A) := w_s,$$
$$\mathrm{down}(A) := \sum_{\substack{1 \le j \le s \\ w_{j-1} \to w_j \text{ is a quantum edge}}} |\gamma_{i_j}|^\vee.$$

Let $\mathcal{A}(w, \Gamma)$ denote the set of w-admissible subsets associated to Γ.

We regard a w-admissible subset $A \in \mathcal{A}(w,\Gamma)$ as a subset of Γ, and write it as $A = \{\gamma_{i_1}, \dots, \gamma_{i_s}\}$. Also, for a tuple $(A_1, \dots, A_r)$ of admissible subsets $A_j \in \mathcal{A}(w_j, \Gamma_j)$, $1 \le j \le r$, with $w_j \in W$ and Γ_j a sequence of roots, we set

$$\mathrm{down}(A_1, \dots, A_r) := \mathrm{down}(A_1) + \cdots + \mathrm{down}(A_r).$$

In the following, we assume that $G = \mathrm{Sp}_{2n}(\mathbb{C})$. We use admissible subsets associated to the following two types of sequences of roots for $1 \le k \le n$:

$$\begin{aligned}\Theta_k &:= (-(1,k), \dots, -(k-1,k));\\ \Gamma_k(k) &:= (-(1,\overline{k}), \dots, -(k-1,\overline{k}),\\ &\qquad -(k,\overline{k+1}), \dots, -(k,\overline{n}),\\ &\qquad -(k,\overline{k}),\\ &\qquad -(k,n), \dots -(k,k+1)).\end{aligned}$$

Let us briefly recall from [13, §4] the “second half” of the inverse Chevalley formula in type C. Following [13, §4.1], for $j, m \in [1, \overline{1}]$ with $j < m$, we set

$$S_{m,j} := \{(j_1, \dots, j_r) \mid r \ge 1,\ j_1, \dots, j_r \in [1, \overline{1}],\ m > j_1 > \cdots > j_r = j\}.$$

Also, for $w \in W$ and $1 \le k \le l \le n$, we set

$$\mathcal{A}_w^{k,l} := \{A \in \mathcal{A}(w, \Theta_k) \setminus \{\emptyset\} \mid \mathrm{end}(A)^{-1} w \varepsilon_k = \varepsilon_l\},$$

while for $w \in W$, and $1 \le k \le n$, $1 \le l \le \overline{k}$, we set

$$\mathcal{A}_w^{\overline{k},l} := \{A \in \mathcal{A}(w, \Gamma_k(k)) \setminus \{\emptyset\} \mid \mathrm{end}(A)^{-1} w(-\varepsilon_k) = \varepsilon_l\}.$$

Theorem A.4 [13, Theorem 4.3] combined with [10, Theorem 5.8]. *For $w \in W$ and $m = 1, \dots, n$, the following identity holds in $K_{T \times \mathbb{C}^*}(\mathbf{Q}_G)$:*

$$\begin{aligned}
&\mathbf{e}^{-w\varepsilon_m}[\mathcal{O}_{\mathbf{Q}_G(w)}]\\
&= \sum_{B \in \mathcal{A}(w,\Theta_m)} (-1)^{|B|} \Big[\mathcal{O}_{\mathbf{Q}_G(\mathrm{end}(B) t_{\mathrm{down}(B)})}(-\varepsilon_m)\Big]\\
&\quad + \sum_{j=m+1}^{n} \sum_{(j_1,\dots,j_r) \in S_{\overline{m},\overline{j}}} \sum_{A_1 \in \mathcal{A}_w^{\overline{m},j_1}} \cdots \sum_{A_r \in \mathcal{A}_{\mathrm{end}(A_{r-1})}^{j_{r-1},j_r}} (-1)^{|A_1|+\cdots+|A_r|-r} q^{-\langle \varepsilon_j, \mathrm{down}(A_1,\dots,A_r)\rangle}\\
&\qquad\qquad \times \sum_{B \in \mathcal{A}(\mathrm{end}(A_r),\Theta_j)} (-1)^{|B|} \Big[\mathcal{O}_{\mathbf{Q}_G(\mathrm{end}(B) t_{\mathrm{down}(A_1,\dots,A_r,B)})}(-\varepsilon_j)\Big]\\
&\quad + \sum_{j=1}^{n} \sum_{(j_1,\dots,j_r) \in S_{\overline{m},j}} \sum_{A_1 \in \mathcal{A}_w^{\overline{m},j_1}} \cdots \sum_{A_r \in \mathcal{A}_{\mathrm{end}(A_{r-1})}^{j_{r-1},j_r}} (-1)^{|A_1|+\cdots+|A_r|-r} q^{\langle \varepsilon_j, \mathrm{down}(A_1,\dots,A_r)\rangle}\\
&\qquad\qquad \times \sum_{B \in \mathcal{A}(\mathrm{end}(A_r),\Gamma_j(j))} (-1)^{|B|} \Big[\mathcal{O}_{\mathbf{Q}_G(\mathrm{end}(B) t_{\mathrm{down}(A_1,\dots,A_r,B)})}(\varepsilon_j)\Big].
\end{aligned} \tag{A.1}$$

We will prove Proposition 4.4 by using Theorem A.4; since the proof of Proposition 4.3 is similar (use [13, Theorem 4.5] instead of [13, Theorem 4.3]) and easier, we leave it to the reader (cf. the proof of [21, Proposition 4.5]).

Proof of Proposition 4.4. We put $w = s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_k$ and $m = k$ in Theorem A.4. Note that $w = [2, 3, \ldots, k, \overline{1}, k+1, \ldots, n]$ in "window notation." Then, by Proposition A.2, we see that

$$\mathcal{A}(s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_k, \Theta_k) = \{\emptyset, \{-(k-1, k)\}\}. \tag{A.2}$$

Hence, the first sum on the RHS of (A.1) is

$$\begin{aligned} &\sum_{B \in \mathcal{A}(s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_k, \Theta_k)} (-1)^{|B|} \Big[\mathcal{O}_{\mathbf{Q}_G(\mathrm{end}(B) t_{\mathrm{down}(B)})}(-\varepsilon_k) \Big] \\ &= \underbrace{[\mathcal{O}_{\mathbf{Q}_G(s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_k)}(-\varepsilon_k)]}_{B=\emptyset} - \underbrace{[\mathcal{O}_{\mathbf{Q}_G(s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_{k-1})}(-\varepsilon_k)]}_{B=\{-(k-1,k)\}}. \end{aligned} \tag{A.3}$$

In addition, we see that

$$\mathcal{A}(s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_k, \Gamma_k(k)) = \{\emptyset, \{-(k, k+1)\}, \{-(k, \overline{k})\}, \{-(k, \overline{k}), -(k, k+1)\}\}, \tag{A.4}$$

with

$$\begin{aligned} \mathrm{end}(\{-(k, k+1)\}) &= s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_{k+1}, \\ \mathrm{end}(\{-(k, \overline{k})\}) &= s_1 \cdots s_{k-1}, \\ \mathrm{end}(\{-(k, \overline{k}), -(k, k+1)\}) &= s_1 \cdots s_k, \end{aligned}$$

and

$$\begin{aligned} \mathrm{down}(\{-(k, k+1)\}) &= \alpha_k^\vee, \\ \mathrm{down}(\{-(k, \overline{k})\}) &= \alpha_k^\vee + \cdots + \alpha_n^\vee, \\ \mathrm{down}(\{-(k, \overline{k}), -(k, k+1)\}) &= \alpha_k^\vee + \cdots + \alpha_n^\vee. \end{aligned}$$

From these, we deduce that the remaining terms in the second sum on the RHS of (A.1) correspond to the tuples $(j, (j_1, \ldots, j_r), A_1, \ldots, A_r)$ such that

- $k+1 \le j \le n$,
- $(j_1, \ldots, j_r) = (\overline{k}, \overline{k+1}, \ldots, \overline{j})$, and
- $(A_1, \ldots, A_r) = (\{-(k, k+1)\}, \{-(k+1, k+2)\}, \ldots, \{-(j-1, j)\})$, with

$$\begin{aligned} \mathrm{end}(A_r) &= s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_j, \\ \mathrm{down}(A_1, \ldots, A_r) &= \alpha_k^\vee + \cdots + \alpha_{j-1}^\vee \\ |A_1| + \cdots + |A_r| &= r = j - k. \end{aligned}$$

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Therefore, the second sum on the RHS of (A.1) is

$$\sum_{j=k+1}^{n} q^{-\langle \varepsilon_j, \alpha_k^\vee + \cdots + \alpha_{j-1}^\vee \rangle} \sum_{B \in \mathcal{A}(s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_j, \Theta_j)} (-1)^{|B|} \left[\mathcal{O}_{\mathbf{Q}_G\left(\mathrm{end}(B) t_{\alpha_k^\vee + \cdots + \alpha_{j-1}^\vee}\right)}(-\varepsilon_j) \right]$$

$$= q \sum_{j=k+1}^{n} \left(\underbrace{\left[\mathcal{O}_{\mathbf{Q}_G\left(s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_j t_{\alpha_k^\vee + \cdots + \alpha_{j-1}^\vee}\right)}(-\varepsilon_j) \right]}_{B=\emptyset} - \underbrace{\left[\mathcal{O}_{\mathbf{Q}_G\left(s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_{j-1} t_{\alpha_k^\vee + \cdots + \alpha_{j-1}^\vee}\right)}(-\varepsilon_j) \right]}_{B=\{-(j-1,j)\}} \right), \tag{A.5}$$

where we have used (A.2) for the above equality.

Also, for $1 \le i \le n$, we see that

$$\mathcal{A}(s_1 \cdots s_i, \Theta_i) = \{\emptyset, \{-(i, i+1)\}\},$$

with

$$\mathrm{end}(\{-(i,i+1)\}) = s_1 \cdots s_{i-1}, \quad \mathrm{down}(\{-(i,i+1)\}) = \alpha_i^\vee.$$

By combining this with (A.4), we deduce that the remaining terms in the third sum on the RHS of (A.1) correspond to the tuples $(j, (j_1, \dots, j_r), A_1, \dots, A_r)$ such that

- ◦ $j = 1, \dots, n$,
- ◦ $(j_1, \dots, j_r) = (\overline{k}, \overline{k+1}, \dots, \overline{l}, l, l-1, \dots, j)$ for some $k \le l \le n$ with $j \le l$, or $(j_1, \dots, j_r) = (\overline{k}, \overline{k+1}, \dots, \overline{l-1}, l, l-1, \dots, j)$ for some $k < l \le n$ with $j \le l$,
- ◦ if $(j_1, \dots, j_r) = (\overline{k}, \overline{k+1}, \dots, \overline{l}, l, l-1, \dots, j)$, then

$$(A_1, \dots, A_r) = (\{-(k,k+1)\}, \dots, \{-(l-1,l)\}, \{-(l,\overline{l})\}, \{-(l-1,l)\}, \dots, \{-(j,j+1)\}),$$

  with

$$\begin{aligned} \mathrm{end}(A_r) &= s_1 \cdots s_{j-1}, \\ \mathrm{down}(A_1, \dots, A_r) &= (\alpha_k^\vee + \cdots \alpha_{l-1}^\vee) + (\alpha_l^\vee + \cdots + \alpha_n^\vee) + (\alpha_{l-1}^\vee + \cdots + \alpha_j^\vee), \\ |A_1| + \cdots + |A_r| &= r = (l-k) + 1 + (l-j) = 2l - k - j + 1, \end{aligned}$$

- ◦ if $(j_1, \dots, j_r) = (\overline{k}, \overline{k+1}, \dots, \overline{l-1}, l, l-1, \dots, j)$, then

$$(A_1, \dots, A_r) = (\{-(k,k+1)\}, \dots, \{-(l-2,l-1)\}, \{-(l-1,\overline{l-1}), -(l-1,l)\}, \{-(l-1,l)\}, \dots, \{-(j,j+1)\}),$$

  with

$$\begin{aligned} \mathrm{end}(A_r) &= s_1 \cdots s_{j-1}, \\ \mathrm{down}(A_1, \dots, A_r) &= (\alpha_k^\vee + \cdots \alpha_{l-2}^\vee) + (\alpha_{l-1}^\vee + \cdots + \alpha_n^\vee) + (\alpha_{l-1}^\vee + \cdots + \alpha_j^\vee), \\ |A_1| + \cdots + |A_r| &= r + 1 = (l-k-1) + 2 + (l-j+1) = 2l - k - j + 2. \end{aligned}$$

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Here, observe that for fixed $1 \le j \le n$ and $k < l \le n$ with $j \le l$, the term in the third sum on the RHS of (A.1) corresponding to $(j_1, \dots, j_r) = (\overline{k}, \overline{k+1}, \dots, \overline{l}, l, l-1, \dots, j)$ and that corresponding to $(j_1, \dots, j_r) = (\overline{k}, \overline{k+1}, \dots, \overline{l-1}, l, l-1, \dots, j)$ cancel out. Hence, only the terms corresponding to $(j_1, \dots, j_r) = (\overline{k}, k, k-1, \dots, j)$, with $j \le k$, remain uncancelled. Therefore, the third sum on the RHS of (A.1) is

$$\sum_{j=1}^{k} q^{\langle \varepsilon_j, \alpha_j^\vee + \cdots + \alpha_n^\vee \rangle} \sum_{B \in \mathcal{A}(s_1 \cdots s_{j-1}, \Gamma_j(j))} (-1)^{|B|} \left[\mathcal{O}_{\mathbf{Q}_G\left(\mathrm{end}(B) t_{\mathrm{down}(B) + \alpha_j^\vee + \cdots + \alpha_n^\vee}\right)}(\varepsilon_j) \right]$$

$$= q \sum_{j=1}^{k} \left(\underbrace{\left[\mathcal{O}_{\mathbf{Q}_G\left(s_1 \cdots s_{j-1} t_{\alpha_j^\vee + \cdots + \alpha_n^\vee}\right)}(\varepsilon_j) \right]}_{B = \emptyset} - \underbrace{\left[\mathcal{O}_{\mathbf{Q}_G\left(s_1 \cdots s_j t_{\alpha_j^\vee + \cdots + \alpha_n^\vee}\right)}(\varepsilon_j) \right]}_{B = \{-(j, j+1)\}} \right); \tag{A.6}$$

for this equality, we have used the fact that

$$\mathcal{A}(s_1, \dots, s_{j-1}, \Gamma_j(j)) = \{\emptyset, \{-(j, j+1)\}\},$$

which can be shown by using Proposition A.2.

By combining (A.3), (A.5), and (A.6), the proof of the theorem is completed. □

B. Proof of Proposition 5.7

In this appendix, we give a proof of Proposition 5.7.

Definition B.1. For $A, B \subset [1, n]$ such that $A \cap B = \emptyset$, and for $0 \le k \le 2n$, we set

$$\mathcal{J}_{A,B}^k := \{ J \subset [1, \overline{1}] \mid \varepsilon_J = \varepsilon_A - \varepsilon_B \text{ and } |J| = k \}.$$

We have

$$\mathfrak{F}_k = \sum_{\substack{A, B \subset [1,n] \\ A \cap B = \emptyset \\ k - (|A| + |B|) \in 2\mathbb{Z}_{\ge 0}}} \left(\sum_{J \in \mathcal{J}_{A,B}^l} \left(\prod_{1 \le j \le \overline{1}} \psi_J(j) \right) \right) [\mathcal{O}_{\mathbf{Q}_G}(-\varepsilon_A + \varepsilon_B)].$$

If $k < n$, then we can construct a bijection $\mathcal{J}_{A,B}^k \to \mathcal{J}_{A,B}^{2n-k}$ as follows. Let $A, B \subset [1, n]$ be such that $A \cap B = \emptyset$. We set $s := |A|$, $t := |B|$, and write A, B as:

$$A = \{i_1 < \cdots < i_s\}, \quad B = \{j_1 < \cdots < j_t\}.$$

We consider only the case $\mathcal{J}_{A,B}^k \neq \emptyset$. Let $k < n$ be such that $s + t \le k$ and $k - (s + t) \in 2\mathbb{Z}$; we set $r := (k - (s + t))/2$. If $J \in \mathcal{J}_{A,B}^k$, then there exist $1 \le k_1 < \cdots < k_r \le n$ such that

$$J = \{i_1, \dots, i_s\} \sqcup \{\overline{j_1}, \dots, \overline{j_t}\} \sqcup \{k_1, \dots, k_r\} \sqcup \{\overline{k_1}, \dots, \overline{k_r}\}. \tag{B.1}$$

Let us take $m_1, \dots, m_u \in [1, n]$ such that

$$\{m_1 < \cdots < m_u\} = [1, n] \setminus (A \sqcup B \sqcup \{k_1, \dots, k_r\}),$$

and define $J^* \subset [1, \overline{1}]$ by

$$J^* := \{i_1, \dots, i_s\} \sqcup \{\overline{j_1}, \dots, \overline{j_t}\} \sqcup \{m_1, \dots, m_u\} \sqcup \{\overline{m_1}, \dots, \overline{m_u}\}.$$

It is easy to check that $J^* \in \mathcal{J}^{2n-k}_{A,B}$.

Example B.2. Let $n = 7$ and $J = \{2, 4, 5, \overline{6}, \overline{5}, \overline{2}\}$. We see that $J \in \mathcal{J}^6_{\{4\},\{6\}}$. By decomposing J as:

$$J = \{4\} \sqcup \{\overline{6}\} \sqcup \{2, 5\} \sqcup \{\overline{2}, \overline{5}\},$$

we have

$$J^* = \{4\} \sqcup \{\overline{6}\} \sqcup \{1, 3, 7\} \sqcup \{\overline{1}, \overline{3}, \overline{7}\} \in \mathcal{J}^8_{\{4\},\{6\}}.$$

We can easily verify the following lemma.

Lemma B.3. *The assignment $J \mapsto J^*$ for $J \in \mathcal{J}^k_{A,B}$ gives a bijection $\mathcal{J}^k_{A,B} \xrightarrow{\sim} \mathcal{J}^{2n-k}_{A,B}$.*

To prove Proposition 5.7, we need the following lemma.

Lemma B.4. *Let $J \subset [1, \overline{1}]$ be decomposed as in (B.1). Assume that $k_r < i_s$, $k_r < j_t$, or $\{\max\{i_s, j_t\}+1, \dots, n\} \subset J$. Then, we have*

$$\prod_{1 \le j \le \overline{1}} \psi_J(j) = \prod_{\substack{1 \le j \le n \\ j \notin J,\ j+1 \in J}} (1 - \mathsf{t}_j) \prod_{\substack{2 \le j \le n \\ \overline{j} \notin J,\ \overline{j-1} \in J}} (1 - \mathsf{t}_{j-1}) = \prod_{1 \le j \le \overline{1}} \psi_{J^*}(j),$$

where we understand that $n + 1 := \overline{n}$.

Proof. The first equality follows from the definition of ψ_J. We show the second equality. For $1 \le j \le n - 1$, we see that $j \notin J^*$ is equivalent to $\overline{j} \in J$, while $j + 1 \in J^*$ is equivalent to $\overline{j + 1} \notin J$. Also, the condition that $n \notin J^*$ and $\overline{n} \in J^*$ is equivalent to the condition that $n \notin J$ and $\overline{n} \in J$. In addition, for $2 \le j \le n$, we see that $\overline{j} \notin J^*$ is equivalent to $j \in J$, while $\overline{j - 1} \in J^*$ is equivalent to $j - 1 \notin J$. This shows the second equality. This proves the lemma. □

Sketch of the proof of Proposition 5.7. Take $A, B \subset [1, n]$ such that $A \cap B = \emptyset$. We set $s := |A|$, $t := |B|$, and assume that $s + t \le k$, $k - (s + t) \in 2\mathbb{Z}$ (so that $\mathcal{J}^k_{A,B} \ne \emptyset$). We write A and B as:

$$A = \{i_1 < \cdots < i_s\}, \quad B = \{j_1 < \cdots < j_t\},$$

and set $M := \max\{i_s, j_t\}$ (if $A = B = \emptyset$, then we set $M := 0$). It suffices to prove that

$$\sum_{J \in \mathcal{J}^k_{A,B}} \left(\prod_{1 \le j \le \overline{1}} \psi_J(j) \right) = \sum_{K \in \mathcal{J}^{2n-k}_{A,B}} \left(\prod_{1 \le j \le \overline{1}} \psi_K(j) \right). \tag{B.2}$$

For $p \ge 0$, we set $\mathcal{J}^k_{A,B}(p)$

$$\mathcal{J}^k_{A,B}(p) := \{J \in \mathcal{J}^k_{A,B} \mid |J \cap \{M + 1, \dots, n\}| = p\}.$$

If we can show that for $0 \le p \le n - M$,

$$\sum_{J \in \mathcal{J}^k_{A,B}(p)} \left(\prod_{1 \le j \le \overline{1}} \psi_J(j) \right) = \sum_{J \in \mathcal{J}^k_{A,B}(p)} \left(\prod_{1 \le j \le \overline{1}} \psi_{J^*}(j) \right), \tag{B.3}$$

then (B.2) follows.

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If $J \in \mathcal{J}^k_{A,B}$ is decomposed as in (B.1) in such a way that $k_r < i_s$, $k_r < j_t$, or $\{M+1,\ldots,n\} \subset J$, then it follows from Lemma B.4 that

$$\prod_{1 \le j \le \overline{1}} \psi_J(j) = \prod_{1 \le j \le \overline{1}} \psi_{J^*}(j).$$

This shows (B.3) for $p = 0, n - M$. Let us consider the case that $1 \le p \le n - M - 1$. We see that

$$\sum_{J \in \mathcal{J}^k_{A,B}(p)} \left(\prod_{1 \le j \le \overline{1}} \psi_J(j) \right) = \sum_{K \in \mathcal{J}^{k-2p}_{A,B}(0)} \sum_{M < k_1 < \cdots < k_p \le n} \left(\prod_{1 \le j \le \overline{1}} \psi_{K \sqcup \{k_1,\ldots,k_p,\overline{k_1},\ldots,\overline{k_p}\}}(j) \right),$$

$$\sum_{J \in \mathcal{J}^k_{A,B}(p)} \left(\prod_{1 \le j \le \overline{1}} \psi_{J^*}(j) \right) = \sum_{K \in \mathcal{J}^{k-2p}_{A,B}(0)} \sum_{M < k_1 < \cdots < k_p \le n} \left(\prod_{1 \le j \le \overline{1}} \psi_{(K \sqcup \{k_1,\ldots,k_p,\overline{k_1},\ldots,\overline{k_p}\})^*}(j) \right).$$

For $1 \le p \le n - M - 1$ and $K \in \mathcal{J}^{k-2p}_{A,B}(0)$, we set

$$S(K,p) := \sum_{M < k_1 < \cdots < k_p \le n} \left(\prod_{1 \le j \le \overline{1}} \psi_{K \sqcup \{k_1,\ldots,k_p,\overline{k_1},\ldots,\overline{k_p}\}}(j) \right),$$

$$T(K,p) := \sum_{M < k_1 < \cdots < k_p \le n} \left(\prod_{1 \le j \le \overline{1}} \psi_{(K \sqcup \{k_1,\ldots,k_p,\overline{k_1},\ldots,\overline{k_p}\})^*}(j) \right).$$

We can show that

$$S(K,p) = T(K,p) \tag{B.4}$$

for all $A, B \subset [1,n]$ such that $A \cap B = \emptyset$, $1 \le p \le n - M - 1$, and $K \in \mathcal{J}^{k-2p}_{A,B}(0)$ in the following steps (the details are left to the reader):

Step 1. Show (B.4) for $A, B \subset [1,n]$ such that $A \cap B = \emptyset$, and $K \in \mathcal{J}^{k-2p}_{A,B}(0)$ with $M \in K$ by induction on $p \ge 1$.
Step 2. Show (B.4) for $A, B \subset [1,n]$ such that $A \cap B = \emptyset$, and $K \in \mathcal{J}^{k-2p}_{A,B}(0)$ with $\overline{M} \in K$ by induction on $p \ge 1$.
Step 3. Show (B.4) for $K = \emptyset$ by induction on $p \ge 1$.

This completes the sketch of the proof of the proposition.

C. Inverse of the line bundle class associated to a fundamental weight

Let $G = \mathrm{Sp}_{2n}(\mathbb{C})$. We can compute the inverse $[\mathcal{O}_{G/B}(-\varpi_j)]^{-1}$ of the line bundle class $[\mathcal{O}_{G/B}(-\varpi_j)]$ for $1 \le j \le n$ with respect to the quantum product $\star$ in $QK_T(G/B)$ by an argument smilar to that for Lemma 6.7.

Proposition C.1. *For $1 \le j \le n$, the line bundle class $[\mathcal{O}_{G/B}(-\varpi_j)] \in QK_T(G/B)$ is invertible with respect to the quantum product $\star$ in $QK_T(G/B)$, and its inverse is given as:*

$$[\mathcal{O}_{G/B}(-\varpi_j)]^{-1} = \frac{1}{1 - Q_j} [\mathcal{O}_{G/B}(\varpi_j)].$$

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Proof. Recall the isomorphism $\Phi : QK_T(G/B) \xrightarrow{\sim} K_T(\mathbf{Q}_G)$ of $R(T)$-modules, given in Theorem 4.1, such that for $1 \le k \le n$ and $\mathcal{Z} \in QK_T(G/B)$,

$$\Phi([\mathcal{O}_{G/B}(-\varpi_k)] \star \mathcal{Z}) = [\mathcal{O}_{\mathbf{Q}_G}(-\varpi_k)] \otimes \Phi(\mathcal{Z}).$$

Let $1 \le j \le n$. By [21, Proposition 5.3], we see that for $\mathcal{Z} \in QK_T(G/B)$,

$$\Phi\left(\mathcal{Z} \star \left(\frac{1}{1-Q_j}[\mathcal{O}_{G/B}(\varpi_j)]\right)\right) = \Phi(\mathcal{Z}) \otimes [\mathcal{O}_{\mathbf{Q}_G}(\varpi_j)].$$

Hence, we deduce that

$$\begin{aligned}\Phi\left([\mathcal{O}_{G/B}(-\varpi_j)] \star \left(\frac{1}{1-Q_j}[\mathcal{O}_{G/B}(\varpi_j)]\right)\right) &= [\mathcal{O}_{\mathbf{Q}_G}(-\varpi_j)] \otimes [\mathcal{O}_{\mathbf{Q}_G}(\varpi_j)] \\ &= [\mathcal{O}_{\mathbf{Q}_G}] \\ &= \Phi(1).\end{aligned}$$

Therefore, we conclude that $[\mathcal{O}_{G/B}(-\varpi_j)] \star \left(\frac{1}{1-Q_j}[\mathcal{O}_{G/B}(\varpi_j)]\right) = 1$, as desired. This proves the proposition. □

Recall that $0 = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_n \subset \mathbb{C}^{2n}$ is the tautological filtration by subbundles of the trivial bundle over G/B, with $\mathrm{rk}(\mathcal{S}_j) = j$ for $1 \le j \le n$. Let $\mathbb{C}^{2n} \twoheadrightarrow \mathcal{S}_n^\vee \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{S}_2^\vee \twoheadrightarrow \mathcal{S}_1^\vee \twoheadrightarrow 0$ be the dual of this sequence of vector bundles. Then, we know that $[\det(\mathcal{S}_j)] = [\mathcal{O}_{G/B}(-\varpi_j)]$ and $[\det(\mathcal{S}_j^\vee)] = [\mathcal{O}_{G/B}(\varpi_j)]$ in $K_T(G/B)$ $(\subset QK_T(G/B))$ for $1 \le j \le n$. Hence, we obtain the following corollary.

Corollary C.2. *For* $1 \le j \le n$*, we have* $[\det(\mathcal{S}_j)]^{-1} = \dfrac{1}{1-Q_j}[\det(\mathcal{S}_j^\vee)]$.

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