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The variety of 2-dimensional algebras over an algebraically closed field*

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Abstract. **The work is devoted to the variety of 2-dimensional algebras over an algebraically closed field. Firstly, we classify such algebras modulo isomorphism. Then we describe the degenerations and the closures of certain algebra series in the variety of 2-dimensional algebras. Finally, we apply our results to obtain analogous descriptions for the subvarieties of flexible, and bicommutative algebras. In particular, we describe rigid algebras and irreducible components for these subvarieties.**

1 Introduction

In this paper, an algebra is simply a vector space over a field with a bilinear binary operation that does not have to be associative.[1] Algebras of a fixed dimension form a variety with a natural action of a general linear group. Orbits under this action correspond to isomorphism classes of algebras. There are many classifications up to isomorphism for varieties of algebras of some fixed dimension satisfying some polynomial identities. For example, there exist such classifications of 3-dimensional Novikov algebras [2], 4-dimensional Leibniz algebras [3], 6-dimensional Lie algebras [6] and many others.

In this paper we classify all 2-dimensional algebras over an algebraically closed field up to isomorphism. It is not the first work devoted to this problem; classifications of different types were made in [4, 5, 1], but none of them are convenient for our main goal, the geometric description of the algebraic variety of 2-dimensional algebras. One of the advantages of our paper is that our approach deals uniformly with all possible characteristics while the authors of [1] do not consider the characteristics 2 and 3 and the authors of [4] consider only the two elements field in the characteristic 2. The authors of [1] in fact do not give an explicit classification of 2-dimensional algebras up to isomorphism because they have other purposes. They describe the moduli space by proving that 2-dimensional algebras can be divided into parts that can be naturally included into projective spaces of different dimensions. The authors claim that classification up to isomorphism is easy and could be extracted from their proofs. The classification is really not very difficult and we believe that one can extract it from [1] after reading the paper, taking parts of the classification from different places and taking into account carefully all the details while for us it was easier to produce this classification from scratch. The

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[1] Sample text for footnote in the opening page.

authors of [4] have produced a full classification. One of the problems is that this classification is extended throughout the whole paper and is mixed with other formulas. To collect all the parts of the classification from [4] in one place and find all the additional conditions for these parts one has to undertake a tedious task. At the same time, [4] contains some inaccuracies. For example, the series μ_{10} parametrized by two scalars has to be divided into two series parametrized by one scalar, the series μ_{11} admits nontrivial isomorphisms, and in the case of a commutative 2-dimensional algebra with one idempotent e it may be impossible to find f linear independent with e such that f^2 and e are linearly dependent. The paper [5] is very nice and gives the full classification of 2-dimensional algebras over any field. Unfortunately, the answer is not given in terms of multiplication tables. The translation of this answer to the language of multiplication tables as well as its direct usage for the description of orbit closures is very difficult and it seems to be easier to produce a new appropriate classification. Also the consideration of arbitrary fields complicates the result and the extraction of the answer for an algebraically closed field becomes tedious. For these reasons, we give a classification that is valid over an algebraically closed field of arbitrary characteristic. In the same part of the paper, we also describe the automorphism groups for all algebras under consideration.

2 Definitions and notation

Throughout the paper we fix an algebraically closed field $\mathbf{k}$, a 2-dimensional $\mathbf{k}$-linear vector space V and a basis $\{e_1, e_2\}$ of V. All spaces in this paper are considered over $\mathbf{k}$, and we write simply dim, Hom and $\otimes$ instead of $dim_{\mathbf{k}}$, $Hom_{\mathbf{k}}$ and $\otimes_{\mathbf{k}}$. An algebra A is a set with a structure of a vector space and a binary operation that induces a bilinear map from $A \times A$ to A.

Since this paper is devoted to 2-dimensional algebras, we give all definitions and notation only for this case, though everything in this section can be rewritten for any dimension.

The set $\mathcal{A}_2 := Hom(V \otimes V, V) \cong V^* \otimes V^* \otimes V$ is a vector space of dimension 8. This space has a structure of the affine variety $\mathbf{k}^8$. Indeed, any $\mu \in \mathcal{A}_2$ is determined by 8 structure constants $c_{ij}^k \in \mathbf{k}$ $(i, j, k = 1, 2)$ such that $\mu(e_i \otimes e_j) = c_{ij}^1 e_1 + c_{ij}^2 e_2$. A subset of $\mathcal{A}_2$ is *Zariski-closed* if it can be defined by a set of polynomial equations in the variables c_{ij}^k.

The general linear group $GL(V)$ acts on $\mathcal{A}_2$ by conjugations:

$$(g * \mu)(x \otimes y) = g\mu(g^{-1}x \otimes g^{-1}y)$$

for $x, y \in V$, $\mu \in \mathcal{A}_2$ and $g \in GL(V)$. Thus, $\mathcal{A}_2$ is decomposed into $GL(V)$-orbits that correspond to the isomorphism classes of 2-dimensional algebras. The classification of 2-dimensional algebras up to isomorphism is equivalent to the classification of $GL(V)$-orbits.

Let $O(\mu)$ denote the orbit of $\mu \in \mathcal{A}_2$ under the action of $GL(V)$ and $\overline{O(\mu)}$ denote the Zariski closure of $O(\mu)$. Let A and B be two 2-dimensional algebras and $\mu, \lambda \in \mathcal{A}_2$ represent A and B respectively. We say that A degenerates to B and write $A \to B$ if $\lambda \in \overline{O(\mu)}$. Note that in this case we have $\overline{O(\lambda)} \subset \overline{O(\mu)}$. Hence, the definition of a degeneration does not depend on the choice of μ and λ. If $A \not\cong B$, then the assertion $A \to B$ is called a *proper degeneration*. We write $A \not\to B$ if $\lambda \notin \overline{O(\mu)}$. Let now $A(*) :=$

$\{A(\alpha)\}_{\alpha\in I}$ be a set of 2-dimensional algebras and $\mu_\alpha \in \mathcal{A}_2$ represent $A(\alpha)$ for $\alpha \in I$. If $\lambda \in \overline{\{O(\mu_\alpha)\}_{\alpha\in I}}$, then we write $A(*) \to B$ and say that $A(*)$ degenerates to B.

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Once data are disseminated, whatever contractual or other obligations are placed on those receiving the data, the data are effectively out of a data providers' control. Data providers must be certain that the data disseminated do not provide a risk of disclosure necessitating a reduction in the detail available, or they are constrained to using a resource intensive auditing regime.

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In the 50 years that the UK Data Archive has been making data available for social and economic research, there have been no damaging disclosures of personal information by academic researchers. While increasing use of detailed and sometimes sensitive[2] data

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can contribute valuable insights for targeting policies, we cannot be complacent. In order to support our policy needs and continue to use data safely and effectively, we need a research infrastructure that protects data confidentiality while enabling researchers to undertake innovative research.

5 Algebraic classification

The first of our aims is to classify all 2-dimensional algebras over $\mathbf{k}$ modulo isomorphism. Our classification is based on the following lemma.

***Lemma* 5.1** *Let A be a 2-dimensional algebra. Then there exists a non-zero element $x \in A$ such that x and x^2 are linearly dependent.*

Proof The required assertion is equivalent to the existence of a 1-dimensional subalgebra in A. Then the lemma follows from the discussion immediately after [1, Proposition 1]. ■

Note that if $x \in A$ and x^2 are linearly dependent, then either $x^2 = 0$ or $x = \alpha e$ for some $\alpha \in \mathbf{k}^*$ and some $e \in A$ such that $e^2 = e$. If $x^2 = 0$, then x is called a 2-*nil element*. An element e such that $e^2 = e$ is called an *idempotent*.

***Corollary* 5.2** *Any 2-dimensional $\mathbf{k}$-algebra belongs to one of the following disjoint classes:*

A. *algebras that do not have nonzero idempotents and have a unique 1-dimensional subspace of 2-nil elements;*
B. *algebras that do not have nonzero idempotents and have two linearly independent 2-nil elements;*
C. *algebras that have a unique nonzero idempotent and do not have nonzero 2-nil elements;*
D. *algebras that have a unique nonzero idempotent and a nonzero 2-nil element;*

Proof The fact that the classes are disjoint is obvious. The fact that any 2-dimensional algebra belongs to one of the classes follows easily from Lemma 5.1. ■

6 Equations

Equations in LaTeX can either be inline or on-a-line by itself. For inline equations use the `$...$` commands. Eg: The equation $H\psi = E\psi$ is written via the command $H\psi = E\psi$.

For on-a-line by itself equations (with auto generated equation numbers) one can use the equation or eqnarray environments D.

$$\mathcal{L} = i\psi\gamma^{\mu} D_{\mu}\psi - \frac{1}{4} F^{a}_{\mu\nu} F^{a\mu\nu} - m\psi\psi \tag{6.1}$$

where,

$$\begin{aligned} D_{\mu} &= \partial_{\mu} - ig\frac{\lambda^{a}}{2} A^{a}_{\mu} \\ F^{a}_{\mu\nu} &= \partial_{\mu} A^{a}_{\nu} - \partial_{\nu} A^{a}_{\mu} + g f^{abc} A^{b}_{\mu} A^{a}_{\nu} \end{aligned} \tag{6.2}$$

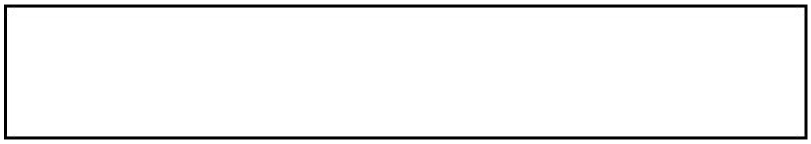

Figure 1: This is a widefig. This is an example of long caption this is an example of long caption this is an example of long caption this is an example of long caption.

Notice the use of `\nonumber` in the align environment at the end of each line, except the last, so as not to produce equation numbers on lines where no equation numbers are required. The `\label{}` command should only be used at the last line of an align environment where `\nonumber` is not used.

$$Y_\infty = \left(\frac{m}{\text{GeV}}\right)^{-3} \left[1 + \frac{3\ln(m/\text{GeV})}{15} + \frac{\ln(c_2/5)}{15}\right] \tag{6.3}$$

The class file also supports the use of `\mathbb{}`, `\mathscr{}` and `\mathcal{}` commands. As such `\mathbb{R}`, `\mathscr{R}` and `\mathcal{R}` produces $\mathbb{R}$, $\mathscr{R}$ and $\mathcal{R}$ respectively.

7 Figures

As per the LaTeX standards eps images in `latex` and pdf/jpg/png images in `pdflatex` should be used. This is one of the major differences between `latex` and `pdflatex`. The images should be single page documents. The command for inserting images for latex and pdflatex can be generalized. The package that should be used is the graphicx package.

8 Tables

Tables can be inserted via the normal table and tabular environment. To put footnotes inside tables one has to use the additional "fntable" environment enclosing the tabular environment. The footnote appears just below the table itself.

Table 1: Tables which are too long to fit, should be written using the "table*" environment as shown here

	Element 1			Element 2[1]		
Projectile	Energy	σ_{calc}	σ_{expt}	Energy	σ_{calc}	σ_{expt}
Element 3	990 A	1168	1547 ± 12	780 A	1166	1239 ± 100
Element 4	500 A	961	922 ± 10	900 A	1268	1092 ± 40

Note: This is an example of table footnote this is an example of table footnote this is an example of table footnote this is an example of table footnote this is an example of table footnote

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9 Cross referencing

Environments such as figure, table, equation, align can have a label declared via the `\label{#label}` command. For figures and table environments one should use the `\label{}` command inside or just below the `\caption{}` command. One can then use the `\ref{#label}` command to cross-reference them. As an example, consider the label declared for Figure 1 which is `\label{fig1}`. To cross-reference it, use the command `Figure \ref{fig1}`, for which it comes up as “Figure 1”. The reference citations should used as per the “natbib” packages. Some sample citations: [2, 3, 4, 5, 6].

10 Lists

List in LaTeX can be of three types: enumerate, itemize and description. In each environments, new entry is added via the `\item` command. Enumerate creates numbered lists, itemize creates bulleted lists and description creates description lists.

(1) First item in the number list.
(2) Second item in the number list.
(3) Third item in the number list.

List in LaTeX can be of three types: enumerate, itemize and description. In each environments, new entry is added via the `\item` command.

- First item in the bullet list.
- Second item in the bullet list.
- Third item in the bullet list.

11 Conclusion

Some Conclusions here.

References

[1] A. Ananin and A. Mironov, The moduli space of 2-dimensional algebras, *Comm. Algebra*, 28 (2000), 9, 4481–4488.
[2] C. Bai and D. Meng, The classification of Novikov algebras in low dimension, *J. Phys. A: Math. Gen.*, 34 (2001), 1581–1594.
[3] E. Cañete and A. Khudoyberdiyev, The classification of 4-dimensional Leibniz algebras, *Linear Algebra and its Applications*, 439 (2013), 1, 273–288.
[4] M. Goze and E. Remm, 2-dimensional algebras, *Afr. J. Math. Phys.*, 10 (2011), 1, 81–91.
[5] H. Petersson, The classification of two-dimensional nonassociative algebras, *Results Math.*, 37 (2000), no. 1-2, 120–154.
[6] L. Snobl and P. Winternitz, Classification and Identification of Lie algebras, *CRM Monograph Series*, 33 (2014).

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