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Newton complementary duals of f-ideals*

Samuel Budd and Adam Van Tuyl

Abstract. **A square-free monomial ideal I of $k[x_1, \ldots, x_n]$ is said to be an f-ideal if the facet complex and non-face complex associated with I have the same f-vector. We show that I is an f-ideal if and only if its Newton complementary dual $\hat{I}$ is also an f-ideal. Because of this duality, previous results about some classes of f-ideals can be extended to a much larger class of f-ideals. An interesting by-product of our work is an alternative formulation of the Kruskal-Katona theorem for f-vectors of simplicial complexes.**

1 Introduction

Let I be a square-free[1] monomial ideal of $R = k[x_1, \ldots, x_n]$ where k is a field. Associated with any such ideal are two simplicial complexes. The *non-face complex*, denoted $\delta_{\mathcal{N}}(I)$, (also called the *Stanley-Reisner complex*) is the simplicial complex whose faces are in one-to-one correspondence with the square-free monomials not in I. Faridi [1] introduced a second complex, the *facet complex* $\delta_{\mathcal{F}}(I)$, where the generators of I define the facets of the simplicial complex (see the next section for complete definitions). In general, the two simplicial complexes $\delta_{\mathcal{N}}(I)$ and $\delta_{\mathcal{F}}(I)$ can be very different. For example, the two complexes may have different dimensions; as a consequence, the f*-vectors* of $\delta_{\mathcal{F}}(I)$ and $\delta_{\mathcal{N}}(I)$, which enumerate all the faces of a given dimension, may be quite different.

If I is a square-free monomial ideal with the property that the f-vectors of $\delta_{\mathcal{F}}(I)$ and $\delta_{\mathcal{N}}(I)$ are the same, then I is called an f*-ideal*. The notion of an f-ideal was first introduced by Abbasi, Ahmad, Anwar, and Baig [2]. It is natural to ask if it is possible to classify all the square-free monomial ideals that are f-ideals. Abbasi, et. al classified all the f-ideals generated in degree two. This result was later generalized by Anwar, Mahmood, Binyamin, and Zafar [3] which classified all the f-ideals I that are unmixed and generated in degree $d \geq 2$. An alternative proof for this result was found by Guo and Wu [4]. Gu, Wu, and Liu [5] later removed the unmixed restriction of [3]. Other work related to f-ideals includes the papers [6, 7].

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2 Background

In this section, we review the required background results.

Let $X = \{x_1, \ldots, x_n\}$ be a set of vertices. A *simplicial complex* Δ on X is a subset of the power set of X that satisfies: (i) if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$, and (ii) $\{x_i\} \in \Delta$ for $i = 1, \ldots, n$. An element $F \in \Delta$ is called a *face*; maximal faces with respect to inclusion are called *facets*. If $F_1, \ldots, F_r$ are the facets of Δ, then we write $\Delta = \langle F_1, \ldots, F_r \rangle$.

For any face $F \in \Delta$, the *dimension* of F is given by $\dim(F) = |F| - 1$. Note that $\emptyset \in \Delta$ and $\dim(\emptyset) = -1$. The *dimension* of Δ is given by $\dim(\Delta) = \max\{\dim(F) \mid F \in \Delta\}$. If $d = \dim(\Delta)$, then the f*-vector* of Δ is the $d + 2$ tuple

$$f(\Delta) = (f_{-1}, f_0, f_1, \ldots, f_d)$$

where f_i is number of faces of dimension i in Δ. We write $f_i(\Delta)$ if we need to specify the simplicial complex.

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Once data are disseminated, whatever contractual or other obligations are placed on those receiving the data, the data are effectively out of a data providers' control. Data providers must be certain that the data disseminated do not provide a risk of disclosure necessitating a reduction in the detail available, or they are constrained to using a resource intensive auditing regime, and are likely to discover any data misuse only after it has happened.

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In the 50 years that the UK Data Archive has been making data available for social and economic research, there have been no damaging disclosures of personal information by academic researchers. While increasing use of detailed and sometimes sensitive[2] data can contribute valuable insights for targeting policies, we cannot be complacent. In order to support our policy needs and continue to use data safely and effectively, we need a research infrastructure that protects data confidentiality while enabling researchers to undertake innovative research.

5 The Newton complementary dual and f-vectors

We introduce the generalized Newton complementary dual of a monomial ideal as defined by [8] (based upon [9]). We then show how the f-vector behaves under this operation.

Definition 5.1 Let $I \subseteq R = k[x_1, \ldots, x_n]$ be a monomial ideal with $\mathcal{G}(I) = \{m_1, \ldots, m_p\}$. Suppose that $m_i = x_1^{\alpha_{i,1}} x_2^{\alpha_{i,2}} \cdots x_n^{\alpha_{i,n}}$ for all $i = 1, \ldots, p$. Let $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$ be a vector such that $\beta_i \geq \alpha_{k,l}$ for all $l = 1, \ldots, n$ and $k = 1, \ldots, p$. The *generalized Newton complementary dual of I* determined by β is the ideal

$$\hat{I}^{[\beta]} = \left\langle \frac{x^\beta}{m} \;\middle|\; m \in \mathcal{G}(I) \right\rangle = \left\langle \frac{x^\beta}{m_1}, \frac{x^\beta}{m_2}, \ldots, \frac{x^\beta}{m_p} \right\rangle \quad \text{where } x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}.$$

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Remark 5.1 If $I \subseteq R = k[x_1, \ldots, x_n]$ is a square-free monomial ideal, then one can take $\beta = (1, \ldots, 1) = \mathbf{1}$, i.e., $x^\beta = x_1 \cdots x_n$. For simplicity, we denote $\hat{I}^{[1]}$ by $\hat{I}$ and call it the *complementary dual* of I. Note that we have $\hat{\hat{I}} = I$.

Example 5.2 We return to the ideal I of example. For this ideal we have

$$\hat{I} = \left\langle \frac{x_1 \cdots x_5}{x_1 x_4}, \frac{x_1 \cdots x_5}{x_2 x_5}, \frac{x_1 \cdots x_5}{x_1 x_2 x_3}, \frac{x_1 \cdots x_5}{x_3 x_4 x_5} \right\rangle = \langle x_2 x_3 x_5, x_1 x_3 x_4, x_4 x_5, x_1 x_2 \rangle.$$

The next lemma is key to understanding how the f-vector behaves under the duality.

***Lemma* 5.3** *Let $I \subseteq R = k[x_1, \ldots, x_n]$ be a square-free monomial ideal. For all integers $j = -1, \ldots, n-1$, there is a bijection*

$$\left\{m \in R_{j+1} \mid m \text{ a square-free monomial that divides some } p \in \mathcal{G}(I)\right\}$$
$$\leftrightarrow \left\{m \in \hat{I}_{n-j-1} \mid m \text{ a square-free monomial}\right\}.$$

Proof Fix a $j \in \{-1, \ldots, n-1\}$ and let A denote the first set, and let B denote the second set. We claim that the map $\varphi : A \longrightarrow B$ given by

$$\varphi(m) = \frac{x_1 x_2 \cdots x_n}{m}$$

gives the desired bijection. This map is defined because if $m \in A$, there is a generator $p \in \mathcal{G}(I)$ such that $m|p$. But that then means that $\frac{x_1 \cdots x_n}{p}$ divides $\varphi(m) = \frac{x_1 \cdots x_n}{m}$, and consequently, $\varphi(m) \in \hat{I}$. Moreover, since $\deg(m) = j+1$, we have $\deg(\varphi(m)) = n-j-1$. Finally, since m is a square-free monomial, so is $\varphi(m)$.

It is immediate that the map is injective. For surjectivity, let $m \in B$. It suffices to show that the square-free monomial $m' = \frac{x_1 \cdots x_n}{m} \in A$ since $\varphi(m') = m$. By our construction of m' it follows that $\deg(m') = j + 1$. Also, because $m \in B$, there is some $p \in \mathcal{G}(I)$ such that $\frac{x_1 \cdots x_n}{p}$ divides m. But this then means that m' divides p, i.e., $m' \in A$. ∎

6 Equations

Equations in LaTeX can either be inline or on-a-line by itself. For inline equations use the `$...$` commands. Eg: The equation $H\psi = E\psi$ is written via the command $H\psi = E\psi$.

For on-a-line by itself equations (with auto generated equation numbers) one can use the equation or eqnarray environments D.

$$\mathcal{L} = i\bar{\psi}\gamma^\mu D_\mu \psi - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - m\bar{\psi}\psi \tag{6.1}$$

where,

$$D_\mu = \partial_\mu - ig\frac{\lambda^a}{2} A_\mu^a$$
$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^a \tag{6.2}$$

Figure 1: This is a widefig. This is an example of long caption this is an example of long caption this is an example of long caption this is an example of long caption.

Notice the use of `\nonumber` in the align environment at the end of each line, except the last, so as not to produce equation numbers on lines where no equation numbers are required. The `\label{}` command should only be used at the last line of an align environment where `\nonumber` is not used.

$$Y_\infty = \left(\frac{m}{\text{GeV}}\right)^{-3} \left[1 + \frac{3\ln(m/\text{GeV})}{15} + \frac{\ln(c_2/5)}{15}\right] \tag{6.3}$$

The class file also supports the use of `\mathbb{}`, `\mathscr{}` and `\mathcal{}` commands. As such `\mathbb{R}`, `\mathscr{R}` and `\mathcal{R}` produces $\mathbb{R}$, $\mathscr{R}$ and $\mathcal{R}$ respectively.

7 Figures

As per the LaTeX standards eps images in `latex` and pdf/jpg/png images in `pdflatex` should be used. This is one of the major differences between `latex` and `pdflatex`. The images should be single page documents. The command for inserting images for latex and pdflatex can be generalized. The package that should be used is the graphicx package.

8 Tables

Tables can be inserted via the normal table and tabular environment. To put footnotes inside tables one has to use the additional “fntable" environment enclosing the tabular environment. The footnote appears just below the table itself.

Table 1: Tables which are too long to fit, should be written using the “table*” environment as shown here

	Element 1			Element 2[1]		
Projectile	Energy	σ_{calc}	σ_{expt}	Energy	σ_{calc}	σ_{expt}
Element 3	990 A	1168	1547 ± 12	780 A	1166	1239 ± 100
Element 4	500 A	961	922 ± 10	900 A	1268	1092 ± 40

Note: This is an example of table footnote this is an example of table footnote this is an example of table footnote this is an example of table footnote this is an example of table footnote
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9 Cross referencing

Environments such as figure, table, equation, align can have a label declared via the `\label{#label}` command. For figures and table environments one should use the `\label{}` command inside or just below the `\caption{}` command. One can then use the `\ref{#label}` command to cross-reference them. As an example, consider the label declared for Figure 1 which is `\label{fig1}`. To cross-reference it, use the command `Figure \ref{fig1}`, for which it comes up as “Figure 1”. The reference citations should used as per the “natbib” packages. Some sample citations: [1, 2, 3, 4, 5, 6, 7].

10 Lists

List in LaTeX can be of three types: enumerate, itemize and description. In each environments, new entry is added via the `\item` command. Enumerate creates numbered lists, itemize creates bulleted lists and description creates description lists.

(1) First item in the number list.
(2) Second item in the number list.
(3) Third item in the number list.

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- First item in the bullet list.
- Second item in the bullet list.
- Third item in the bullet list.

11 Conclusion

Some Conclusions here.

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Department of Mathematics and Statistics, McMaster University, Hamilton, ON, L8S 4L8
e-mail: *budds2@mcmaster.ca*, *sjbudd3@gmail.com*, *vantuyl@math.mcmaster.ca*.