An introduction to Category Theory

The Solutions

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Introduction

The book An introduction to Category Theory contains over 200 exercises of varying degree of difficulty. However, to keep down the size no solutions are included in the book. These pages contain a more or less complete set of solutions.

These solutions are arrange by chapter and section to match the exercises in the book. Thus if you want the solution to exercise $X.Y.Z$ (the Z^{th} exercise in Section Y of Chapter X of the book), simply go to Chapter X, Section Y here and look at the Z^{th} solution.

These pages are in a larger format than the book. I have taken some care with the pages breaks so that where possible a solution, or a batch of solutions, fits on a double page. Of course, that is not always possible, and it means that some page are longer than others.

No doubt these solutions still contain typos, garbled bits, and perhaps even mistakes. I will update this document every so often. The date on the front page and below indicates when this version was produced.

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1
Categories

1.1 Categories defined

1.1.1 Not needed? $\square$

1.1.2 These examples are dealt with in Section 1.5. $\square$

1.2 Categories of structured sets

1.2.1 (c) Consider the function

$$f(r) = \alpha^r(a)$$

which sends each $r \in \mathbb{N}$ to the r^{th} iterate of α applied to a. A simple calculation shows this is a $\boldsymbol{Pno}$-arrow. A proof by induction shows this is the only possible arrow. $\square$

1.2.2 Consider a pair

$$(A, X) \xrightarrow{\quad f \quad} (B, Y) \xrightarrow{\quad g \quad} (C, Z)$$

of such morphisms. We show the function composite $g \circ f$ is also a morphism, that is

$$x \in X \Longrightarrow g(f(x)) \in Z$$

for each element x of A. The morphism property of f and then g gives

$$x \in X \Longrightarrow y = f(x) \in Y \Longrightarrow g(f(x)) = g(y) \in Z$$

as required. This doesn't yet prove we have a category, but the other requirements – arrow composition is associative, and there are identity arrows – are easy. $\square$

1.2.3 The appropriate notion of arrow

$$(A, R) \xrightarrow{\; f \;} (B, S)$$

is a function between the carrying sets such that

$$(x, y) \in R \Longrightarrow (f(x), f(y)) \in S$$

for all $x, y \in A$. This generalizes the idea used in $\boldsymbol{Pre}$ and $\boldsymbol{Pos}$. $\square$

1.2.4 Consider a pair of continuous maps

$$R \xrightarrow{\quad \psi \quad} S \xrightarrow{\quad \phi \quad} T$$

between topological spaces. A simple calculation gives

$$(\phi \circ \psi)^{\leftarrow} = \psi^{\leftarrow} \circ \phi^{\leftarrow}$$

which is the required property. $\square$

1.2.5 Let $R = \boldsymbol{C}[A, A]$. We have a binary operation $\circ$ on R, namely arrow composition. This operation is associative (by one of the axioms of being a category). We also have a distinguished element id_A of R, the identity arrow on A. This is the required unit.

(Strictly speaking, this do not show that R is a monoid, for we don't know that R is a set. There are some categories for which $\boldsymbol{C}[A, A]$ is so large it is not a set. This is rather weired but it shouldn't worry us.) $\square$

1.2.6 To show that $\boldsymbol{Pfn}$ is a category we must at least show that composition of arrows is associative.

Consider three composible partial functions

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r, "h"] & D \\
X \arrow[u, hook] \arrow[ur, "\overline{f}"'] & Y \arrow[u, hook] \arrow[ur, "\overline{g}"'] & Z \arrow[u, hook] \arrow[ur, "\overline{h}"'] &
\end{tikzcd}$$

as indicated. We must describe

$$h \circ (g \circ f) \qquad (h \circ g) \circ f$$

and show that they are the same. We need

$$\begin{tikzcd}
A \arrow[r, "g \circ f"] & C \arrow[r, "h"] & D \\
U \arrow[u, hook] \arrow[ur, "\overline{g} \circ \overline{f}_{|U}"'] & Y \arrow[u, hook] \arrow[ur, "h"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, "h \circ g"] & D \\
X \arrow[u, hook] \arrow[ur, "\overline{f}"'] & V \arrow[u, hook] \arrow[ur, "\overline{h} \circ \overline{g}_{|V}"'] &
\end{tikzcd}$$

where

$$a \in U \iff a \in X \text{ and } \overline{f}(a) \in Y \qquad b \in V \iff b \in Y \text{ and } \overline{g}(b) \in Z$$

for $a \in A$ and $b \in B$. We also need

$$\begin{tikzcd}
A \arrow[r, "h \circ (g \circ f)"] & D \\
L \arrow[u, hook] \arrow[ur, "\overline{h} \circ (\overline{g} \circ \overline{f}_{|U})_{|L}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "(h \circ g) \circ f"] & D \\
R \arrow[u, hook] \arrow[ur, "\overline{h} \circ (\overline{g} \circ \overline{f}_{|V})_{|R}"'] &
\end{tikzcd}$$

where

$$a \in L \iff \left\{ \begin{array}{l} a \in U \\ \text{and} \\ (\overline{g} \circ \overline{f}_{|U})(a) \in Z \end{array} \right\} \qquad a \in R \iff \left\{ \begin{array}{l} a \in X \\ \text{and} \\ \overline{f}(a) \in V \end{array} \right\}$$

for $a \in A$. We show $L = R$ and the two function composites are equal.

For $a \in L$ we have $a \in U$, so that $\overline{f}_{|U}(a) = \overline{f}(a)$. Thus, remembering the definition of U we have

$$a \in L \iff a \in X \text{ and } \overline{f}(a) \in Y \text{ and } (\overline{g} \circ \overline{f}_{|U})(a) \in Z$$

for $a \in A$. Remembering the definition of V we have

$$\overline{f}(a) \in V \iff \overline{f}(a) \in Y \text{ and } \overline{g}(\overline{f}(a)) \in Z$$

and hence

$$a \in R \iff a \in X \text{ and } \overline{f}(a) \in Y \text{ and } (\overline{g} \circ \overline{f}_{|U})(a) \in Z$$

for $a \in A$. This shows that $L = R$.

Consider any $a \in L = R$. We have

$$a \in U \qquad \overline{f}(a) \in V$$

so that

$$\big(\overline{g} \circ \overline{f}_{|U}\big)_{|L}(a) = \big(\overline{g} \circ \overline{f}_{|U}\big)(a) = \overline{g}(\overline{f}(a))$$

to give

$$\overline{h} \circ \big(\overline{g} \circ \overline{f}_{|U}\big)_{|L}(a) = \overline{h}(\overline{g}(\overline{f}(a)))$$

and

$$\big((\overline{h} \circ \overline{g}_{|V}) \circ \overline{f}_{|R}\big)(a)(\overline{h} \circ \overline{g}_{|V})(\overline{f}(a)) = \overline{h}(\overline{g}(\overline{f}(a)))$$

that is

$$\overline{h} \circ \big(\overline{g} \circ \overline{f}_{|U}\big)_{|L}(a) = \overline{h}(\overline{g}(\bar{(}a))) = \big((\overline{h} \circ \overline{g}_{|V}) \circ \overline{f}_{|R}\big)(a)$$

to show that the two function composites are the same.

What about identity arrows? Every total function is also a partial function. Each set A carries an identity function id_A which is

$$\begin{tikzcd} A & A \\ A \arrow[u, hook, "="] \arrow[ur, "\overline{id_A}"'] & \end{tikzcd}$$

when viewed as a partial function.

Consider the composites

$$\begin{tikzcd} A \arrow[r, "id_A"] & A \arrow[r, "f"] & B \\ A \arrow[u, hook, "="] \arrow[ur, "\overline{id_A}"'] & X \arrow[u, hook] \arrow[ur, "\overline{f}"'] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "id_A"] & B \\ X \arrow[u, hook] \arrow[ur, "\overline{f}"'] & B \arrow[u, hook, "="] \arrow[ur, "\overline{id_B}"'] & \end{tikzcd}$$

where f is an arbitrary partial function. To compute these composites we first use

$$a \in L \Longleftrightarrow a \in A \text{ and } \overline{id_A}(a) \in X \qquad a \in R \Longleftrightarrow a \in X \text{ and } \overline{f}(a) \in B$$

(for $a \in A$) to extract $L, R \subseteq A$. Notice that, in fact

$$L = X = R$$

but for different reasons. The arrow composites are

$$\begin{tikzcd} A \arrow[r, "f \circ id_A"] & B \\ X \arrow[u, hook] \arrow[ur, "\overline{f} \circ \overline{id_A}_{|X}"'] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "id_B \circ f"] & B \\ X \arrow[u, hook] \arrow[ur, "\overline{id_B} \circ \overline{f}_{|X}"'] & \end{tikzcd}$$

and these function composites are

$$\overline{f} \circ \overline{id_A}_{|X} = \overline{f} \circ \overline{id_X} = \overline{f} \qquad \overline{id_X} \circ \overline{f}_X = \overline{id_B} \circ \overline{f} = \overline{f}$$

to show

$$f \circ id_A = f = id_B \circ f$$

as required. $\square$

1.2.7 We set up a pair of translations between the two categories

$$\boldsymbol{Pfn} \underset{M}{\overset{L}{\rightleftarrows}} \boldsymbol{Set}_{\perp}$$

and then show that each 2-step goes back to where it started.

We set

$$LA = A \cup \{\perp\} \qquad MS = S - \{\perp\}$$

for each object A of $\boldsymbol{Pfn}$ and each object S of $\boldsymbol{Set}_{\perp}$. In other words, L attaches the distinguished point, and M removes the distinguished point. Almost trivially we have

$$(M \circ L)A = A \qquad (L \circ M)S = S$$

for each such A and S.

The way we deal with arrows is more intricate. For each arrow of $\boldsymbol{Pfn}$

$$\begin{tikzcd} A \arrow[r, "f"] & B \\ X \arrow[u, hook] \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

we let

$$\begin{array}{rcll} LA & \xrightarrow{L(f)} & LB & \\ a & \longmapsto & \overline{f}(a) & \text{for } a \in X \\ a & \longmapsto & \perp & \text{for } a \in A - X \\ \perp & \longmapsto & \perp & \end{array}$$

to obtain an arrow of $\boldsymbol{Set}_{\perp}$. In other words we set

$$L(f)(a) = \begin{cases} \overline{f}(a) & \text{if } a \in X \\ \perp & \text{if } a \notin X \end{cases}$$
$$L(f)(\perp) = \perp$$

for each $a \in A$. By the lower clause, this is an arrow of $\boldsymbol{Set}_{\perp}$.

Consider any arrow

$$S \xrightarrow{\phi} T$$

of $\boldsymbol{Set}_{\perp}$. We extract

$$X \subseteq MS = S - \{\perp\} \quad \text{by} \quad X = S - \phi^{\leftarrow}(\perp)$$

that is

$$s \in X \Longleftrightarrow \phi(s) \neq \perp$$

for $s \in S$. In particular, $\perp \notin X$. Thus we have a partial function

$$\begin{tikzcd} MS \arrow[r, "M(\phi)"] & MT \\ X \arrow[u, hook] \arrow[ur, "\phi|_X"'] & \end{tikzcd}$$

controlled by the restriction of ϕ to X.

These constructions give

$$\begin{array}{rcll} S & \xrightarrow{L(M(\phi)f)} & T & \\ s & \longmapsto & \phi(s) & \text{for } s \in X \\ s & \longmapsto & \perp & \text{for } s \in MS - X \\ \perp & \longmapsto & \perp & \end{array}$$

so that

$$L(M(\phi) = \phi$$

which is what we want.

For the other way round each partial function

$$A \xrightarrow{f} B$$

as above gives a pointed arrow

$$LA \xrightarrow{\phi = L(f)} LB$$

which we convert back into a partial function. To do that we set

$$W = LA = \phi^{\leftarrow}(\perp)$$

so that

$$a \in W \iff \phi(a) \neq \perp \iff a \in X$$

for each $a \in A$. Thus

$$W = X$$

with

$$\phi|_X = \overline{f}$$

and hence

$$M(L(f)) = f$$

as required.

Do you think that $\boldsymbol{Pfn}$ and $\boldsymbol{Set}_\perp$ are 'essentially the same'? □

1.2.8 Showing that each of

$$R\text{-}\boldsymbol{Set} \qquad \boldsymbol{Set}\text{-}R$$

is a category is easy. For both categories an object is a structured set

$$\big(A\ (\alpha_r \mid r \in R)\big)$$

a set A furnished with an R-indexed family of 1-placed operations on A. This family must satisfy

$$\alpha_s \circ \alpha_r = \alpha_{rs} \qquad \alpha_s \circ \alpha_r = \alpha_{sr}$$

for $r, s \in R$. Note the

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here. For both categories an arrow

$$\big(A\ (\alpha_r \mid r \in R)\big) \xrightarrow{f} \big(B\ (\beta_r \mid r \in R)\big)$$

is a function f between the carriers such that

$$f \circ \alpha_r = \beta_r \circ f$$

for each $r \in R$. □

1.3 An arrow need not be a function

1.3.1 Let $\mathbb{R}^m$ be the vector space of column vectors with m real components. Each $m \times n$ matrix A gives a linear transformation

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^m \\ x \arrow[r, mapsto] & Ax \end{tikzcd}$$

and every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ arises in this way from a unique $m \times n$ matrix A. The composite of two linear transformations

$$\mathbb{R}^n \xrightarrow{B} \mathbb{R}^k \xrightarrow{A} \mathbb{R}^m$$

is linear, and corresponds to the matrix product AB. □

1.3.2 The main problem is to define the composition of graph morphisms, and to show that this composition is associative.

Consider a pair of graph morphisms.

$$(N, E) \xrightarrow{f} (M, F) \xrightarrow{g} (L, G)$$

Thus, writing ρ for σ or for τ throughout we have two commuting squares

$$\begin{tikzcd} E \arrow[r, "f_1"] \arrow[d, "\rho_E"'] & F \arrow[r, "g_1"] \arrow[d, "\rho_F"] & G \arrow[d, "\rho_G"] \\ N \arrow[r, "f_0"'] & M \arrow[r, "g_0"'] & L \end{tikzcd}$$

that is

$$f_0 \circ \rho_E = \rho_F \circ f_1 \qquad g_0 \circ \rho_F = \rho_G \circ g_1$$

for each of the two cases of ρ. But now the outside square commutes

$$\begin{tikzcd} E \arrow[r, "g_1 \circ f_1"] \arrow[d, "\rho_E"'] & G \arrow[d, "\rho_G"] \\ N \arrow[r, "g_0 \circ f_0"'] & L \end{tikzcd}$$

that is

$$g_0 \circ f_0 \circ \rho_E = g_0 \circ \rho_F \circ f_1 = \rho_G \circ g_1 \circ f_1$$

and so obtain a graph morphism. We take this as the composite of graph morphisms. An easy exercise shows this composite is associative. □

1.3.3 Consider a composible pair of arrows of this category.

$$(A, R) \xrightarrow{(f, \phi)} (B, S) \xrightarrow{(g, \psi)} (C, T)$$

In other words we have two pairs of composible functions.

$$A \xrightarrow{f} B \xrightarrow{g} C \qquad\qquad R \xleftarrow{\phi} S \xrightarrow{\psi} T$$

The pair

$$(A,R) \xrightarrow{(g\circ f,\,\phi\circ\psi)} (C,T)$$

is the composite

$$(A,R) \xrightarrow{(f,\phi)\circ(g,\psi)} (C,T)$$

in the category. Verifying the axioms is more or less trivial.

This is essentially the same as Example 1.3.4. The category $\boldsymbol{S}$ has been replaced by its opposite $\boldsymbol{S}^{\mathrm{op}}$. See Example 1.5.3. □

1.3.4 Consider a pair of arrows in this category

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ R & \xleftarrow[\phi]{} & S & \xleftarrow[\psi]{} & T \end{array}$$

where we have separated the two components. We show that the pair

$$(A,R) \xrightarrow{(g\circ f,\,\phi\circ\psi)} (C,T)$$

is an arrow, that is

$$\big(g\circ f\big)\big(a(\phi\circ\psi)(t)\big) = \big(g\circ f\big)(a)t$$

for each $a\in A$ and $t\in T$. Consider such a pair a,t and let

$$b = f(a) \qquad s = \psi(t)$$

to produce $b\in B$ and $s\in S$. Then

$$\begin{aligned} \big(g\circ f\big)\big(a(\phi\circ\psi)(t)\big) &= g\big(f(a\phi(s)\big) \\ &= g\big(f(a)s\big) \\ &= g\big(b\psi(t)\big) \\ &= g(b)t \qquad\qquad = \big(g\circ f\big)(a)t \end{aligned}$$

for the required result. □

1.3.5 To show the composition is associative consider three relations

$$A \xrightarrow{F} B \xrightarrow{G} C \xrightarrow{H} D$$

between sets. For $a\in A$ and $d\in D$ we have

$$\begin{array}{l|l} \quad d\big(H\circ(G\circ F)\big)a & \quad d\big((H\circ G)\circ F\big)a \\ \iff (\exists c\in C)[dHc(G\circ F)a] & \iff (\exists b\in B)[d(H\circ G)bFa] \\ \iff (\exists c\in C, b\in B)[dHcGbFa] & \iff (\exists b\in B, c\in C)[dHcGbFa] \end{array}$$

so that a flip of quantifiers gives the required result.

Consider a pair of functions with associated graphs.

$$A \xrightarrow[\Gamma(f)]{f} B \xrightarrow[\Gamma(g)]{g} C$$

For $a \in A$ and $c \in C$ we have

$$\begin{aligned} c\big(\Gamma(g) \circ \Gamma(f)\big)a &\Longleftrightarrow (\exists b \in B)[c\Gamma(g)b\Gamma(f)a] \\ &\Longleftrightarrow (\exists b \in B)[c = g(b) \ \&\ b = f(a)] \\ &\Longleftrightarrow c = g(f(a)) \qquad\qquad \Longleftrightarrow c\Gamma(g \circ f)a \end{aligned}$$

for the required result. □

1.3.6 (a) Suppose first that $f \dashv g$, that is

$$f(a) \leq b \Longleftrightarrow a \leq g(b)$$

for $a \in S$ and $b \in T$. Since

$$f(a) \leq f(a) \qquad g(b) \leq g(b)$$

a use of this equivalence one way or the other gives

$$a \leq (g \circ f)(a) \qquad (f \circ g)(b) \leq b$$

for the first required result.

Conversely, suppose the two comparisons hold (for all $a \in S$ and $b \in T$), and suppose

$$f(a) \leq b \qquad a \leq g(b)$$

(for some $a \in S$ and $b \in T$). Since both f and g are monotone we have

$$a \leq (g \circ f)(a) \leq g(b) \qquad f(a) \leq (f \circ g)(b) \leq b$$

to verify the equivalence.

(b) For each $a \in S$ we have

$$a \leq (g \circ f)(a)$$

and hence

$$f(a) \leq (f \circ g \circ f)(a)$$

since f is monotone. For each $b \in T$ we have

$$(f \circ g)(b) \leq b$$

and hence

$$(f \circ g \circ f)(a) \leq f(a)$$

as a particular case. This gives

$$f \circ g \circ f = f$$

and the other equality follows in a similar fashion. □

1.3.7 We are given two projection pairs

$$A \xrightarrow[f_* \circ f^* = id_A]{f^* \dashv f_*} B \xrightarrow[g_* \circ g^* = id_B]{g^* \dashv g_*} C$$

that is two adjunctions with the indicated equalities. These certainly give a composite adjunction

$$A \xrightarrow{h^* \dashv h_*} C$$

where

$$h^* = g^* \circ f^* \qquad h_* = f_* \circ g_*$$

are the two components. But now

$$h_* \circ h^* = f_* \circ g_* \circ g^* \circ f^* = f_* \circ id_B \circ f^* = f_* \circ f^* = id_A$$

to show that $h^* \dashv h_*$ is a projection pair. □

1.3.8 (a) The two required adjunction properties are

$$\mathbb{Z} \xrightarrow{\iota \dashv \rho} \mathbb{R} \qquad\qquad \mathbb{R} \xrightarrow{\lambda \dashv \iota} \mathbb{Z}$$

with

$$m \le \rho(x) \iff m \le x \qquad \lambda(x) \le m \iff x \le m$$

for $x \in \mathbb{R}, m \in \mathbb{Z}$. From these we see that

$$\lambda(x) = \lfloor x \rfloor \qquad \rho(x) = \lceil x \rceil$$

the integer

floor ceiling

of x are the only possible functions and these do form adjunctions.

(b) For each $m \in \mathbb{Z}$ we have

$$\lambda(m) = m = \rho(m)$$

which gives all the required composite properties.

(c) The condition $\iota \circ \rho = id_{\mathbb{Z}}$ follows by the observation of part (b). Also

$$(\lambda \circ \iota)(1/2) = \lambda(1/2) = 0$$

so that $\lambda \circ \iota \neq id_{\mathbb{R}}$. □

1.3.9 For a given monotone map

$$T \xrightarrow{\ \phi\ } S$$

we require monotone maps

$$\begin{tikzcd} \mathcal{L}T \arrow[r, shift left=6, "f^\sharp"] \arrow[r, shift right=6, "f_\flat"'] & \mathcal{L}S \arrow[l, "\phi^\leftarrow" description] \end{tikzcd}$$

with

$$f^\sharp(Y) \subseteq X \Longleftrightarrow Y \subseteq \phi^\leftarrow(X)$$

$$\phi^\leftarrow(X) \subseteq Y \Longleftrightarrow X \subseteq f_\flat(Y)$$

for each $X \in \mathcal{L}S$ and $Y \in LT$.

For $Y \in \mathcal{L}T$ let

$$f^\sharp(Y) = \downarrow\phi[Y]$$

the lower section of S generated by the direct image of Y across ϕ. For $X \in \mathcal{L}S$ and $Y \in \mathcal{L}T$ we have

$$\begin{aligned} f^\sharp(Y) \subseteq X &\Longleftrightarrow \phi[Y] \subseteq X \\ &\Longleftrightarrow (\forall t \in T)[t \in Y \Longrightarrow \phi(t) \in X] \\ &\Longleftrightarrow (\forall t \in T)[t \in Y \Longrightarrow t \in \phi^\leftarrow(X)] \quad \Longleftrightarrow Y \subseteq \phi^\leftarrow(X) \end{aligned}$$

as required.

For $Y \in \mathcal{L}T$ let

$$f_\flat(Y) = \big(\uparrow\phi[Y']\big)'$$

the complement of the upper section of S generated by the direct image of the complement of Y across ϕ. For $X \in \mathcal{L}S$ and $Y \in \mathcal{L}T$ we have

$$\begin{aligned} X \subseteq f_\flat(Y) &\Longleftrightarrow \uparrow\phi[Y'] \subseteq X' \\ &\Longleftrightarrow \phi[Y'] \subseteq X' \\ &\Longleftrightarrow (\forall t \in T)[t \in Y' \Longrightarrow \phi(t) \in X'] \\ &\Longleftrightarrow (\forall t \in T)[\phi(t) \in X \Longrightarrow t \in Y] \\ &\Longleftrightarrow (\forall t \in T)[t \in \phi^\leftarrow(X) \Longrightarrow t \in Y] \quad \Longleftrightarrow \phi^\leftarrow(X) \subseteq Y \end{aligned}$$

as required. □

1.3.10 You will find it instructive to go through the following solution of a more general exercise.

Let

$$\nabla = (N, E)$$

be a graph (in the sense of Exercise 1.3.2). We let

$$i, j, k, \ldots \text{ range over } N \qquad e, f, g, \ldots \text{ range over } E$$

and think of these as stocks of indexes. As with any graph there is some source and target data, namely

$$\sigma(e) \xrightarrow{\ e\ } \tau(e)$$

for each $e \in E$. We view ∇ as a template. For an arbitrary category $\boldsymbol{C}$ we produce a new category $\boldsymbol{C}^\nabla$, the category of ∇-diagrams on $\boldsymbol{C}$.

An object of $\boldsymbol{C}^\nabla$ is a pair

$$\mathsf{A} = \big(A(i) \,|\, i \in N\big) \qquad \mathcal{A} = \big(A(e) \,|\, e \in E\big)$$

an

N-indexed family of objects $\qquad$ E-indexed family of arrows

of $\boldsymbol{C}$, respectively. These families must satisfy

$$A(\sigma(e)) \xrightarrow{A(e)} A(\tau(e))$$

for each $e \in E$.

An arrow of $\boldsymbol{C}^\nabla$

$$(\mathsf{A}, \mathcal{A}) \xrightarrow{\phi} (\mathsf{B}, \mathcal{B})$$

is an N-indexed family of arrows of $\boldsymbol{C}$

$$A(i) \xrightarrow{\phi_i} B(i)$$

such that the $\boldsymbol{C}$-square

$$\begin{tikzcd}
A(\sigma(e)) \arrow[r, "\phi_{\sigma(e)}"] \arrow[d, "A(e)"'] & B(\sigma(e)) \arrow[d, "B(e)"] \\
A(\tau(e)) \arrow[r, "\phi_{\tau(e)}"'] & B(\tau(e))
\end{tikzcd}$$

commutes for each $e \in E$.

Composition of arrows is done componentwise. Given arrows

$$(\mathsf{A}, \mathcal{A}) \xrightarrow{\phi} (\mathsf{B}, \mathcal{B}) \xrightarrow{\psi} (\mathsf{C}, \mathcal{C})$$

in $\boldsymbol{C}^\nabla$, we have components

$$A(i) \xrightarrow{\phi_i} B(i) \xrightarrow{\psi_i} C(i)$$

for each $i \in N$. We take this composite as the i^{th} component of $\psi \circ \phi$.

$$A(i) \xrightarrow{(\psi \circ \phi)_i = \psi_i \circ \phi_i} C(i)$$

Of course, we need to show that this does produce an arrow of $\boldsymbol{C}^\nabla$, in other words that

$$\begin{tikzcd}
A(\sigma(e)) \arrow[r, "(\psi \circ \phi)_{\sigma(e)}"] \arrow[d, "A(e)"'] & C(\sigma(e)) \arrow[d, "C(e)"] \\
A(\tau(e)) \arrow[r, "(\psi \circ \phi)_{\tau(e)}"'] & C(\tau(e))
\end{tikzcd}$$

commutes for each $e \in E$. This square can be decomposed as

$$\begin{tikzcd} A(\sigma(e)) \arrow[r, "\phi_{\sigma(e)}"] \arrow[d, "A(e)"'] & B(\sigma(e)) \arrow[r, "\psi_{\sigma(e)}"] \arrow[d, "B(e)"] & C(\sigma(e)) \arrow[d, "C(e)"] \\ A(\tau(e)) \arrow[r, "\phi_{\tau(e)}"'] & B(\tau(e)) \arrow[r, "\psi_{\tau(e)}"'] & C(\tau(e)) \end{tikzcd}$$

and hence the required result is immediate.

A similar argument shows that this composition is associative. □

1.3.11 We have

$(\boldsymbol{Set} \downarrow \mathsf{1})$ is essentially $\boldsymbol{Set}$

$(\mathsf{1} \downarrow \boldsymbol{Set})$ is essentially $\boldsymbol{Set}_\perp$

$(\boldsymbol{Set} \downarrow \mathsf{2})$ is essentially $\boldsymbol{SetD}$, Sets with a distinguished subset but with a restricted family of arrows

$(\mathsf{2} \downarrow \boldsymbol{Set})$ is essentially Sets with two distinguished points

where the third uses the correspondence between subsets and characteristic functions. Let's look at this third example.

Let

$$\mathsf{2} = \{0, 1\}$$

where here it is useful to think of 1 as 'true' and 0 as 'false'. An object of $\boldsymbol{Set} \downarrow \mathsf{2}$ is a set A with a carried characteristic function α.

$$A \xrightarrow{\alpha} \mathsf{2}$$

This function α gives a subset $X \subseteq A$ where

$$a \in X \Longleftrightarrow \alpha(a) = 1$$

for each $a \in A$. Furthermore this set X determines α since

$$\alpha(a) = \begin{cases} 1 \text{ if } a \in X \\ 0 \text{ if } a \notin X \end{cases}$$

for each $a \in A$. There is a bijective correspondence between characteristic functions carried by A and subsets of A. (If you have never seen this trick before, then take note. This and various generalizations are used throughout mathematics.) This shows that the objects of $\boldsymbol{Set} \downarrow \mathsf{2}$ are precisely the sets with distinguished subset.

What is an arrow of $\boldsymbol{Set} \downarrow \mathsf{2}$?

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & B \arrow[dl, "\beta"] \\ & \mathsf{2} & \end{tikzcd} \qquad (A, X) \xrightarrow{f} (B, Y)$$

On the left we have the official version. It is a function f for which

$$\alpha = \beta \circ f$$

holds. On the right we have the unofficial version. It is a function f with

$$a \in X \Longleftrightarrow \alpha(a) = 1 \Longleftrightarrow \beta(f(a)) = 1 \Longleftrightarrow f(a) \in Y$$

that is an equivalence

$$a \in X \Longleftrightarrow f(a) \in Y$$

for each $a \in A$. An arrow of $\boldsymbol{SetD}$ is a function f with an implication

$$a \in X \Longrightarrow f(a) \in Y$$

for each $a \in A$. Thus the two categories have the same objects but $\boldsymbol{Set} \downarrow \mathsf{2}$ has a more restrictive kind of arrow. □

1.3.12 (a) Consider a composible pair of arrows of $(S \downarrow \boldsymbol{C} \downarrow T)$ as on the left. This gives a commuting diagram as indicated.

$$\begin{tikzcd}
& S \arrow[dl, "\alpha^S"'] \arrow[d, "\beta^S"] \arrow[dr, "\gamma^S"] & \\
A \arrow[r, "f"] \arrow[dr, "\alpha_T"'] & B \arrow[r, "g"] \arrow[d, "\beta_S"] & C \arrow[dl, "\gamma_T"] \\
& T &
\end{tikzcd}
\qquad
\begin{tikzcd}
& S \arrow[dl, "\alpha^S"'] \arrow[dr, "\gamma^S"] & \\
A \arrow[rr, "h"] \arrow[dr, "\alpha_T"'] & & C \arrow[dl, "\gamma_T"] \\
& T &
\end{tikzcd}$$

Let

$$h = g \circ f$$

be the function composite of f and g. To show that is an arrow of $(S \downarrow \boldsymbol{C} \downarrow T)$ we must check that the diagram on the right commutes. This is a simple exercise in diagram chasing (which we look at in more detail in Section 2.1).

(b) An object I of a category is initial if for each object A there is a unique arrow $I \longrightarrow A$. Not every category has such an object, but many do. (The category $\boldsymbol{Set}$ has an initial object, and you might worry a bit about what it is.) If I is an initial object of $\boldsymbol{C}$ then

$$(I \downarrow \boldsymbol{C} \downarrow T) \qquad (\boldsymbol{C} \downarrow T)$$

are essentially the same category.

An object F of a category is final if for each object A there is a unique arrow $A \longrightarrow F$. Not every category has such an object, but many do. (The category $\boldsymbol{Set}$ has a final object, and it is pretty obvious what it is.) If F is a final object of C then

$$(S \downarrow \boldsymbol{C} \downarrow F) \qquad (S \downarrow \boldsymbol{C})$$

are essentially the same category.

We look at initial and final objects in Section 2.4. □

1.4 More complicated categories

1.4.1 Let's look at the composition of arrows. Consider a pair of arrows

$$\mathsf{A} \xrightarrow{\ f\ } \mathsf{B} \xrightarrow{\ g\ } \mathsf{C}$$

of $\widehat{S}$. How might we produce the composite arrow

$$\mathsf{A} \xrightarrow{h = g \circ f} \mathsf{C}$$

in $\widehat{S}$? For each index $i \in S$ we have a pair of functions

$$A(i) \xrightarrow{\ f_i\ } B(i) \xrightarrow{\ g_i\ } C(i)$$

between sets, and we can certainly form the function composite

$$A(i) \xrightarrow{h_i = g_i \circ f_i} C(i)$$

at the index. We show this gives an arrow in $\widehat{S}$.

Consider any pair $j \le i$ of comparable indexes. We have a pair of commuting squares, as on the left

$$\begin{tikzcd}
A(i) \arrow[r, "f_i"] \arrow[d, "A(j,i)"'] & B(i) \arrow[r, "g_i"] \arrow[d, "B(j,i)"] & C(i) \arrow[d, "C(j,i)"] \\
A(j) \arrow[r, "f_j"'] & B(j) \arrow[r, "g_j"'] & C(j)
\end{tikzcd}
\qquad
\begin{tikzcd}
A(i) \arrow[r, "h_i"] \arrow[d, "A(j,i)"'] & C(i) \arrow[d, "C(j,i)"] \\
A(j) \arrow[r, "h_j"'] & C(j)
\end{tikzcd}$$

and we require a commuting square, as on the right. This is a simple exercise in diagram chasing.

There are several more little bits to be done, but all are just as easy.

This is an example of how convenient arrow-theoretic methods can be. If we always had to expose the inner details of these objects and arrows then some calculations would be a mess. By hiding these parts we get a clearer picture of what is going on. There are times when we have to get inside a presheaf, but that doesn't mean we should do it all the time. □

1.4.2 Observe that a chain complex is a special kind of presheaf with $\mathbb{Z}$ as the indexing poset. The connecting arrows are module morphisms with the extra requirement is that if $m + 2 \le n$ then the connecting morphism

$$A_n \longrightarrow A_m$$

is zero. Category theory can bring out similarities that are not so obvious when we have to carry around lots of details. □

1.5 Two simple categories and a bonus

1.5.1 (a) The product as categories is the cartesian product as algebras.
(b) The product as categories is the cartesian product as presets. □

1.5.2 The category $(S \downarrow s)$ is the principal upper section of S above s.
The category $(s \downarrow S)$ is the principal lower section of S below s.
The category $(s \downarrow S \downarrow t)$ is the convex section of S between s and t. This could be empty if $s \ngeq t$. □

1.5.3 The category S^{op} is the poset S turned upside-down.
The category R^{op} is the same set with a new operation $\bar{\star}$ given by

$$r \mathbin{\bar{\star}} s = s \star r$$

for $r, s \in R$. Here $\star$ is the old operation. □

1.5.4 It is the category $\boldsymbol{A} \times \boldsymbol{S}^{\mathrm{op}}$. □

2
Basic gadgetry

2.1 Diagram chasing

2.1.1 For the equational reasoning we need to label more arrows.

$$\begin{tikzcd}
& \bullet \arrow[rr, "g"] \arrow[dr, "p"] & & \bullet \arrow[dr, "h"] & \\
\bullet \arrow[ur, "f"] \arrow[rr, "r"] \arrow[drr, "k"'] & & \bullet \arrow[ur, "q"] \arrow[rr, "s"] & & \bullet \\
& & \bullet \arrow[urr, "l"'] & &
\end{tikzcd}$$

The calculation on the left gives the equational version.

$$h \circ g \circ f = h \circ q \circ p \circ f = s \circ r = l \circ k$$

$$\begin{tikzcd}
& \bullet \arrow[rr, "g"] & & \bullet \arrow[dr, "h"] & \\
\bullet \arrow[ur, "f"] & & & & \bullet \\
& \bullet \arrow[dr, "p"] & & \bullet \arrow[dr, "h"] & \\
\bullet \arrow[ur, "f"] & & \bullet \arrow[ur, "q"] & & \bullet \\
\bullet \arrow[rr, "r"] & & \bullet \arrow[rr, "s"] & & \bullet \\
\bullet \arrow[drr, "k"'] & & & & \bullet \\
& & \bullet \arrow[urr, "l"'] & &
\end{tikzcd}$$

The diagram chase on the right gives the same result. □

2.1.2 We label the arrows as follows with q for the unlabelled arrow.

$$\begin{tikzcd}
& \bullet \arrow[dl, "f"'] \arrow[drr, "h"] \arrow[dr] & & \\
\bullet \arrow[rrr, "g"] \arrow[drr, "p"'] & & & \bullet \\
& & \bullet \arrow[ur, "r"'] &
\end{tikzcd}$$

$$g \circ f = r \circ p \circ f = r \circ q = h$$

Then the calculation on the right gives the required result. □

2.1.3 A trip twice round the pentagram is given by the sequence

$$12345123451$$

of corners. Because various triangles collapse the result is given by

$$\begin{array}{r} \underline{123}45123451 \\ 1\underline{345}123451 \\ 13\underline{512}3451 \\ 135\underline{234}51 \\ 1352\underline{451} \\ 135241 \end{array}$$

where at each step the underline indicates the triangle that collapses. □

2.2 Monics and epics

2.2.1 (a) For instance, consider a section s which is also epic. Since s is a section we have a composite

$$B \xrightarrow{s} A \xrightarrow{r} B \qquad r \circ s = id_B$$

which is an identity. This also shows that the parallel pair of arrows

$$\begin{tikzcd} B \arrow[r, "s"] & A \arrow[r, "r"] \arrow[rr, bend right, "id_A"'] & B \arrow[r, "s"] & A \end{tikzcd}$$

agree, and hence

$$s \circ r = id_A$$

since s is epic. This show that r is the inverse of s.

(b) Simplify $h \circ f \circ g$ in two ways. □

2.2.2 (a) In a preset there is no more than one arrow

$$i \longrightarrow j$$

between a given pair of elements. Thus for any parallel pair

$$\begin{tikzcd} i \arrow[r, shift left] \arrow[r, shift right] & j \end{tikzcd}$$

the two arrows are equal. This shows that every arrow is monic and epic.

(b) A poset is balanced precisely when it is discrete. A preset is balanced precisely when the comparison is an equivalence. □

2.2.3 An element is monic or epic if it is cancellable on the appropriate side.

An element is a retraction or a section if it has a one sided inverse on the appropriate side.

An element is an isomorphism if it has a two sided inverse

A monoid is balanced precisely when the set of cancellable elements is a group. □

2.2.4 To help with both parts it is convenient to use a slightly different notation. We re-name the arrows

$$A \xrightarrow{f} B \xrightarrow{g} C$$

and we verify the following.

(m, i) If both f, g are monic then so is $g \circ f$.

(m, ii) If $g \circ f$ is monic then so is f.

(m, iii) If both f, g are split monics then so is $g \circ f$.

(m, iv) If $g \circ f$ is a split monic then so is f.

(e, i) If both f, g are epic then so is $g \circ f$.

(e, ii) If $g \circ f$ is epic then so is g.

(e, iii) If both f, g are split epics then so is $g \circ f$.

(e, iv) If $g \circ f$ is a split epic then so is g.

(c) There is an example where $g \circ f$ is an isomorphism but where f is not epic and g is not monic.

Because of the symmetry it is sufficient to verify an appropriate half of (m, i-iv) and (e, i-iv).

(m, i) Suppose both f, g are monic and consider a parallel pair of arrows

$$\begin{tikzcd} X \arrow[r, shift left, "k"] \arrow[r, shift right, "l"'] & A \end{tikzcd}$$

where the composites

$$\begin{tikzcd} X \arrow[r, shift left, "g \circ f \circ k"] \arrow[r, shift right, "g \circ f \circ l"'] & A \end{tikzcd}$$

agree. Since g is monic the parallel pair

$$\begin{tikzcd} X \arrow[r, shift left, "f \circ k"] \arrow[r, shift right, "f \circ l"'] & A \end{tikzcd}$$

agree, and hence since f is monic we have $k = l$, as required.

(e, ii) Suppose $g \circ f$ is epic and consider a parallel pair

$$C \overset{k}{\underset{l}{\rightrightarrows}} X$$

where the composites

$$B \overset{k \circ g}{\underset{l \circ g}{\rightrightarrows}} X$$

agree. Then the composites

$$B \overset{k \circ g \circ f}{\underset{l \circ g \circ f}{\rightrightarrows}} X$$

agree, and hence $k = l$ since $g \circ f$ is epic.

(m, iii) Suppose both f, g are split monics. Thus we have

$$r \circ f = id_A \qquad s \circ g = id_B$$

for arrows

$$\begin{tikzcd} A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "r"] \arrow[r, shift left, "g"] & C \arrow[l, shift left, "s"] \end{tikzcd}$$

as indicated. But now

$$r \circ s \circ g \circ f = r \circ id_B \circ f = r \circ f = id_A$$

to provide the required one-sided inverse of $g \circ f$.

(e, iv) Suppose $g \circ f$ is a split epic. Thus

$$g \circ f \circ s = id_B$$

for some arrow

$$A \xleftarrow[s]{} C$$

as indicated. But now $f \circ s$ provides the required one-sided inverse of g.

(c) We work in the category ***Set*** of sets. Consider any functions

$$A \xrightarrow{f} B \xrightarrow{g} C$$

where A, C are singleton sets but B is larger. In particular, f is an embedding and hence monic, g is surjective and hence epic, and $g \circ f$ is an isomorphism. Because of the size of B we see that f is not surjective and g is not injective. But, by Exercise 2.2.7 (slightly later), in ***Set*** we have 'monic=injective' so that g is not monic. Also, by a simple argument (which you should sort out) in ***Set*** we have 'epic=surjective' so that f is not epic. □

2.2.5 It will help if we get a bit of notation sorted out. Let

$$(A, \cdot, \iota)$$

be an arbitrary monoid written multiplicatively. Here we will display the operation symbol. Consider any monoid morphism

$$(\mathbb{Z}, +, 0) \xrightarrow{f} (A, \cdot, \iota)$$

from the additively written monoid $\mathbb{Z}$. Thus

$$f(0) = \iota \qquad f(m+n) = f(m) \cdot f(n)$$

for all $m, n \in \mathbb{Z}$. Consider any situation

$$\mathbb{N} \overset{e}{\hookrightarrow} \mathbb{Z} \underset{g}{\overset{f}{\rightrightarrows}} A \quad \text{where} \qquad f \circ e = g \circ e$$

that is

$$f(m) = g(m)$$

for all $m \in \mathbb{N}$. We require $f = g$, that is

$$f(-m) = g(-m)$$

for all $m \in \mathbb{N}$. But, taking it slowly, for $m \in \mathbb{N}$ we have

$$\begin{aligned}
g(-m) &= g(-m) \cdot \iota \\
&= g(-m) \cdot f(0) \\
&= g(-m) \cdot f(m + (-m)) \\
&= g(-m) \cdot f(m) \cdot f(-m)) \\
&= g(-m) \cdot g(m) \cdot f(-m)) \\
&= g(-m + m) \cdot f(-m)) \\
&= g(0) \cdot f(-m)) \\
&= \iota \cdot f(-m)) \qquad\qquad = f(-m)
\end{aligned}$$

as required. Of course, the central equality is the crucial step. □

2.2.6 The format for this solution is like that of Solution 2.2.5, but now we have more algebraic identities we can use. Consider a situation

$$\mathbb{Z} \overset{e}{\hookrightarrow} \mathbb{Q} \underset{g}{\overset{f}{\rightrightarrows}} A \quad \text{where} \qquad f \circ e = g \circ e$$

in ***Rng***, that is

$$f(m) = g(m)$$

for all $m \in \mathbb{Z}$. We require $f = g$, that is

$$f\left(\frac{m}{n}\right) = g\left(\frac{m}{n}\right)$$

for all $m, n \in \mathbb{Z}$ with $n \neq 0$. Consider any non-zero $n \in \mathbb{Z}$. We have

$$g\left(\frac{1}{n}\right) = g\left(\frac{1}{n}\right) \cdot f(n) \cdot f\left(\frac{1}{n}\right) = g\left(\frac{1}{n}\right) \cdot g(n) \cdot f\left(\frac{1}{n}\right) = f\left(\frac{1}{n}\right)$$

and hence

$$g\left(\frac{m}{n}\right) = g(m) \cdot g\left(\frac{1}{n}\right) = f(m) \cdot f\left(\frac{1}{n}\right) = f\left(\frac{m}{n}\right)$$

as required. □

2.2.7 Suppose the category $\boldsymbol{C}$ of structured sets has a selector $(S, \star)$. Consider any monic in $\boldsymbol{C}$.

$$A \xrightarrow{m} B$$

Consider $a_1, a_2 \in A$ with $m(a_1) = m(a_2)$. We show $a_1 = a_2$, and hence show that m is injective. Consider the parallel pair

$$\begin{tikzcd} S \arrow[r, shift left, "\alpha_1"] \arrow[r, shift right, "\alpha_2"'] & A \end{tikzcd}$$

with $\alpha_1(\star) = a_1$ and $\alpha_2(\star) = a_2$. Each of the two composites

$$\begin{tikzcd} S \arrow[r, shift left, "m \circ \alpha_1"] \arrow[r, shift right, "m \circ \alpha_2"'] & B \end{tikzcd}$$

is uniquely determined by its value at $\star$. But

$$(m \circ \alpha_1)(\star) = m(a_1) = m(a_2) = (m \circ \alpha_2)(\star)$$

so that

$$m \circ \alpha_1 = m \circ \alpha_2 \quad \text{to give} \quad \alpha_1 = \alpha_2$$

(since m is monic), and hence

$$a_1 = \alpha_1(\star) = \alpha_2(\star) = a_2$$

as required.

(b) It suffices to exhibit a selector for each of the categories.
For $\boldsymbol{Set}$, $\boldsymbol{Pos}$, $\boldsymbol{Top}$ the 1-element structure will do.
For $\boldsymbol{Mon}$ we use the monoid $(\mathbb{N}, +, 0)$ with $\star = 0$.
For $\boldsymbol{Grp}$ we use the group $(\mathbb{Z}, +, 0)$ with $\star = 1$.
For $\boldsymbol{Rng}$ we use the ring of polynomials $\mathbb{Z}[X]$ with $\star = X$.
For $\boldsymbol{Set}$-R we use R itself with $\star = 1$. □

2.2.8 (a) An isomorphism in $\boldsymbol{Top}$ is usually called a homeomorphism.
See Exercise 2.2.7.

Consider a topological space S and let S^d be the set S as a discrete space. The identity function on the set S is a bijective continuous map

$$S^d \longrightarrow S$$

but is not a homeomorphism (unless S is discrete).

(b) Let's prove the general result. Consider any situation in $\boldsymbol{Top}_2$

$$\begin{tikzcd} T \arrow[r, "\epsilon"] & S \arrow[r, shift left, "\phi"] \arrow[r, shift right, "\psi"'] & R \end{tikzcd}$$

where $\epsilon[T]$ is dense in S and where

$$\phi \circ \epsilon = \psi \circ \epsilon$$

holds. We require $\phi = \psi$.

By way of contradiction suppose $\phi \neq \psi$ so that $\phi(s) \neq \psi(s)$ for some $s \in S$. Since R is T_2, this gives

$$\phi(s) \in U \qquad \psi(s) \in V \qquad U \cap V = \emptyset$$

for some pair U, V of open sets of S. We have

$$s \in \phi^{\leftarrow}(U) \cap \psi^{\leftarrow}(V)$$

and both these sets are open in S. The intersection is non-empty, and so must meet $\epsilon[T]$ (since $\epsilon[T]$ is dense in S). This gives some $t \in T$ with

$$\epsilon(t) \in \phi^{\leftarrow}(U) \cap \psi^{\leftarrow}(V) \quad \text{that is} \quad (\phi \circ \epsilon)(t) \in U \qquad (\psi \circ \epsilon)(t) \in V$$

which is the contradiction since $\phi \circ \epsilon = \psi \circ \epsilon$ and $U \cap V = \emptyset$. $\square$

2.2.9 (e) We produce a sequence of equalities

$$j \circ f \circ e = j \circ g \circ b = \cdots = m \circ l \circ e$$

by passing across each face in turn. Since e is epic this gives

$$j \circ f = m \circ l$$

are required.

(m) A dual version of (e). $\square$

2.2.10 We label the arrows and various cells as shown.

$$\begin{tikzcd}
\bullet \arrow[rrr, "f"] \arrow[ddd, "k"'] \arrow[dr, "e"] & & & \bullet \arrow[ddd, "g"] \arrow[dl, "p"'] \\
& \bullet \arrow[r, "a"] \arrow[d, "d"] \arrow[ur, phantom, "(1)"] \arrow[dl, phantom, "(4)"] \arrow[dr, phantom, "(5)"] & \bullet \arrow[d, "b"] \arrow[dr, phantom, "(2)"] & \\
& \bullet \arrow[r, "c"] \arrow[dr, phantom, "(3)"] & \bullet \arrow[dr, "m"] & \\
\bullet \arrow[rrr, "h"'] \arrow[ur, "q"] & & & \bullet
\end{tikzcd}$$

We are given that cells (1, 2, 3, 4) commute.

(a) We are also given that (5) commutes. Then (2, 1, 5, 4, 3) gives

$$g \circ f = m \circ b \circ p \circ f = m \circ c \circ d \circ f = m \circ c \circ q \circ k = h \circ k$$

to show that the outer cell commutes. You should also look at this in the form of a diagram chase

(b) We have

$$m \circ b \circ a \circ e = m \circ b \circ p \circ f = g \circ f \qquad m \circ c \circ d \circ e = m \circ c \circ q \circ k = h \circ k$$

using (1, 2) on the left hand side and (4, 3) on the right hand side. Assuming the outer square commutes this gives

$$m \circ b \circ a \circ e = g \circ f = h \circ k = m \circ c \circ d \circ e$$

and hence

$$b \circ a = c \circ d$$

by the assumed cancellative properties of m and e. □

2.3 Simple limits and colimits

2.3.1 For an arbitrary subset X of a poset the

limit colimit

is denoted

$$\bigwedge X \qquad \bigvee X$$

and called the

greatest lower bound or infimum least upper bound or supremum

of X, provided these exists, of course.

When X is empty we have

$$\bigwedge \emptyset = \top \text{ (top)} \qquad \bigvee \emptyset = \bot \text{ (bottom)}$$

of the poset.

When X is a singleton we have

$$\bigwedge X = \{s\} = \bigvee X$$

where s is the unique member of X.

When $X = \{a, b\}$ we have

$$\bigwedge X = a \wedge b \qquad \bigvee X = a \vee b$$

the

meet join

of the pair.

When S is a preset each of these notions may not determine a unique element, only a family of equivalent elements. □

2.4 Initial and final objects

2.4.1 Consider any initial object I in a category. Since I is initial there is a unique endo-arrow

$$I \longrightarrow I$$

on I. We already know one example of such an arrow, namely the identity arrow id_I. Thus this is the only endo-arrow on I.

Consider any pair I, J of initial objects. There are unique arrows

$$I \xrightarrow{f} J \qquad\qquad J \xrightarrow{g} I$$

since

I is initial $\qquad$ J is initial

respectively. The composite

$$g \circ f \qquad\qquad f \circ g$$

is an endo-arrow on

$$I \qquad\qquad J$$

respectively. By the previous observation we have

$$g \circ f = id_I \qquad\qquad f \circ g = id_J$$

and hence f, g are an inverse pair of isomorphisms.

If F, G are two final objects, then there is a unique arrow $F \longrightarrow G$, and this is an isomorphism. This is proved in exactly the same way, we simply think of the arrows as pointing in the other direction. Equivalently, we apply the 'initial' result to the opposite category. □

2.4.2 By the uniqueness of mediators the only endo-arrow of I is id_I.

Consider any arrow

$$A \xrightarrow[r]{} I$$

and let

$$A \xleftarrow{s} I$$

be the unique arrow given by the initial property of I. Then $r \circ s$ is an endo-arrow of I, and hence

$$r \circ s = id_I$$

by the remark in Solution 2.4.1.

By duality, each arrow

$$F \xrightarrow{s} F$$

from a final object is a section.

Consider any arrow

$$F \xrightarrow{\quad f \quad} I$$

passing from a final object to an initial object. From above we know there are arrows

$$f \circ s = id_I \qquad \begin{tikzcd} F & I \arrow[l, shift right, "r"'] \arrow[l, shift left, "s"] \end{tikzcd} \qquad r \circ f = id_F$$

and the usual argument gives $s = r$, so that f is an isomorphism. □

2.4.3 Exercise 1.2.3 shows that $(\mathbb{N}, \mathrm{succ}, 0)$ is the initial object of $\boldsymbol{Pno}$. This fact is equivalent to the Peano axioms.

The trivial object, with just one element, is final. □

2.4.4 The trivial group is both initial and final in $\boldsymbol{Grp}$.

The ring $\mathbb{Z}$ of integers is initial in $\boldsymbol{Rng}$. The trivial ring, with $1 = 0$, is final.

The ring $\mathbb{Z}$ of integers is initial in $\boldsymbol{Idm}$. There is no final object (assuming that $1 \neq 0$ must hold in an integral domain).

There is neither an initial object not a final object in $\boldsymbol{Fld}$. However, if we fix the characteristic then there is an initial object. □

2.4.5 In $\boldsymbol{Set}$ the final object $\mathsf{1}$ is the singleton set. It doesn't matter what its unique element is, so let

$$\mathsf{1} = \{\star\}$$

here. Each function

$$\mathsf{1} \xrightarrow{\quad \alpha \quad} A$$

is uniquely determined by its only value

$$\alpha(\star)$$

which is an element of A, and every element is the unique value of some such function. Thus we have a bijection between

$$\boldsymbol{Set}[\mathsf{1}, A] \qquad A$$

as required. Now consider any composite

$$\mathsf{1} \xrightarrow{\quad \alpha \quad} A \xrightarrow{\quad f \quad} B$$

where α corresponds to the element $a \in A$, that is $\alpha(\star) = a$. The composite $f \circ \alpha$ corresponds to the element

$$(f \circ \alpha)(\star) \in B$$

and this is just

$$f(\alpha(\star)) = f(a)$$

as required. □

2.4.6 (a) The presheaf with a singleton for each component set is the final object $\mathsf{1}$.

(b) A global element

$$\mathsf{1} \longrightarrow A$$

of a presheaf $A = (\mathsf{A}, \mathcal{A})$ selects an element

$$a(i) \in A(i)$$

from each component set. This choice function $a(\cdot)$ must satisfy

$$A(j,i)\big(a(i)\big) = a(j)$$

for each $j \leq i$. □

2.5 Products and coproducts

2.5.1 The 'algebraic' categories are straight forward. In each case we take the cartesian product of the two carrying sets and then furnish this in a fairly obvious way.

You may not have seen products in $\boldsymbol{Pos}$ before but they are constructed in the obvious way using cartesian products.

You will have seen products on $\boldsymbol{Top}$ before, and may have been puzzled by the strange construction of the product topology. The categorical description explains this. Let S and T be a pair of topological spaces. We require a space $S \times T$ and a pair

$$\begin{tikzcd} & S \times T \arrow[dl, "p"'] \arrow[dr, "q"] & \\ S & & T \end{tikzcd}$$

of continuous maps where this $\boldsymbol{Top}$-wedge has a certain universal property. We take the cartesian product $S \times T$ of the two sets. For the topological furnishings let's try the smallest topology on $S \times T$ for which both projections are continuous. Thus we take the smallest topology on $S \times T$ for which each inverse image

$$p^{\leftarrow}(U) \quad \text{for } U \in \mathcal{O}S \qquad q^{\leftarrow}(V) \quad \text{for } V \in \mathcal{O}T$$

is open. This gives a subbase of the usual product topology. Why does this give a product wedge in $\boldsymbol{Top}$?

Consider any wedge

$$\begin{tikzcd} & R \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ S & & T \end{tikzcd}$$

in $\boldsymbol{Top}$. Forget the topology for a moment. We have a wedge in $\boldsymbol{Set}$ and a product wedge in $\boldsymbol{Set}$. Thus there is a *unique* function θ such that

$$\begin{tikzcd} & R \arrow[dl, "\phi"'] \arrow[d, "\theta"] \arrow[dr, "\psi"] & \\ S & S \times T \arrow[l, "p"] \arrow[r, "q"'] & T \end{tikzcd}$$

commutes. It suffices to show that this function θ is continuous, for then we have a product wedge in $\boldsymbol{Top}$. To do that it suffices to show that $\theta^{\leftarrow}(W)$ is open in R for each subbasic open set W of $S \times T$. There are two kinds of such sets, and both are dealt with by the same argument. For instance consider $W = p^{\leftarrow}(U)$ for some $U \in \mathcal{O}S$. For each point $r \in R$ we have

$$\begin{aligned} r \in \theta^{\leftarrow}(W) &\Longleftrightarrow \theta(r) \in W = p^{\leftarrow}(U) \\ &\Longleftrightarrow (p \circ \theta)(r) \in U \\ &\Longleftrightarrow \phi(r) \in U \qquad\qquad \Longleftrightarrow r \in \phi^{\leftarrow}(U) \end{aligned}$$

so that

$$\theta^{\leftarrow}(W) = \phi^{\leftarrow}(U)$$

which is open in R. □

2.5.2 The coproduct for each of

$$\boldsymbol{Set},\ \boldsymbol{Pos},\ \boldsymbol{Set}\text{-}R,\ \boldsymbol{Top}$$

can be obtained as a furnished disjoint union with the obvious insertions.

The coproduct for each of

$$\boldsymbol{CMon},\ \boldsymbol{AGrp},\ \boldsymbol{Mod}\text{-}R$$

can be obtained as a furnished cartesian product with the obvious insertions.

The coproduct for each of

$$\boldsymbol{Mon},\ \boldsymbol{Grp},\ \boldsymbol{CRng},\ \boldsymbol{Rng}$$

is formed by a more complicated construction. Details are given in Exercise 4.7.3 □

2.5.3 For two elements a, b of a poset (with arrows pointing upwards) the

meet $a \wedge b$ join $a \vee b$

is the

product   coproduct

of the pair. □

2.5.4 For $\boldsymbol{Set}_\perp$-objects A and B the product in $\boldsymbol{Set}_\perp$ is given by the cartesian product $A \times B$ with the obvious projections. The distinguished element of $A \times B$ is $(\perp, \perp)$. The proof of this is easier than, for instance, the $\boldsymbol{Mon}$ case.

The coproduct is more interesting. Let

$$A \amalg B = \big((A - \{\perp\}) + (B - \{\perp\}\big) \cup \{\perp\}$$

the disjoint union of the two point depleted sets with a point attached. This set has three kinds of elements

$$(a, 0) \text{ for } a \in A - \{\perp\} \qquad (b, 1) \text{ for } b \in B - \{\perp\} \qquad \perp$$

and, of course, $\perp$ is the distinguished point. The function

$$\begin{aligned} A &\xrightarrow{\ i\ } A \amalg B \\ a &\longmapsto (a, 0) \qquad \text{for } a \in A - \{\perp\} \\ \perp &\longmapsto \perp \end{aligned}$$

is an arrow of $\boldsymbol{Set}_\perp$, and there is a similar arrow

$$B \xrightarrow{j} A \amalg B$$

from B. These furnish $A \amalg B$ as the coproduct. The proof is similar to that for $\boldsymbol{Set}$. $\square$

2.5.5 Let $\boldsymbol{SetD}$ be the category of sets each with a distinguished subset. The product is constructed in routine way using cartesian products. However, it is worth looking at some of the details.

This is one of the places where it is useful to distinguish between a structure and its carrying set. Thus let

$$\mathcal{A} = (A, X) \qquad \mathcal{B} = (B, Y)$$

be a pair of objects of $\boldsymbol{SetD}$. Let

$$\mathcal{A} \times \mathcal{B} = (A \times B, X \times Y)$$

so this is certainly an object of $\boldsymbol{SetD}$. By dropping down to $\boldsymbol{Set}$ consider the two projection *functions*, as on the left. This is a wedge in $\boldsymbol{Set}$.

$$\begin{tikzcd} & A \\ A \times B \arrow[ur, "p"] \arrow[dr, "q"'] & \\ & B \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{A} \\ \mathcal{A} \times \mathcal{B} \arrow[ur, "p"] \arrow[dr, "q"'] & \\ & \mathcal{B} \end{tikzcd}$$

In fact, it is a product wedge in $\boldsymbol{Set}$. We easily check that p and q are arrows of $\boldsymbol{SetD}$, so we have a wedge in $\boldsymbol{SetD}$, as on the right. We show this is a product wedge in $\boldsymbol{SetD}$.

Consider any object $\mathcal{C} = (C, Z)$ of $\boldsymbol{SetD}$ and wedge of $\boldsymbol{SetD}$ arrows, as on the left.

$$\begin{tikzcd} & & \mathcal{A} \\ \mathcal{C} \arrow[urr, "f"] \arrow[drr, "g"'] & \mathcal{A} \times \mathcal{B} \arrow[ur, "p"] \arrow[dr, "q"'] & \\ & & \mathcal{B} \end{tikzcd} \quad \begin{tikzcd} & & A \\ C \arrow[urr, "f"] \arrow[drr, "g"'] & A \times B \arrow[ur, "p"] \arrow[dr, "q"'] & \\ & & B \end{tikzcd} \quad \begin{tikzcd} & & A \\ C \arrow[urr, "f"] \arrow[drr, "g"'] \arrow[r, "h"] & A \times B \arrow[ur, "p"] \arrow[dr, "q"'] & \\ & & B \end{tikzcd}$$

By forgetting the carried structure we obtain a wedge of $\boldsymbol{Set}$ arrows, as in the middle. But (p, q) are a product wedge in $\boldsymbol{Set}$, so we obtain a unique mediating $\boldsymbol{Set}$-arrow, a function h, as on the right. It suffices to show that h is a $\boldsymbol{SetD}$ arrow. That is a routine calculation.

The construction of the coproduct is not so obvious, but once we have seen the product construction we can dualize. Let

$$\mathcal{A} = (A, X) \qquad \mathcal{B} = (B, Y)$$

be a pair of objects of $\boldsymbol{SetD}$. Recall that in $\boldsymbol{Set}$ the coproduct

$$A + B = \big(A \times \{0\}\big) \cup \big(B \times \{1\}\big)$$

is the union of A and B where these sets have been tagged to make them disjoint. The union

$$X + Y = \big(X \times \{0\}\big) \cup \big(Y \times \{1\}\big)$$

is a subset of $A + B$, and so

$$\mathcal{A} + \mathcal{B} = (A + B, X + Y)$$

is an object of $\boldsymbol{SetD}$. Consider the two insertions as on the left.

$$\begin{tikzcd}
A \arrow[dr, "i"] & \\
& A+B \\
B \arrow[ur, "j"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A} \arrow[dr, "i"] & \\
& \mathcal{A}+\mathcal{B} \\
\mathcal{B} \arrow[ur, "j"'] &
\end{tikzcd}$$

This is a coproduct wedge in $\boldsymbol{Set}$. It is easy to check that i and j are arrow of $\boldsymbol{SetD}$, so we have a wedge in $\boldsymbol{SetD}$, as on the right. By mimicking the proof for the product wedge with the arrows reversed, we see that we have a coproduct wedge in $\boldsymbol{SetD}$. □

2.5.6 This is a teaser which almost everyone gets wrong the first time. For sets A and B the product and coproduct in $\boldsymbol{RelA}$ are both carried by the same set, but this is *not* the cartesian product $A \times B$. It is

$$A + B$$

the disjoint union of the sets. The members of $A + B$ are tagged members of A and B. Thus $A + B$ has two kinds of elements

$$(a, 0) \text{ for } a \in A \qquad (b, 1) \text{ for } b \in B$$

where the tag records where the element came from. We set up relations

$$\begin{tikzcd}
A \arrow[r, shift right, "I"'] & A+B \arrow[l, shift right, "P"'] \arrow[r, shift left, "Q"] & B \arrow[l, shift left, "J"]
\end{tikzcd}$$

and show that P, Q form a product wedge, and I, J form a coproduct wedge. With z ranging over $A + B$ and $a \in A, b \in B$ we let

$$aPz \iff z = (a, 0) \iff zIa \qquad bQz \iff z = (b, 1) \iff zJb$$

to produce the relations.

To show that P, Q form a product wedge consider any wedge from an arbitrary set X, as on the left. We require a pair of commuting triangles

$$\begin{tikzcd}
& A \\
X \arrow[ur, "F"] \arrow[dr, "G"'] & \\
& B
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \\
X \arrow[ur, "F"] \arrow[dr, "G"'] \arrow[r, "M"] & A+B \arrow[u, "P"'] \arrow[d, "Q"] \\
& B
\end{tikzcd}$$

for some unique relation M, as on the right.

Remembering that $z \in A + B$ can have only two forms, we see that

$$zMx \iff \begin{cases} (\exists a \in A)[z = (a, 0) \;\&\; aFx] \\ \qquad\qquad \text{or} \\ (\exists b \in B)[z = (b, 1) \;\&\; bGx] \end{cases}$$

gives a relation M of the correct type. For $a \in A$ and $x \in X$ we have

$$a(P \circ M)x \iff (\exists z)[aPzMx] \iff (a,0)Mx \iff aFx$$

to show that

$$P \circ M = F$$

and hence the top triangle commutes. A similar argument shows that the bottom triangle commutes.

To show the uniqueness of this mediating relation consider any relation

$$X \xrightarrow{N} A + B$$

where both

$$P \circ N = F \qquad Q \circ N = G$$

hold. For $a \in A$ and $x \in X$ we have

$$(a,0)Nx \iff aP(a,0)Nx \iff (\exists z)[aPzNx] \iff a(P \circ N)x \iff aFx$$

and for $b \in B$ we have

$$(b,1)Nx \iff bGx$$

by a similar argument. This gives

$$N = M$$

for the required uniqueness.

The verification that I, J form a coproduct wedge is similar. $\square$

2.5.7 (a) We first make a general observation. Consider two instances of the same product wedge, as on the left.

$$\begin{tikzcd}
 & A \\
A \times B \arrow[ur, "p"] \arrow[dr, "q"'] & A \times B \arrow[u, "p"'] \arrow[d, "q"] \\
 & B
\end{tikzcd}
\qquad\qquad
A \times B \xrightarrow{r} A \times B$$

The only arrow r, as on the right, to make the triangles commute is $id_{A\times B}$. This is because mediators are unique.

With this we can show that $A \times \mathsf{1}$ and A are isomorphic.

Consider the diagram on the left. Here p, q form the product wedge and g is the unique arrow to $\mathsf{1}$. There is a unique mediator m such that

$$\begin{tikzcd}
 & A \\
A \arrow[ur, "p"] \arrow[dr, "g"'] & A \times \mathsf{1} \arrow[u, "p"'] \arrow[d, "q"] \\
 & \mathsf{1}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & & A \\
A \times \mathsf{1} \arrow[r, "p"] \arrow[urr, "p"] \arrow[drr] & A \arrow[r, "m"] \arrow[ur, "id_A"] \arrow[dr, "g"'] & A \times \mathsf{1} \arrow[u, "p"'] \arrow[d, "q"] \\
 & & \mathsf{1}
\end{tikzcd}$$

the equalities

$$p \circ m = id_A \qquad q \circ m = g$$

hold. Now consider the commuting diagram on the right. (The bottom cell commutes by the nature of $\mathbf{1}$.) By the first observation we have

$$m \circ p = id_{A \times \mathbf{1}}$$

and hence p, m are an inverse pair of isomorphisms.

(b) Consider the diagram

$$\begin{tikzcd}
A \times B \arrow[rr, "\alpha"] \arrow[dr, "\beta"] & & A \\
(A \times B) \times C = L \arrow[u, "\lambda"] \arrow[d, "\mu"'] & B & R = A \times (B \times C) \arrow[u, "\sigma"'] \arrow[d, "\rho"] \\
C & & B \times C \arrow[ll, "\gamma"] \arrow[ul, "\delta"']
\end{tikzcd}$$

where L, R are the two objects of interest and each arrow is one of the structuring projections of one of the product wedges. There are no commuting cells in this diagram. We insert four mediating arrows.

Firstly we obtain

$$L \xrightarrow{\eta} B \times C \qquad\qquad A \times B \xleftarrow{\zeta} R$$

with

$$\begin{array}{llll} (1) & \delta \circ \eta = \beta \circ \lambda & (3) & \beta \circ \zeta = \delta \circ \rho \\ (2) & \gamma \circ \eta = \mu & (4) & \alpha \circ \zeta = \sigma \end{array}$$

respectively. Notice that (1, 2) uniquely determine η, and (3, 4) uniquely determine ζ. Secondly we obtain

$$L \xrightarrow{\phi} R \qquad\qquad L \xleftarrow{\psi} R$$

with

$$\begin{array}{llll} (5) & \sigma \circ \phi = \alpha \circ \lambda & (7) & \mu \circ \psi = \gamma \circ \rho \\ (6) & \rho \circ \phi = \eta & (8) & \lambda \circ \psi = \zeta \end{array}$$

respectively. Notice that (5,6) uniquely determine ϕ, and (7, 8) uniquely determine ψ. We show that ϕ and ψ are an inverse pair of isomorphisms.

For the diagram

$$\begin{tikzcd}
 & A \\
L \arrow[ur, "\alpha \circ \lambda"] \arrow[dr, "\beta \circ \lambda"'] & A \times B \arrow[u, "\alpha"'] \arrow[d, "\beta"] \\
 & B
\end{tikzcd}$$

the unique mediator must be λ. But with

$$\xi = \psi \circ \phi$$

we have

$$\begin{aligned} \alpha \circ \xi &= \alpha \circ \lambda \circ \psi \circ \phi = \alpha \circ \zeta \circ \phi = \sigma \circ \phi \qquad = \alpha \circ \lambda \\ \beta \circ \lambda \circ \xi &= \beta \circ \lambda \circ \psi \circ \phi = \beta \circ \zeta \circ \phi = \delta \circ \rho \circ \phi = \delta \circ \eta = \beta \circ \lambda \end{aligned}$$

using (8, 4, 5) on the top line, and (8, 3, 6, 1) on the bottom line. Thus

$$\lambda \circ \xi = \lambda$$

since we have just verified that $\lambda \circ \xi$ has the required mediating property.

For the diagram on the left the unique mediator must be id_L.

$$\begin{tikzcd} & A \times B \\ L \arrow[ur, "\lambda"] \arrow[dr, "\mu"'] & L \arrow[u, "\lambda"'] \arrow[d, "\mu"] \\ & C \end{tikzcd} \qquad \begin{aligned} \mu \circ \xi &= \mu \circ \psi \circ \phi \\ &= \gamma \circ \rho \circ \phi \\ &= \gamma \circ \eta \qquad = \mu \end{aligned}$$

But using (7, 6, 2) we have the equalities on the right. This with the previous equality gives

$$\psi \circ \phi = \xi = id_L$$

which is half of what we want. The other required equality

$$\phi \circ \psi = id_R$$

follows by a similar argument. □

2.5.8 Let

$$\begin{aligned} L_1 &= A \times C \qquad & R_1 &= A + B \\ L_2 &= B \times C \qquad & R_2 &= C \end{aligned}$$

so that

$$L = L_1 + L_2 \qquad R = R_1 \times R_2$$

are the two component objects. Let

$$\begin{array}{cc} L_1 \xrightarrow{\alpha} A & A \xrightarrow{\iota_A} R_1 \\ L_1 \xrightarrow{\gamma_1} C & B \xrightarrow{\iota_B} R_1 \\ L_2 \xrightarrow{\beta} B & \\ L_2 \xrightarrow{\gamma_2} C & \\ R \xrightarrow{\rho_1} R_1 & L_1 \xrightarrow{\lambda_1} L \\ R \xrightarrow{\rho_2} R_2 & L_2 \xrightarrow{\lambda_2} L \end{array}$$

be the

projections insertions

which structure the various objects as

products coproducts

respectively. We can fit these arrows together in two ways. Let's look at both possibilities in parallel.

We have arrows

$$L_1 \xrightarrow{\delta_{11} = \iota_A \circ \alpha} R_1 \qquad L_2 \xrightarrow{\delta_{12} = \iota_B \circ \beta} R_1$$
$$L_1 \xrightarrow{\delta_{21} = \gamma_1} R_2 \qquad L_2 \xrightarrow{\delta_{22} = \gamma_2} R_2$$

which give commuting triangles

$$\begin{tikzcd} L_1 \arrow[d, "\lambda_1"'] \arrow[dr, "\delta_{j1}"] & \\ L \arrow[r, "\mu_j"] & R_j \\ L_2 \arrow[u, "\lambda_2"] \arrow[ur, "\delta_{j2}"'] & \end{tikzcd} \qquad \begin{tikzcd} & R_1 \\ L_i \arrow[ur, "\delta_{1i}"] \arrow[r, "\nu_i"] \arrow[dr, "\delta_{2i}"'] & R \arrow[u, "\rho_1"'] \arrow[d, "\rho_2"] \\ & R_2 \end{tikzcd}$$

for $i, j \in \{1, 2\}$. In other words

We use the coproduct properties of L to produce a unique mediator μ_j

We use the product properties of R to produce a unique mediator ν_j

for the various cases. Observe that

$$\mu_j \circ \lambda_i = \delta_{ji} = \rho_j \circ \nu_i$$

(for $i, j \in \{1, 2\}$) uniquely determined μ_j and ν_i in terms of the δ_{ji}, and these in turn are determined by the given structuring arrows.

Next we swap the roles of L and R to obtain commuting triangles

$$\begin{tikzcd} & R_1 \\ L \arrow[ur, "\mu_1"] \arrow[r, "\mu"] \arrow[dr, "\mu_2"'] & R \arrow[u, "\rho_1"'] \arrow[d, "\rho_2"] \\ & R_2 \end{tikzcd} \qquad \begin{tikzcd} L_1 \arrow[d, "\lambda_1"'] \arrow[dr, "\nu_1"] & \\ L \arrow[r, "\nu"] & R \\ L_2 \arrow[u, "\lambda_2"] \arrow[ur, "\nu_2"'] & \end{tikzcd}$$

for unique mediators μ and ν. These are determined by

$$\mu_j = \rho_j \circ \mu \qquad \nu_i = \nu \circ \lambda_i$$

respectively. Either μ or ν does the required job. In fact

$$\mu = \nu$$

as we now show. We have

$$\rho_j \circ \mu \circ \lambda_i = \mu_j \circ \lambda_i = \delta_{ji}$$

for each i and j, so that

$$\nu_i = \mu \circ \lambda_i$$

for each i, and hence $\mu = \nu$.

For the counterexample consider the lantern poset

$$\begin{tikzcd} & \top & \\ a \arrow[ur] & c \arrow[u] & b \arrow[ul] \\ & \bot \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}$$

viewed as a category (with arrows pointing upwards). Then

$$a \wedge c = \bot = b \wedge c \qquad a \vee b = \top$$

to give

$$l = (a \wedge c) \vee (b \wedge c) = \bot \qquad r = (a \vee b) \wedge c = c$$

and hence $r \not\leq l$. $\square$

2.5.9 The answer to both questions is 'No'.

Consider a pair of abelian groups A, B with the cartesian product $A \times B$ of these. We know this gives the categorical product of the two in both $\boldsymbol{AGrp}$ and $\boldsymbol{Grp}$. We also know that the two canonical insertions

$$\begin{tikzcd} A \arrow[r, "i"] & A \times B & B \arrow[l, "j"'] \end{tikzcd}$$
$$i(a) = (a, 1) \qquad (1, b) = j(b)$$

gives the coproduct in $\boldsymbol{AGrp}$. We show that for certain A, B this is *not* the coproduct in $\boldsymbol{Grp}$.

For both A, B we take a copy of the 2-element group. We let

$$A = \{1, a\} \quad \text{with } a^2 = 1 \qquad B = \{1, b\} \quad \text{with } b^2 = 1$$

for the two groups. Also let

$$C = \langle a, b \,|\, a^2 = 1 = b^2 \rangle$$

that is C is the group of all words in the two letters a, b where both aa and bb collapse to the empty word. The group operation is concatenation followed by a successive collapsing of similar letters. For instance

$$ababab \cdot babab = abababbabab = ababaabab = ababbab = abaab = abb = a$$

and each word does have an inverse. The group C is not commutative since $ab \neq ba$.

Consider the diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, "f"'] & A \times B & B \arrow[l, "j"'] \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

where each of f, g sends the letter to the corresponding word of length 1. Observe that each of f, g is a group morphism (in fact, an embedding). We show there is no morphism

$$A \times B \xrightarrow{h} C$$

which makes the diagram commute. Thus $A \times B$ is not the coproduct of A, B in $\boldsymbol{Grp}$. Observe that

$$(a,1)(1,b) = (a,b) = (1,b)(a,1)$$

in $A \times B$. If there is such a morphism h then

$$\begin{aligned} ab &= h(a,1)h(1,b) \\ &= h\big((a,1)(1,b)\big) \\ &= h(a,b) \\ &= h\big((1,b)(a,1)\big) \\ &= h(1,b)h(a,1) \quad = ba \end{aligned}$$

which is contradictory. In fact, C is the coproduct of A, B in $\boldsymbol{Grp}$. □

2.6 Equalizers and coequalizers

2.6.1 The two parts of Lemma 2.6.3 are proved in the same way. Let's show that each equalizer is monic.

Consider an arrow m, as on the left, which is the equalizer of a

$$A \xrightarrow{m} B \qquad\qquad B \underset{q}{\overset{p}{\rightrightarrows}} C$$

parallel pair, as on the right. Thus

$$p \circ m = q \circ m$$

with the appropriate universal property. To show that m is monic consider any parallel pair

$$X \underset{g}{\overset{f}{\rightrightarrows}} A$$

with

$$m \circ f = h = m \circ g$$

where h is the common composite. We require $f = g$.

$$X \underset{g}{\overset{f}{\rightrightarrows}} A \xrightarrow{m} B \underset{q}{\overset{p}{\rightrightarrows}} C$$

From the diagram above we have

$$p \circ h = p \circ m \circ f = q \circ m \circ f = q \circ m \circ g = q \circ h$$

so that the universal property of m gives

$$h = m \circ k$$

for some unique arrow k. This uniqueness ensure that $f = g$.

The proof of the equalizer version of Lemma 2.6.4 is the mirror image of the coequalizer version. The proof can be obtained from the coequalizer version by changing one or two words and remembering that arrows now point the other way.

The arrow l makes equal f and g. The arrow k is the equalizer of f and g. Thus there is a unique mediator m satisfying (1). By reversing the roles of l and k we see there is a unique mediator n satisfying (2). From (1, 2) we have

$$k \circ m \circ n = l \circ n = k = k \circ id_T \quad \text{and hence} \quad m \circ n = id_T$$

since k is monic. Similarly

$$n \circ m = id_S$$

to show that m and n are an inverse pair of isomorphisms. □

2.6.2 We have a particular insertion

$$S \xhookrightarrow{\ i\ } A$$

which automatically satisfies

$$i(s) = s$$

for each $s \in S$. For the given function

$$X \xrightarrow{\ h\ } A$$

we have set up a triangle, as on the left

$$m(x) = h(x) \qquad \begin{tikzcd} S \arrow[r, hook, "i"] & A \\ X \arrow[u, "m"] \arrow[ur, "h"'] & \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r, hook, "i"] & A \\ X \arrow[u, "n"] \arrow[ur, "h"'] & \end{tikzcd}$$

for a certain function m as indicated. Trivially, for $x \in X$ we have

$$i(m(x)) = m(x) = h(x)$$

so the triangle does commute.

Conversely, suppose we have some function n to make the triangle commute, as on the right. Then for each $x \in X$ we have

$$n(x) = i(n(x)) = h(x)$$

to show that $n = m$, and hence m has the required uniqueness. □

2.6.3 (a) Making use of Example 2.6.5 we have

$$\begin{tikzcd} E \arrow[r, hook, "j"] & S \arrow[r, hook, "i"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

where $i \circ j$ is the equalizer of the pair f, g in $\boldsymbol{Set}$, and where i is group embedding and hence monic in $\boldsymbol{Grp}$.

Consider any group morphism

$$X \xrightarrow{h} A$$

which does make equal f and g. Working first in $\boldsymbol{Set}$ we have a commuting triangle for some unique function n, as on the left.

$$\begin{tikzcd} E \arrow[r, hook, "j"] & S \arrow[r, hook, "i"] & A \\ X \arrow[u, "n"] \arrow[urr, "h"'] & & \end{tikzcd} \qquad m = j \circ n \qquad \begin{tikzcd} E \arrow[r, hook, "j"] & S \arrow[r, hook, "i"] & A \\ X \arrow[u, "n"] \arrow[ur, "m"] \arrow[urr, "h"'] & & \end{tikzcd}$$

With the composite arrow m we obtain the diagram, as on the right, where by construction the left hand triangle commutes in $\boldsymbol{Set}$, and

$$h = i \circ j \circ n = i \circ m$$

so the right hand triangle commutes, again in $\boldsymbol{Set}$. We show that m is a group morphism, so that the right hand triangle commutes in $\boldsymbol{Grp}$.

Consider any $x, y \in X$. We require

$$m(xy) = m(x)m(y)$$

in A. But, i and h are group morphisms so that

$$i\big(m(xy)\big) = h(xy) = h(x)h(y) = i(m(x))i(m(y)) = i\big(m(x)m(y)\big)$$

and i is an injection (monic in $\boldsymbol{Grp}$), to give the required result.

This shows that h does factorize through i via some group morphism m. We show that this is the only possible factorization. Thus suppose

$$h = i \circ k$$

for some group morphism k. Then

$$i \circ k = h = i \circ j \circ n = i \circ m$$

so that

$$k = m$$

since i is monic in $\boldsymbol{Grp}$.

(b) We have a diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "k"] & B/K \end{tikzcd}$$

where k is the canonical quotient. For each $a \in A$ we have

$$k\big(f(a)g(a)^{-1}\big) = k(1) = 1$$

which leads to

$$k(f(a)) = k(g(a))$$

and hence k does make equal f and g. We show that k is the coequalizer of f and g.

Consider any group morphism

$$B \xrightarrow{h} X$$

which does make equal f and g. For each $a \in A$ we have

$$h(f(a)) = h(g(a))$$

so that

$$h\big(f(a)g(a)^{-1}\big) = h(1) = 1$$

and hence

$$f(a)g(a)^{-1} \in \ker(h)$$

to show

$$F \subseteq \ker(h)$$

and hence

$$K \subseteq \ker(h)$$

by the construction of K. This shows there is a unique morphism m for which the triangle

$$\begin{tikzcd} B \arrow[rr, "h"] \arrow[dr, "k"'] & & X \\ & B/K \arrow[ur, "m"'] & \end{tikzcd}$$

commutes, and this is precisely the mediating property we require. □

2.6.4 Since the function σ is surjective there can be at most one function $h^\sharp$ to make the diagram commute. For $s_1, s_2 \in S$ we have

$$[s_1] = [s_2] \Longrightarrow s_1 \sim s_2 \Longrightarrow h(s_1) = h(s_2)$$

to show that the suggested function $h^\sharp$ is well defined. For $s \in S$ we have

$$(h^\sharp \circ \sigma)(s) = h^\sharp\big([s]\big) = h(s)$$

to show that the triangle commutes. □

2.6.5 Let

$$\begin{array}{ccc} B & \xrightarrow{\ \beta\ } & B/\sim \\ b & \longmapsto & [b] \end{array}$$

be the constructed quotient.

Consider any $a \in A$ and let

$$b_1 = f(a) \qquad b_2 = g(a)$$

to obtain to elements of b with $b_1 \rightsquigarrow b_2$. In particular we have

$$b_1 \sim b_2$$

so that

$$(\beta \circ f)(a) = \beta(b_1) = [b_1] = [b_2] = \beta(b_2) = (\beta \circ g)(a)$$

to show that β does make equal f and g.

Consider any function h which does make equal f and g. We show that h factorizes uniquely through β. Since β is surjective, there can be at most one such factorization, so it suffices to show that one does exist.

We use Example 2.6.6 and Exercise 2.6.4. Suppose

$$b_1 \rightsquigarrow b_2$$

for $b_1, b_2 \in B$. Then

$$b_1 = f(a) \qquad b_2 = g(a)$$

for some $a \in A$. This gives

$$h(b_1) = h(f(a)) = h(g(a)) = h(b_2)$$

and hence Example 2.6.6 gives a function $h^\sharp$ for the factorization. □

2.6.6 For the given continuous maps

$$S \underset{\psi}{\overset{\phi}{\rightrightarrows}} T$$

we let

$$T \xrightarrow{\ \alpha\ } A$$

be the coequalizer of the two functions ϕ and ψ in $\boldsymbol{Set}$. This α is surjective. We furnish A with the smallest topology $\mathcal{O}A$ for which α is continuous. This is precisely the set of all $W \subseteq A$ for which $\alpha^{\leftarrow}(W) \in \mathcal{O}T$. (You should check this. The topology is sometimes called the final topology or the quotient topology on A.) We show that this continuous map α is the coequalizer of the two maps ϕ and ψ in $\boldsymbol{Top}$.

Trivially, α does make equal ϕ and ψ.

Consider any continuous map

$$T \xrightarrow{\theta} R$$

which makes equal ϕ and ψ. At the ***Set*** level there is a unique function μ such that the following triangle commutes.

$$\begin{tikzcd}
S \arrow[r, shift left, "\phi"] \arrow[r, shift right, "\psi"'] & T \arrow[r, "\alpha"] \arrow[d, "\theta"'] & A \arrow[dl, "\mu"] \\
& R &
\end{tikzcd}$$

It suffices to show that μ is continuous. Consider any $U \in \mathcal{O}R$. We require

$$\mu^{\leftarrow}(U) \in \mathcal{O}A$$

that is

$$(\alpha^{\leftarrow} \circ \mu^{\leftarrow})(U) = \alpha^{\leftarrow}(\mu^{\leftarrow}(U)) \in \mathcal{O}T$$

(by the definition of $\mathcal{O}A$). But

$$\alpha^{\leftarrow} \circ \mu^{\leftarrow} = (\mu \circ \alpha)^{\leftarrow} = \theta^{\leftarrow}$$

and $\theta^{\leftarrow}(U) \in \mathcal{O}T$ since θ is continuous. □

2.6.7 (a) Since the comparison $\leq$ is reflexive, the defined relation $\sim$ is reflexive. By rephrasing the definition as

$$a \sim b \Longleftrightarrow a \leq b \text{ and } b \leq a$$

we see that $\sim$ is symmetric. If

$$a \sim b \sim c$$

then

$$a \leq b \leq c \text{ and } c \leq b \leq a$$

so that

$$a \leq c \text{ and } c \leq a$$

to give

$$a \sim c$$

to show that $\sim$ is transitive.

The preset S is a poset precisely when

$$a \leq b \leq a \Longrightarrow a = b$$

that is

$$a \sim b \Longrightarrow a = b$$

and the converse implication always holds.

(b) To show that the comparison on $S/\sim$ is well-defined suppose

$$[a_1] = [a_2] \qquad [b_1] = [b_2]$$

for elements $a_1, a_2, b_1, b_2 \in S$. We require

$$a_1 \leq b_1 \Longleftrightarrow a_2 \leq b_2$$

and clearly, by symmetry, a proof of one of the implications will do.

From the two assumed equalities we have

$$a_1 \sim a_2 \qquad b_1 \sim b_2$$

and hence

$$a_1 \leq b_1 \Longrightarrow a_2 \leq a_1 \leq b_1 \leq b_2 \Longrightarrow a_2 \leq b_2$$

as required.

This shows that

$$\begin{tikzcd} S \arrow[r, "\eta"] & S/\sim \\ a \arrow[r, mapsto] & {[a]} \end{tikzcd}$$

is well-defined and, trivially, it is monotone.

(c) Consider a monotone map

$$S \xrightarrow{f} T$$

from the preset S to a poset T.

For $a, b \in S$ we have

$$a \sim b \Longrightarrow a \leq b \leq a \Longrightarrow f(a) \leq f(b) \leq f(a) \Longrightarrow f(a) = f(b)$$

where the last step holds since T is a poset. Since η is surjective there is at most one monotone map $f^\sharp$ such that

$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr, "\eta"'] & & T \\ & S/\sim \arrow[ur, "f^\sharp"'] & \end{tikzcd}$$

commutes. Thus it suffices to show that

$$f^\sharp([a]) = f(a)$$

(for $a \in S$) gives a well-defined monotone function.

The implications above show that $f^\sharp$ is well-defined, and a similar argument shows that f is monotone. The universal property gives the required function. For the general argument see Solution 3.3.18. □

2.6.8 We are given that e does make equal f and g. Consider any other arrow h which makes equal f and g, as on the left. We must show that h factorizes uniquely through e.

$$\begin{tikzcd}
 & \bullet \arrow[d, "h"] & \\
\bullet \arrow[r, "e"] & \bullet \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & \bullet \arrow[d, "h"] \arrow[dr, "m"] & \\
 & \bullet \arrow[r, "e"] \arrow[d, "e"'] & \bullet \arrow[r, "p"] \arrow[d, "f"] & \bullet \arrow[d, "e"] \\
\bullet \arrow[r, "h"'] & \bullet \arrow[r, "g"'] & \bullet \arrow[r, "q"'] & \bullet
\end{tikzcd}$$

Let

$$m = p \circ h$$

and consider the right hand diagram. We have

$$e \circ m = e \circ p \circ h = q \circ f \circ h = q \circ g \circ h = h$$

to show that h does factorize through e.

Conversely, suppose

$$h = e \circ n$$

for some arrow n. Then

$$n = p \circ e \circ n = p \circ h = m$$

to show the required uniqueness. $\square$

2.7 Pullbacks and pushouts

2.7.1 (a) Consider any wedge in $\boldsymbol{C}$, as in the center. Consider also the

$$\begin{tikzcd}[column sep=small]
 & A \\
P \arrow[ur, "p"] \arrow[dr, "q"'] & \\
 & B
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
A \arrow[dr, "a"] & \\
 & C \\
B \arrow[ur, "b"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & A \arrow[dr, "a"] & \\
P \arrow[ur, "p"] \arrow[dr, "q"'] & & C \\
 & B \arrow[ur, "b"'] &
\end{tikzcd}$$

product wedge of the two objects A, B, as at the left. These compose to give a square, as at the right. Of course, this square need not commute. Let

$$S \xrightarrow{\ e\ } P$$

be the equalizer of the parallel pair

$$\begin{tikzcd}
P \arrow[r, shift left, "a \circ p"] \arrow[r, shift right, "b \circ q"'] & C
\end{tikzcd}$$

obtained from the square. We show that

$$\begin{tikzcd}
 & A \arrow[dr, "a"] & \\
S \arrow[ur, "p \circ e"] \arrow[dr, "q \circ e"'] & & C \\
 & B \arrow[ur, "b"'] &
\end{tikzcd}$$

is a pullback square.

Consider any commuting square

$$
\begin{tikzcd}
& A \arrow[dr, "a"] & \\
X \arrow[ur, "f"] \arrow[dr, "g"'] & & C \\
& B \arrow[ur, "b"'] &
\end{tikzcd}
$$

where the right hand side is the given wedge. Using the product property we have

$$
\begin{array}{c} f = p \circ h \\[2em] g = q \circ h \end{array}
\qquad
\begin{tikzcd}
& A \arrow[dr, "a"] & \\
X \arrow[ur, "f"] \arrow[r, "h"] \arrow[dr, "g"'] & P \arrow[u, "p"'] \arrow[d, "q"] & C \\
& B \arrow[ur, "b"'] &
\end{tikzcd}
$$

for some *unique* arrow h. But now

$$a \circ p \circ h = a \circ f = b \circ g = b \circ q \circ h$$

to show that h makes equal the parallel pair, and hence

$$h = e \circ m$$

for some unique arrow

$$X \xrightarrow{m} S$$

by the equalizing property. In particular, we have a commuting diagram

$$
\begin{tikzcd}
& & A \arrow[dr, "a"] & \\
X \arrow[urr, "f"] \arrow[r, "m"] \arrow[drr, "g"'] & S \arrow[r, "e"] & P \arrow[u, "p"'] \arrow[d, "q"] & C \\
& & B \arrow[ur, "b"'] &
\end{tikzcd}
$$

to show that the arbitrary square from X does factorize via m through the constructed square from S. We must show that this is the only possible factorization.

Suppose

$$f = p \circ e \circ n \qquad g = q \circ e \circ n$$

for some arrow

$$X \xrightarrow{n} S$$

in place of m. Then

$$e \circ n = h \qquad n = m$$

by the uniqueness of h followed by the uniqueness of m.

(b) This follows by a dual argument to that of (a). $\square$

2.7.2 Let the arrows point up the poset. All pushouts precisely when it has joins of those pairs of elements which have a lower bound. ☐

2.7.3 For the first part we are given a pair of pullbacks

$$
\begin{tikzcd}
\bullet \arrow[r, "c"] \arrow[d, "r"] & \bullet \arrow[r, "a"] \arrow[d, "q"] & \bullet \arrow[d, "p"] \\
\bullet \arrow[r, "d"] & \bullet \arrow[r, "b"] & \bullet
\end{tikzcd}
$$

where we have labelled he arrows. Consider a pair of arrows

$$
\begin{tikzcd}
\bullet \arrow[rrrd, "f"] \arrow[rdd, "g"'] & & & \\
& \bullet \arrow[r, "c"] \arrow[d, "r"] & \bullet \arrow[r, "a"] \arrow[d, "q"] & \bullet \arrow[d, "p"] \\
& \bullet \arrow[r, "d"] & \bullet \arrow[r, "b"] & \bullet
\end{tikzcd}
\qquad p \circ f = b \circ d \circ g
$$

with the indicated commuting properties. Using the right hand pullback there is a *unique* arrow h for which

$$
\begin{tikzcd}
\bullet \arrow[rrd, "h"] \arrow[rrrd, "f"] \arrow[rdd, "g"'] & & & \\
& & \bullet \arrow[r, "a"] \arrow[d, "q"] & \bullet \arrow[d, "p"] \\
& \bullet \arrow[r, "d"] & \bullet \arrow[r, "b"] & \bullet
\end{tikzcd}
\qquad f = a \circ h \quad d \circ g = q \circ h
$$

commutes, as indicated. This gives us a commuting diagram

$$
\begin{tikzcd}
\bullet \arrow[rrd, "h"] \arrow[rdd, "g"'] & & \\
& \bullet \arrow[r, "c"] \arrow[d, "r"] & \bullet \arrow[d, "q"] \\
& \bullet \arrow[r, "d"] & \bullet
\end{tikzcd}
\qquad d \circ g = q \circ h
$$

and the left hand pullback provides a *unique* arrow m for which

$$
\begin{tikzcd}
\bullet \arrow[rd, "m"] \arrow[rrd, "h"] \arrow[rdd, "g"'] & & \\
& \bullet \arrow[r, "c"] \arrow[d, "r"] & \bullet \arrow[d, "q"] \\
& \bullet \arrow[r, "d"] & \bullet
\end{tikzcd}
\qquad h = c \circ m \quad g = r \circ m
$$

commutes, as indicated.

From the equalities obtained we have

$$f = a \circ h = a \circ c \circ m \qquad g = r \circ m$$

to show that we have produced a factorization of f and g through a common arrow m. It remains to show that m is the only arrow that does this job.

Consider any arrow n for which

$$f = a \circ c \circ n \qquad g = r \circ n$$

holds. Then

$$f = a \circ c \circ n \qquad d \circ g = d \circ r \circ n = q \circ c \circ n$$

and hence

$$c \circ n = h$$

by the uniqueness of h. But now

$$c \circ n = h \qquad r \circ n = g$$

to give

$$n = m$$

by the uniqueness of m.

For the second part suppose we have commuting squares

$$\begin{tikzcd}
\bullet \arrow[r, "c"] \arrow[d, "r"] & \bullet \arrow[r, "a"] \arrow[d, "q"] & \bullet \arrow[d, "p"] \\
\bullet \arrow[r, "d"] & \bullet \arrow[r, "b"] & \bullet
\end{tikzcd}$$

where the outer cell and the right hand square are pullbacks. Suppose also we have a commuting diagram as on the left. We must produce a

$$\begin{tikzcd}
\bullet \arrow[drr, "f"] \arrow[ddr, "g"'] & & \\
& \bullet \arrow[r, "c"] \arrow[d, "r"] & \bullet \arrow[d, "q"] \\
& \bullet \arrow[r, "d"] & \bullet
\end{tikzcd}
\qquad q \circ f = d \circ g \qquad
\begin{tikzcd}
\bullet \arrow[drr, "f"] \arrow[ddr, "g"'] \arrow[dr, "l"] & & \\
& \bullet \arrow[r, "c"] \arrow[d, "r"] & \bullet \arrow[d, "q"] \\
& \bullet \arrow[r, "d"] & \bullet
\end{tikzcd}$$

commuting diagram, as on the right, and show that there is only one possible arrow l. Thus we want

$$f = c \circ l \quad (?) \quad g = r \circ l$$

for some unique arrow l.

Let's look at what the two pullback squares give us.

Consider the right hand square with connecting arrows, as indicated.

$$
\begin{tikzcd}
\bullet \arrow[drr, "a\circ f"] \arrow[ddr, "d\circ g"'] & & \\
& \bullet \arrow[r, "a"] \arrow[d, "q"] & \bullet \arrow[d, "p"] \\
& \bullet \arrow[r, "b"] & \bullet
\end{tikzcd}
\qquad
\begin{array}{c} a\circ f = a\circ m \\[2ex] d\circ g = q\circ m \end{array}
\qquad
\begin{tikzcd}
\bullet \arrow[drr, "a\circ f"] \arrow[ddr, "d\circ g"'] \arrow[dr, "m"] & & \\
& \bullet \arrow[r, "a"] \arrow[d, "q"] & \bullet \arrow[d, "p"] \\
& \bullet \arrow[r, "b"] & \bullet
\end{tikzcd}
$$

Since the square commutes we have

$$p\circ a\circ f = b\circ q\circ f = b\circ d\circ g$$

and hence the pullback property gives a unique arrow m with the two central equalities. In fact, since

$$a\circ f = a\circ f \qquad d\circ g = q\circ f$$

we see that $m = f$.

Consider the outer cell with connecting arrows, as indicated.

$$
\begin{tikzcd}
\bullet \arrow[drr, "a\circ f"] \arrow[ddr, "g"'] & & \\
& \bullet \arrow[r, "a\circ c"] \arrow[d, "r"] & \bullet \arrow[d, "p"] \\
& \bullet \arrow[r, "b\circ d"] & \bullet
\end{tikzcd}
\qquad
\begin{array}{c} a\circ f = a\circ c\circ n \\[2ex] g = r\circ n \end{array}
\qquad
\begin{tikzcd}
\bullet \arrow[drr, "a\circ f"] \arrow[ddr, "g"'] \arrow[dr, "n"] & & \\
& \bullet \arrow[r, "a\circ c"] \arrow[d, "r"] & \bullet \arrow[d, "p"] \\
& \bullet \arrow[r, "b\circ d"] & \bullet
\end{tikzcd}
$$

Using the calculations above we see the pullback property gives a unique arrow n with the two central equalities. In due course we show that the required arrow l is n.

We compare the properties of m and n. We have

$$a\circ f = a\circ c\circ n \qquad d\circ g = d\circ r\circ n = q\circ c\circ n$$

and hence

$$c\circ n = m = f$$

by the uniqueness of m. Since

$$f = c\circ n \qquad g = r\circ n$$

we see that $l = n$ makes the required triangles commute. Comversely, if

$$f = c\circ l \quad g = r\circ l \quad \text{then} \quad a\circ f = a\circ c\circ l \quad g = r\circ l$$

and hence $l = n$ by the uniqueness of n. $\square$

2.7.4 Consider any parallel pair p, q of arrows that h makes equal.

$$h \circ p = h \circ q = l \qquad \begin{tikzcd} \bullet \arrow[d, shift right, "p"'] \arrow[d, shift left, "q"] & \\ \bullet \arrow[r, "k"] \arrow[d, "h"'] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "g"'] & \bullet \end{tikzcd}$$

We must show that $p = q$. By going round the square we find that

$$f \circ k \circ p = f \circ k \circ q$$

and hence, since f is monic, we have

$$k \circ p = k \circ q = r$$

say. This shows that both p and q make

$$\begin{tikzcd} \bullet \arrow[dr] \arrow[drr, "r"] \arrow[ddr, "l"'] & & \\ & \bullet \arrow[r, "k"] \arrow[d, "h"] & \bullet \arrow[d, "f"] \\ & \bullet \arrow[r, "g"] & \bullet \end{tikzcd}$$

commutes, and hence $p = q$ since the given square is a pullback. □

2.7.5 Suppose that f is the equalizer of the parallel pair p, q as indicated

$$\begin{tikzcd} \bullet \arrow[r, "k"] \arrow[d, "h"'] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "g"'] & \bullet \arrow[d, shift right, "p"'] \arrow[d, shift left, "q"] \\ & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, shift left, "p \circ g"] \arrow[r, shift right, "q \circ g"'] & \bullet \end{tikzcd}$$

on the left. Since the given square does commute, we see that h makes equal the composite parallel pair on the right, and we show that h actually equalizes this pair.

$$
\begin{tikzcd}
\bullet \arrow[ddr, "l"'] & & \\
& \bullet \arrow[r, "k"] \arrow[d, "h"] & \bullet \arrow[d, "f"] \\
& \bullet \arrow[r, "g"] & \bullet \arrow[d, shift right, "p"'] \arrow[d, shift left, "q"] \\
& & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[ddr, "l"'] \arrow[drr, "r"] & & \\
& \bullet \arrow[r, "k"] \arrow[d, "h"] & \bullet \arrow[d, "f"] \\
& \bullet \arrow[r, "g"] & \bullet \arrow[d, shift right, "p"'] \arrow[d, shift left, "q"] \\
& & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[ddr, "l"'] \arrow[drr, "r"] \arrow[dr, "m"] & & \\
& \bullet \arrow[r, "k"] \arrow[d, "h"] & \bullet \arrow[d, "f"] \\
& \bullet \arrow[r, "g"] & \bullet \arrow[d, shift right, "p"'] \arrow[d, shift left, "q"] \\
& & \bullet
\end{tikzcd}
$$

Consider any other arrow l which does make equal this pair, as on the left. Since f is the equalizer of p, q there is a *unique* arrow r which makes the central diagram commute. Since the given square is a pullback there is a *unique* arrow m which makes the right hand diagram commute. From these diagrams we have

$$g \circ l = f \circ r \qquad l = h \circ m \qquad r = k \circ m$$

and it is the central equality that most interests us. It suffices to show that this is the only possible factorization of l through h.

Consider any arrow n for which

$$l = h \circ n$$

holds. It suffices to show

$$r = k \circ n$$

for then $n = m$ by the uniqueness of m. Since the given square commutes we have

$$f \circ k \circ n = g \circ h \circ n = g \circ l$$

and hence the uniqueness of r gives the required result. a □

2.8 Using the opposite category

2.8.1 No solution needed? □

3
Functors and natural tansformations

3.1 Functors defined

3.1.1 A covariant functor from S to T is simply a monoid morphism from S to T. A contravariant functor f from S to T is monoid 'morphism' that flips the elements, that is

$$f(rs) = f(s)f(r)$$

for $r, s \in S$. $\square$

3.1.2 A covariant functor from S to T is simply a monotone map. A contravariant functor f from S to T is an antitone map, that is

$$r \leq s \Longrightarrow f(s) \leq f(r)$$

for $r, s \in S$. $\square$

3.1.3 Consider a covariant functor

$$\boldsymbol{Src}^{\mathrm{op}} \xrightarrow{F} \boldsymbol{Trg}$$

using the opposite on the source. Consider any arrow

$$A \xrightarrow{f} B$$

of $\boldsymbol{Src}$. This is an arrow

$$B \xrightarrow{f} A$$

of $\boldsymbol{Src}^{\mathrm{op}}$, and the functor F sends it to an arrow

$$FB \xrightarrow{F(f)} FA$$

of $\boldsymbol{Trg}$. Thus F has flipped the direction of f. The other required properties (preservation of composition and identity arrows) are immediate, to show that

$$\boldsymbol{Src} \xrightarrow{F} \boldsymbol{Trg}$$

is a contravariant functor.

The other part is just as easy. $\square$

3.1.4 The composite of two functors of the same variance produces a covariant functor.

The composite of two functors of opposite variance is a contravariant functor. $\square$

3.2 Some simple functors

3.2.1 The functors S and T select the source and target of the arrow, respectively. The diagonal functor Δ send each object A to the identity arrow id_A. □

3.2.2 The $\boldsymbol{Set}$-valued contravariant functors on the poset S are the presheaves on S. □

3.2.3 The

covariant contravariant

$\boldsymbol{Set}$-valued functors from R are precisely the

left right

R-sets. It is worth looking at the details of this, and the contravariant case is potentially more interesting.

Consider such a contravariant functor. This must send each object of R to some set. But since R (when viewed as a category) has just one object, this object assignment produces a set, A say.

The functor must send each arrow of R (element of R) to a function from A to A.

$$R \xrightarrow{\;r\;} R \qquad \longmapsto \qquad A \xleftarrow{\;\alpha_r\;} A$$

Let α_r be the 1-placed operation on A assigned to $r \in R$. The contravariance

$$\begin{tikzcd} & R \arrow[dr, "r"] & \\ R \arrow[ur, "s"] \arrow[rr, "sr"'] & & R \end{tikzcd} \qquad \longmapsto \qquad \begin{tikzcd} & A \arrow[dl, "\alpha_s"'] & \\ A & & A \arrow[ul, "\alpha_r"'] \arrow[ll, "\alpha_{sr}"] \end{tikzcd}$$

gives

$$\alpha_{sr} = \alpha_s \circ \alpha_r \qquad \alpha_1 = id_A$$

for each $r, s \in R$, where the right hand equality is the identity requirement. We now write each operation α as a right action

$$\begin{array}{ccc} A & \xrightarrow{\;\alpha_r\;} & A \\ a & \longmapsto & ar \end{array}$$

to get

$$a(sr) = (as)r \qquad a1 = a$$

and so produce a right R-set. □

3.2.4 This is more or less proved by Exercise 1.3.5. □

3.2.5 Remember that $\boldsymbol{C}[-,-]$ is contravariant in the left argument and covariant in the right argument. So what we have here is really a covariant functor

$$\boldsymbol{C}^{\mathrm{op}} \times \boldsymbol{C} \xrightarrow{\;H\;} \boldsymbol{Set}$$

from the 'twisted product' category. In detail, for arrows

$$B \xrightarrow{\ f\ } A \qquad\qquad S \xrightarrow{\ g\ } T$$

of $\boldsymbol{C}$ we have

$$H(f,g) = g \circ - \circ f$$

for the arrow behaviour of H. □

3.3 Some less simple functors

3.3.1 Three power set functors

3.3.1 Only $\forall(\cdot)$ is liable to cause trouble. Given functions between sets

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C$$

for each $X \in \mathcal{P}A$ we have

$$\begin{aligned}\big(\forall(g) \circ \forall(f)\big)(X) &= \forall(g)\big(\forall(f)(X)\big)\\ &= \forall(g)\big(f[X']'\big)\\ &= g[f[X']'']'\\ &= g[f[X']]'\\ &= (g \circ f)[X']' \qquad = \forall(g \circ f)(X)\end{aligned}$$

as required. A similar proof can be done using the quantifier characterization. □

3.3.2 This is simpler than Exercise 1.3.9. A set is a discrete poset. □

3.3.2 Spaces, presets, and posets

3.3.3 (a) The specialization order of $\Uparrow A$ is the given comparison on A.

(b) Almost be defintion each open set $U \in \mathcal{O}S$ is an upper section of the specialization order. □

3.3.4 (a) Consider a monotone map

$$A \xrightarrow{\ f\ } B$$

between two presets, and an upper section $V \in \Upsilon B$ of the target. We require $f^{\leftarrow}(V) \in \Upsilon A$. Consider any $x \leq y$ in A with $x \in f^{\leftarrow}(V)$. Then

$$f(x) \leq f(y) \qquad f(x) \in V$$

so that $f(y) \in V$ and hence $y \in f^{\leftarrow}(V)$.

We have an object and an arrow assignment

$$\boldsymbol{Pre} \xrightarrow{\ \Uparrow\ } \boldsymbol{Top}$$

which is trivial on arrows, so we do have a functor.

(b) Consider a continuous map between spaces

$$S \xrightarrow{\ \phi\ } T$$

and consider a comparison $x \le y$ in S. We require $\phi(x) \le \phi(y)$ in T.

Consider $V \in \mathcal{O}T$ with $\phi(x) \in V$. We required $\phi(y) \in V$. But

$$x \in \phi^{\leftarrow}(V) \in \mathcal{O}S \qquad x \le y$$

so that $y \in \phi^{\leftarrow}(V)$, as required.

We have an object and an arrow assignment

$$\boldsymbol{Top} \xrightarrow{\ \Downarrow\ } \boldsymbol{Pre}$$

which is trivial on arrows, so we do have a functor. □

3.3.5 Suppose first that θ is monotone and consider any $U \in \mathcal{O}S$. We require $\theta^{\leftarrow}(U) \in \Upsilon A$. Consider elements x, y of A with

$$x \in \theta^{\leftarrow}(U) \qquad x \le y$$

so that $y \in \theta^{\leftarrow}(U)$ is required. We have

$$\theta(x) \in U \qquad \theta(x) \le \theta(y)$$

(since θ is monotone), and hence

$$y \in \theta(U)$$

since each open set of S is an upper section of S.

Secondly, suppose that θ is continuous and consider elements $x \le y$ of A. We require $\theta(x) \le \theta(y)$. For each $U \in \mathcal{O}S$ we have

$$\theta(x) \in U \Longrightarrow x \in \theta^{\leftarrow}(U) \Longrightarrow y \in \theta^{\leftarrow}(U) \Longrightarrow \theta(y) \in U$$

where the central implication holds since θ is continuous and hence $\theta^{\leftarrow}(U) \in \Upsilon A$.

These two implications show that the hom-sets

$$\boldsymbol{Pre}[A, \Downarrow S] \qquad \boldsymbol{Top}[\Uparrow A, S]$$

contain exactly the same functions. There is a trivial bijection between the two sets. □

3.3.6 For $U, V \in \mathcal{O}S$ with $U \subseteq V$ we require $\mathcal{O}(\phi)(U) \subseteq \mathcal{O}(\phi)(V)$. But for $t \in T$ we have

$$t \in \mathcal{O}(\phi)(U) \Longrightarrow \phi(t) \in U \subseteq V \Longrightarrow \phi(t) \in V \Longrightarrow t \in \mathcal{O}(\phi)(V)$$

for the required result.

For each pair of continuous maps

$$T \xrightarrow{\ \phi\ } S \xrightarrow{\ \psi\ } R$$

we require

$$\mathcal{O}(\psi \circ \phi) = \mathcal{O}(\phi) \circ \mathcal{O}(\psi)$$

that is

$$(\psi \circ \phi)^{\leftarrow}(U) = (\phi^{\leftarrow} \circ \psi^{\leftarrow})(U)$$

for $U \in \mathcal{O}R$. But for $t \in T$ we have

$$\begin{aligned} t \in (\psi \circ \phi)^{\leftarrow}(U) &\Longleftrightarrow (\psi \circ \phi)(t) \in U \\ &\Longleftrightarrow \psi(\phi(t)) \in U \\ &\Longleftrightarrow \phi(t) \in \phi^{\leftarrow}(U) \\ &\Longleftrightarrow t \in \psi^{\leftarrow}(\phi^{\leftarrow}(U)) \quad \Longleftrightarrow (\phi^{\leftarrow} \circ \psi^{\leftarrow})(U) \end{aligned}$$

for the required result.

Observe that a character

$$p : S \longrightarrow \mathsf{2}$$

is continuous precisely when

$$p^{\leftarrow}(\{1\})$$

is open in S. This is because

$$p^{\leftarrow}(\emptyset) = \emptyset \qquad p^{\leftarrow}(\mathsf{2}) = S$$

and these are open in S. For each continuous map ϕ and continuous character p on the target we have a continuous character

$$\Xi(\phi)(p) = p \circ \phi$$

since continuous maps are closed under composition.

To show that $\Xi(\phi)$ is monotone consider continuous characters p, q of S with $p \leq q$. Then for each $t \in T$ we have

$$\Xi(\phi)(p)(t) = p(\phi(t)) \leq q(\phi(t))\Xi(\phi)(q)(t)$$

to show

$$\Xi(\phi)(p) \leq \Xi(\phi)(q)$$

as required.

Finally, for each pair of continuous maps

$$T \xrightarrow{\quad\phi\quad} S \xrightarrow{\quad\psi\quad} R$$

we require

$$\Xi(\psi \circ \phi) = \Xi(\phi) \circ \Xi(\psi)$$

that is

$$\Xi(\psi \circ \phi)(r) = \big(\Xi(\phi) \circ \Xi(\psi)\big)(r)$$

for each continuous character r of R. But for such an r we have

$$\begin{aligned} \Xi(\psi \circ \phi)(r) &= r \circ (\psi \circ \phi) \\ &= (r \circ \psi) \circ \phi \\ &= \Xi(\psi)(r) \circ \phi \\ &= \Xi(\phi)\big(\Xi(\psi)(r)\big) = \big(\Xi(\phi) \circ \Xi(\psi)\big)(r) \end{aligned}$$

as required. $\square$

3.3.7 To show $\chi_S(U)$ is continuous (for $U \in \mathcal{O}S$) we require

$$\chi_S(U)^{\leftarrow}(W) \in \mathcal{O}S$$

for each $W \in \mathcal{O}\mathbf{2}$. Trivially we have

$$\chi_S(U)^{\leftarrow}(\emptyset) = \emptyset \qquad \chi_S(U)^{\leftarrow}(\mathbf{2}) = S$$

so it suffices to deal with $W = \{1\}$. For each $s \in S$ we have

$$s \in \chi_S(U)^{\leftarrow}(\{1\}) \iff \chi_S(U)(s) \in (\{1\}) \iff \chi_S(U)(s) = 1 \iff s \in U$$

to give the required result.

For $p \in \Xi S$ with

$$U = p^{\leftarrow}(\{1\}) \in OS$$

we have

$$p = \chi_S(U)$$

and hence

$$\mathcal{O}S \xrightarrow{\chi_S} \Xi S$$

is a bijection. To show it is a poset isomorphism we require

$$U \subseteq V \iff \chi_S(U) \le \chi_S(V)$$

for $U, V \in \mathcal{O}S$. But we have

$$\begin{aligned}
\chi_S(U) \le \chi_S(V) &\iff (\forall s \in S)[\chi_S(U)(s) \le \chi_S(V)(s)] \\
&\iff (\forall s \in S)[\chi_S(U)(s) = 1 \Longrightarrow \chi_S(V)(s) = 1] \\
&\iff (\forall s \in S)[s \in (U \Longrightarrow s \in V] \\
&\iff U \subseteq V
\end{aligned}$$

as required. □

3.3.3 Functors from products

3.3.8 Let $F = - \times R$. By construction, for each arrow

$$A \xrightarrow{f} B$$

the arrow

$$FA \xrightarrow{F(f)} FB$$

is the unique arrow for which

$$\begin{tikzcd}
& A \arrow[rr, "f"] & & B \\
FA \arrow[ur, "p_A"] \arrow[dr, "q_A"'] \arrow[rr] & & FB \arrow[ur, "p_B"'] \arrow[dr, "q_B"] & \\
& R \arrow[rr, "id_R"'] & & R
\end{tikzcd}$$

commutes. For arrows

$$A \xrightarrow{f} B \xrightarrow{g} C$$

we require

$$F(g \circ f) = F(g) \circ F(f)$$

(together with a trivial observation to give $F(id) = id$).

We have several commuting cells

$$\begin{tikzcd}
& A \arrow[rr, "f"] & & B \arrow[rr, "g"] & & C \\
FA \arrow[ur, "p_A"] \arrow[dr, "q_A"'] \arrow[rr, "F(f)"] & & FB \arrow[ur, "p_B"] \arrow[dr, "q_B"] \arrow[rr, "F(g)"] & & FC \arrow[ur, "p_C"] \arrow[dr, "q_C"] & \\
& R \arrow[rr, "id_R"'] & & R \arrow[rr, "id_R"'] & & R
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \arrow[rr, "g \circ f"] & & C \\
FA \arrow[ur, "p_A"] \arrow[dr, "q_A"'] \arrow[rr, "F(g \circ f)"] & & FC \arrow[ur, "p_C"'] \arrow[dr, "q_C"] & \\
& R \arrow[rr, "id_R"'] & & R
\end{tikzcd}$$

and the uniqueness of the central arrows give the required result. □

3.3.9 We have an object assignment

$$\begin{array}{ccc}
\boldsymbol{C} \times \boldsymbol{C} & \longrightarrow & \boldsymbol{C} \\
A_1, A_2 & \longmapsto & A_1 \times A_2
\end{array}$$

so we now require a companion arrow assignment. Consider any arrow of $\boldsymbol{C} \times \boldsymbol{C}$, in other words a pair

$$A_1 \xrightarrow{f_1} B_1 \qquad A_2 \xrightarrow[f_2]{} B_2$$

of arrows of $\boldsymbol{C}$. We have a diagram

$$\begin{tikzcd}
& A \arrow[rr, "f_1"] & & B_1 \\
A_1 \times A_2 \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & & B_1 \times B_2 \arrow[ur, "q_1"'] \arrow[dr, "q_2"] & \\
& R \arrow[rr, "f_2"'] & & B_2
\end{tikzcd}$$

where p_1, p_2, q_1, q_2 are the structuring projections. The product property of the right hand wedge gives a commuting diagram

$$\begin{tikzcd}
 & A_1 \arrow[rr, "f_1"] & & B_1 \\
A_1 \times A_2 \arrow[ur, "p_1"] \arrow[dr, "p_2"'] \arrow[rr, "f_1 \times f_2"] & & B_1 \times B_2 \arrow[ur, "q_1"'] \arrow[dr, "q_2"] & \\
 & A_2 \arrow[rr, "f_2"'] & & B_2
\end{tikzcd}$$

for some *unique* central arrow. This is often written $f_1 \times f_2$, as shown, and then

$$(f_1, f_2) \longmapsto f_1 \times f_2$$

is the arrow assignment.

To verify that we have a functor we need to show that the arrow construction passes across composition. As usual, it is the uniqueness that gives this.

Consider a composible pair

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] & B_1 \arrow[r, "g_1"] & C_1 \\
A_2 \arrow[r, "f_2"'] & B_2 \arrow[r, "g_2"'] & C_2
\end{tikzcd}$$

of arrows of $\boldsymbol{C} \times \boldsymbol{C}$. The commuting diagram

$$\begin{tikzcd}
 & A_1 \arrow[rr, "f_1"] & & B_1 \arrow[rr, "g_1"] & & C_1 \\
A_1 \times A_2 \arrow[ur, "p_1"] \arrow[dr, "p_2"'] \arrow[rr, "f_1 \times f_2"] & & B_1 \times B_2 \arrow[ur, "q_1"] \arrow[dr, "q_2"'] \arrow[rr, "g_1 \times g_2"] & & C_1 \times C_2 \arrow[ur, "r_1"'] \arrow[dr, "r_2"] & \\
 & A_2 \arrow[rr, "f_2"'] & & B_2 \arrow[rr, "g_2"'] & & C_2
\end{tikzcd}$$

ensures that

$$\begin{tikzcd}
 & A_1 \arrow[rr, "g_1 \circ f_1"] & & C_1 \\
A_1 \times A_2 \arrow[ur, "p_1"] \arrow[dr, "p_2"'] \arrow[rr, "(g_1 \times g_2) \circ (f_1 \times f_2)"] & & C_1 \times C_2 \arrow[ur, "r_1"'] \arrow[dr, "r_2"] & \\
 & A_2 \arrow[rr, "g_2 \circ f_2"'] & & C_2
\end{tikzcd}$$

commutes, and hence

$$(g_1 \times g_2) \circ (f_1 \times f_2) = (g_1 \circ f_1) \times (g_2 \circ f_2)$$

by the uniqueness of the mediators. □

3.3.4 Comma category

3.3.10 We produce a composition of arrows in $(U \downarrow L)$ and check it is associative. Consider two arrows of $(U \downarrow L)$

$$\begin{tikzcd}
UA_U \arrow[d, dash] & UB_U \arrow[d, dash] & UC_U \arrow[d, dash] \\
\alpha \arrow[d] \arrow[r, "f"] & \beta \arrow[d] \arrow[r, "g"] & \gamma \arrow[d] \\
LA_L & LB_L & LC_L
\end{tikzcd}$$

which ought to compose. In more detail we have

$$\begin{tikzcd}
A_U \arrow[r, "f_U"] & B_U \arrow[r, "g_U"] & C_U \\
UA_U \arrow[r, "U(f_U)"] \arrow[d, "\alpha"] & UB_U \arrow[r, "U(g_U)"] \arrow[d, "\beta"] & UC_U \arrow[d, "\gamma"] \\
LA_L \arrow[r, "L(f_L)"'] & LB_L \arrow[r, "L(g_L)"'] & LC_L \\
A_L \arrow[r, "f_L"'] & B_L \arrow[r, "g_L"'] & C_L
\end{tikzcd}$$

where the two squares commute. Now consider composite arrows

$$\begin{tikzcd}
A_U \arrow[r, "h_U = g_L \circ f_U"] & C_U & \boldsymbol{U} \\
A_L \arrow[r, "h_L = g_L \circ f_L"'] & C_L & \boldsymbol{L}
\end{tikzcd}$$

in the indicated categories. From above and using the functorial properties of U and L we see that the square on the left commutes

$$\begin{tikzcd}
A_U \arrow[r, "h_U"] & C_U \\
UA_U \arrow[r, "U(h_U)"] \arrow[d, "\alpha"] & UC_U \arrow[d, "\gamma"] \\
LA_L \arrow[r, "L(h_L)"'] & LC_L \\
A_L \arrow[r, "h_L"'] & C_L
\end{tikzcd}
\qquad
\begin{tikzcd}
UA_U \arrow[d, dash] & UC_U \arrow[d, dash] \\
\alpha \arrow[d] \arrow[r, "h"] & \gamma \arrow[d] \\
LA_L & LC_L
\end{tikzcd}$$

and so we have an arrow h of $(U \downarrow L)$ as on the right. We take this as the composite $g \circ f$ of the given arrows. A diagram chase verifies the category axioms. $\square$

3.3.11 (a) $\left(Id_{\boldsymbol{C}} \downarrow Id_{\boldsymbol{C}}\right) = \boldsymbol{C}^{\downarrow}$.
(b) Using

$$\begin{tikzcd} \boldsymbol{C} \arrow[r, "Id_{\boldsymbol{C}}"] & \boldsymbol{C} & \boldsymbol{C} \arrow[l, "K"'] \end{tikzcd} \qquad \begin{tikzcd} \boldsymbol{C} \arrow[r, "K"] & \boldsymbol{C} & \boldsymbol{C} \arrow[l, "Id_{\boldsymbol{C}}"'] \end{tikzcd}$$

where K is the constant functor with

$$KA = S$$

for each $\boldsymbol{C}$-object A, we have

$$(\boldsymbol{C} \downarrow S) = (Id_{\boldsymbol{C}} \downarrow K) \qquad (S \downarrow \boldsymbol{C}) = (K \downarrow Id_{\boldsymbol{C}})$$

respectively. □

3.3.12 For the three cases the object

$$\begin{array}{c} A_U \\ \alpha \\ A_L \end{array}$$

of $\boldsymbol{Com}$ is sent to the object

$$\begin{array}{ccc} \boldsymbol{U} & \boldsymbol{C}^{\downarrow} & \boldsymbol{L} \\ & \begin{tikzcd} UA_U \arrow[d, dash] \\ \alpha \arrow[d] \\ LA_L \end{tikzcd} & \\ UA_U & & LA_L \end{array}$$

of the indicated category. The arrow

$$\begin{tikzcd} UA_U \arrow[d, dash] & UB_U \arrow[d, dash] \\ \alpha \arrow[d] \arrow[r, "f"] & \beta \arrow[d] \\ LA_L & LB_L \end{tikzcd}$$

of $\boldsymbol{Com}$ is sent to the arrow

$$\begin{array}{ccc} \boldsymbol{U} & \boldsymbol{C}^{\downarrow} & \boldsymbol{L} \\ A_U \xrightarrow{f_U} B_U & \begin{tikzcd} UA_U \arrow[r, "U(f_U)"] \arrow[d, dash] & UB_U \arrow[d, dash] \\ \alpha \arrow[d] & \beta \arrow[d] \\ LA_L \arrow[r, "L(f_L)"'] & LB_L \end{tikzcd} & A_L \xrightarrow{f_L} B_L \end{array}$$

of the indicated category. The composition properties are easy. □

3.3.5 Other examples

3.3.13 Consider an arbitrary group A. We assume it is written multiplicatively. A commutator of A is an element

$$[x,y] = xyx^{-1}y^{-1}$$

for arbitrary $x, y \in A$. Thus A is abelian precisely when the unit 1 is the only commutator. Observe that

$$[x,y]^{-1} = [y,x]$$

so the set of products of commutators is a subgroup δA of A. In particular, A is abelian precisely when δA is the trivial subgroup.

To show that the object assignment

$$A \longmapsto \delta A$$

fills out to a functor we do a little bit more. We show there is a unique commuting square

$$\begin{tikzcd} \delta A \arrow[r, "\iota_A"] \arrow[d, "\delta(f)"'] & A \arrow[d, "f"] \\ \delta B \arrow[r, "\iota_B"'] & B \end{tikzcd}$$

for each group morphism f. Here ι_A and ι_B are the two embeddings.

If there is such a morphism $\delta(f)$ then it can only be

$$f|_{\delta A}$$

the restriction of f to δA. We remember that

$$f(x^{-1}) = f(x)^{-1}$$

for each $x \in A$, and hence

$$f([x,y]) = [f(x), f(y)]$$

for each $x, y \in A$, so that

$$a \in \delta A \Longrightarrow f(a) \in \delta B$$

which is what we want.

The uniqueness in the construction of $\delta(\cdot)$ ensures that it passes across composition of morphisms. For each pair of morphisms

$$A \xrightarrow{\quad f \quad} B \xrightarrow{\quad g \quad} C$$

we have a commuting diagram, as on the left

$$\begin{tikzcd} \delta A \arrow[r, "\iota_A"] \arrow[d, "\delta(f)"'] & A \arrow[d, "f"] \\ \delta B \arrow[r, "\iota_B"] \arrow[d, "\delta(g)"'] & B \arrow[d, "g"] \\ \delta C \arrow[r, "\iota_C"'] & C \end{tikzcd} \qquad \begin{tikzcd} \delta A \arrow[r, "\iota_A"] \arrow[d, "\delta(g)\circ\delta(f)"'] & A \arrow[d, "g\circ f"] \\ \delta C \arrow[r, "\iota_C"'] & C \end{tikzcd}$$

to give a commuting square as on the right. Thus

$$\delta(g) \circ \delta(f) = \delta(g \circ f)$$

by the uniqueness of $\delta(g \circ f)$.

For the second part we show there is a unique commuting square

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & A/\delta A \arrow[d, "f/\delta"] \\ B \arrow[r, "\eta_B"'] & B/\delta B \end{tikzcd}$$

for each group morphism f. Here η_A and η_B are the two canonical quotient morphisms.

The notation

$$f/\delta$$

is not to be taken too seriously.

Since η_A is surjective (epic in $\boldsymbol{Grp}$) there can be at most one such morphism f/δ. There is such a morphism precisely when

$$\mathrm{Ker}(\eta_A) \subseteq \mathrm{Ker}(\eta_B \circ f)$$

in other words

$$\delta A \subseteq \mathrm{Ker}(\eta_B \circ f)$$

that is

$$[x, y] \in \mathrm{Ker}(\eta_B \circ f)$$

for each $x, y \in A$. But we know

$$f([x, y]) = [f(x), f)y)] \in \delta B$$

which gives the required result. The uniqueness in the diagram ensures we have a functor. □

3.3.14 (a) Given an R-set A we require

$$(a \star s) \star t = a \star (st)$$

for $s, t \in S$. But ϕ is a monoid morphism, so that

$$\phi(s)\phi(t) = \phi(st)$$

and hence

$$(a \star s) \star t = (a \cdot \phi(s)) \cdot \phi(t) = a \cdot (\phi(s)\phi(t)) = a \cdot \phi(st) = a \star st$$

as required.

(b) We must show that for each R-morphism

$$A \xrightarrow{f} B$$

the function f is also an S-morphism, that is

$$f(a \star s) = f(a) \star s$$

for each $a \in A$ and $s \in S$. But

$$f(a \star s) = f(a \cdot \phi(s)) = f(a) \cdot \phi(s) = f(a) \star s$$

for the required result. □

3.3.15 We have an object assignment and an arrow assignment

$$\begin{array}{ccc} \boldsymbol{Mon} & & \boldsymbol{MON} \\ R & \longmapsto & \boldsymbol{Set}\text{-}R \\ \phi & \longmapsto & \Phi \end{array}$$

so it suffices to check that the arrow assignment passes across composition. For each $\boldsymbol{Mon}$-arrow ϕ the functor Φ is trivial on objects and arrows, so the requirement is satisfied. □

3.3.16 In Solution 1.2.7 we set up an inverse pair of translations

$$\begin{tikzcd} \boldsymbol{Pfn} \arrow[r, shift left, "L"] & \boldsymbol{Set}_\perp \arrow[l, shift left, "M"] \end{tikzcd}$$

on both objects and arrows. We check these each of these is a functor.

We deal first with L, and it is only the passage across composition that requires much thought. Consider a composible pair of arrows

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C$$

in $\boldsymbol{Pfn}$. Thus

$$L(g \circ f) = L(g) \circ L(f)$$

is required. The arrow composite $g \circ f$ is determined by

$$\begin{tikzcd} A & B & C \\ X \arrow[u, hook] \arrow[ur, "\overline{f}"'] & Y \arrow[u, hook] \arrow[ur, "\overline{g}"'] & \\ U \arrow[u, hook] \arrow[ur, "\overline{f}_{|U}"'] & & \end{tikzcd}$$

where X and Y are the respective domains of definition of f and g, and

$$U = \overline{f}^{\leftarrow}(Y) \qquad \text{that is} \qquad a \in U \iff a \in X \text{ and } \overline{f}(a) \in Y$$

for $a \in A$.

We adjoin a bottom $\perp$ to each of A, B, C, and then set

$$L(f)(a) = \begin{cases} \overline{f}(a) & \text{if } a \in X \\ \perp & \text{if } a \notin X \end{cases} \qquad L(g)(b) = \begin{cases} \overline{g}(b) & \text{if } b \in Y \\ \perp & \text{if } b \notin Y \end{cases}$$

for each $a \in A, b \in B$. There is a similar description of $L(g \circ f)$, that is

$$L(g \circ f)(a) = \left\{ \begin{matrix} \overline{g} \circ \overline{f}_{|U}(a) & \text{if } a \in U \\ \perp & \text{if } a \notin U \end{matrix} \right\} = \left\{ \begin{matrix} \overline{g}(\overline{f}(a)) & \text{if } a \in U \\ \perp & \text{if } a \notin U \end{matrix} \right.$$

for each $a \in A$. In all cases $\perp$ is sent to $\perp$.

Observe that

$$L(f)(a) \in Y \iff a \in X \text{ and } \overline{f}(a) \in Y \iff a \in U$$

for each $a \in A$.

With these, for each $a \in A$ we have

$$\begin{aligned} L(g)\big(L(f)(a)\big) &= \left\{ \begin{matrix} \overline{g}\big(L(f)(a)\big) & \text{if } L(f)(a) \in Y \\ \perp & \text{if } L(f)(a) \notin Y \end{matrix} \right\} \\ &= \left\{ \begin{matrix} \overline{g}\big(\overline{f}(a)\big) & \text{if } a \in X \text{ and } \overline{f}(a) \in Y \\ \perp & \text{if not} \end{matrix} \right\} \\ &= \left\{ \begin{matrix} \overline{g}\big(\overline{f}(a)\big) & \text{if } a \in U \\ \perp & \text{if } a \notin U \end{matrix} \right. \end{aligned}$$

where the second equality follows by the observation above. This shows that L passes across composition in the required fashion.

To show that M passes across composition consider a pair of arrows

$$R \xrightarrow{\ \psi\ } S \xrightarrow{\ \phi\ } T$$

in $\boldsymbol{Set}_\perp$. We remove the bottom from each of R, S, T to obtain sets MR, MS, MT and we let

$$M(\psi) = \psi_{|W} \qquad M(\phi) = \phi_{|X}$$

where these domains of definition are given by

$$r \in W \iff \psi(r) \neq \perp \qquad s \in X \iff \phi(s) \neq \perp$$

for $r \in R$ and $s \in S$. Similarly we have

$$M(\phi \circ \psi) = (\phi \circ \psi)_{|U}$$

where U is given by

$$r \in U \iff \phi(\psi(r)) \neq \perp \iff \psi(r) \in X$$

for each $r \in R$.

These constructions give us two arrows in $\boldsymbol{Pfn}$.

$$MR \xrightarrow{\ M(\phi) \circ M(\psi)\ } MT \qquad\qquad MR \xrightarrow{\ M(\phi \circ \psi)\ } MT$$

$$\begin{tikzcd} MR & MS & MT \\ W \arrow[u, hook] \arrow[ur, "\psi_{|W}"'] & X \arrow[u, hook] \arrow[ur, "\phi_{|X}"'] & \\ V \arrow[u, hook] \arrow[ur, "\psi_{|W|V}"'] & & \end{tikzcd} \qquad\qquad \begin{tikzcd} MR & MT \\ U \arrow[u, hook] \arrow[ur, "(\phi \circ \psi)_{|U}"'] & \end{tikzcd}$$

The left hand one is a composite in $\boldsymbol{Pfn}$, whereas the right hand one is the image of a composite in $\boldsymbol{Set}_\perp$.

The domain of definition V is given by

$$r \in V \iff r \in W \text{ and } \psi_{|W}(r) \in X$$

for $r \in R$. Since $\phi(\perp) = \perp$, for each $r \in R$ we have

$$\psi(r) = \perp \implies \phi(\psi(r)) = \perp$$

and hence

$$\psi(r) \in X \implies r \in W$$

to show that $V = U$. Thus the two arrows are

$$\begin{tikzcd} MR & & MT \\ U \arrow[u, hook] \arrow[urr, "\phi_{|X} \circ \psi_{|U}"'] \end{tikzcd} \qquad \begin{tikzcd} MR & & MT \\ U \arrow[u, hook] \arrow[urr, "(\phi \circ \psi)_{|U}"'] \end{tikzcd}$$

which, since $U \subseteq X$, show that they are equal. $\square$

3.3.17 This exercise extends the earlier Exercise 2.6.6.

(a) To show that the comparison on $S/\sim$ is well-defined suppose

$$[s_1] = [s_1'] \qquad [s_2] = [s_2']$$

for elements $s_1, s_1', s_2, s_2' \in S$. We require

$$s_1 \leq s_2 \iff s_2' \leq s_2'$$

and clearly, by symmetry, a proof of one of the implications will do. From the two assumed equalities we have

$$s_1 \sim s_1' \qquad s_2 \sim s_2'$$

and hence

$$s_1 \leq s_2 \implies s_1' \leq s_1 \leq s_2 \leq s_2' \implies s_1' \leq s_2'$$

as required. This shows that

$$\begin{tikzcd} S \arrow[r, "\eta_S"] & S/\sim \\ s \arrow[r, mapsto] & {[s]} \end{tikzcd}$$

is well-defined and, trivially, it is monotone.

(b) Consider a monotone map

$$S \xrightarrow{\ f\ } T$$

from a preset S to a poset T. We check there is a commuting triangle

$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr, "\eta_S"'] & & T \\ & S/\sim \arrow[ur, "f^\sharp"'] & \end{tikzcd}$$

for some *unique* monotone map $f^\sharp$. Since η_S is surjective there is at most one such map $f^\sharp$. Thus it suffices to show that

$$f^\sharp([s]) = f(s)$$

(for $s \in S$) gives a well-defined monotone function. For $s_1, s_2 \in S$, since f is monotone, we have

$$\begin{aligned} [s_1] = [s_2] &\Longrightarrow s_1 \le s_2 \le s_1 \\ &\Longrightarrow f(s_1) \le f(s_2) \le f(s_1) \Longrightarrow f(s_1) = f(s_2) \end{aligned}$$

where the last step holds since T is a poset. This shows that f is well-defined. A similar argument shows that f is monotone.

This universal property induces the required functor. For the general argument see Solution 3.3.18. □

3.3.18 (a) Consider an arbitrary arrow

$$A \xrightarrow{f} B$$

of ***Src***. We must produce an arrow

$$FA \xrightarrow{F(f)} FB$$

and then check that the two assignments form a functor.

Consider the composite arrow of ***Src***.

$$A \xrightarrow{f} B \xrightarrow{\eta_B} (¿ \circ F)B$$

Applying the universal property to this arrow gives a commuting square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta_A"'] & B \arrow[d, "\eta_B"] \\ (¿ \circ F)A \arrow[r, "¿(!)"'] & (¿ \circ F)B \end{tikzcd}$$

for some unique arrow

$$FA \xrightarrow{\;!\;} FB$$

of ***Trg***. We take this arrow for $F(f)$. Thus

$$F(f) = (\eta_B \circ f)^\sharp$$

in terms of the given notation.

To show that F passes across composition consider a pair of arrows

$$A \xrightarrow{f} B \xrightarrow{g} C$$

of ***Src***. Each of these gives a commuting square.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta_A"'] & B \arrow[r, "g"] \arrow[d, "\eta_B"] & C \arrow[d, "\eta_C"] \\ (¿ \circ F)A \arrow[r, "¿(F(f))"'] & (¿ \circ F)B \arrow[r, "¿(F(g))"'] & (¿ \circ F)C \end{tikzcd}$$

And the composite gives a similar commuting square

$$\begin{tikzcd} A \arrow[r, "g \circ f"] \arrow[d, "\eta_A"'] & C \arrow[d, "\eta_C"] \\ (¿ \circ F)A \arrow[r, "¿(F(g \circ f))"'] & (¿ \circ F)C \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "g \circ f"] \arrow[d, "\eta_A"'] & C \arrow[d, "\eta_C"] \\ (¿ \circ F)A \arrow[r, "¿(F(g) \circ F(f))"'] & (¿ \circ F)C \end{tikzcd}$$

as on the left. The two commuting squares above combine to give the commuting square on the right. The uniqueness of the fill-in arrow gives

$$F(g \circ f) = F(g) \circ F(f)$$

as required.

The required identity property is almost trivial.

(b) For each arrow

$$FA \xrightarrow{g} S$$

of ***Trg*** let

$$A \xrightarrow{g_\flat} ¿S$$

be the composite

$$A \xrightarrow{\eta_A} (¿ \circ F)A \xrightarrow{¿(g)} ¿S$$

of ***Src***.

Trivially, the triangle

$$\begin{tikzcd} A \arrow[rr, "g_\flat"] \arrow[dr, "\eta_A"'] & & ¿S \\ & (¿ \circ F)A \arrow[ur, "¿(g)"'] & \end{tikzcd}$$

commutes, to show that

$${g_\flat}^\sharp = g$$

by the uniqueness property of the $(\cdot)^\sharp$ construction.

Similarly, for each arrow

$$A \xrightarrow{f} ¿S$$

of ***Src*** we have

$${f^\sharp}_\flat = f$$

by the given commuting triangle.

This sets up an inverse pair

$$\begin{array}{ccc} f & \longmapsto & f^\sharp \\ \boldsymbol{Src}[A, ¿S] & & \boldsymbol{Trg}[FA, S] \\ g_\flat & \longleftarrow\!\! | & g \end{array}$$

of bijections, and it is not too hard to show that each is natural for variations of A and S. □

3.4 Natural transformations defined

3.4.1 (a) Consider the category

$$(\downarrow) \qquad \begin{array}{c} 0 \\ \downarrow \\ 1 \end{array}$$

where the two identity arrows have been omitted from the picture. Let $\boldsymbol{C}$ be an arbitrary category. A covariant functor

$$(\downarrow) \longrightarrow \boldsymbol{C}$$

must select two objects A_0, A_1 of $\boldsymbol{C}$ and an arrow between them.

$$\begin{tikzcd} A_0 \arrow[d, "\alpha"] \\ A_1 \end{tikzcd}$$

(It also selects the identity arrows on these two objects, but that is not a problem.) There are no non-trivial composition properties here. Thus such a functor is precisely an object of $\boldsymbol{C}^{\downarrow}$.

Consider two such functors

$$\begin{tikzcd} A_0 \arrow[d, "\alpha"] & B_0 \arrow[d, "\beta"] \\ A_1 & B_1 \end{tikzcd}$$

that is two objects of $\boldsymbol{C}^{\downarrow}$. There are just two source objects, namely 0 and 1, so a natural transformation between these functors must select two arrows

$$A_0 \xrightarrow{f_0} B_0 \qquad A_1 \xrightarrow[f_1]{} B_1$$

of $\boldsymbol{C}$. The naturality requires that the square

$$\begin{tikzcd} A_0 \arrow[r, "f_0"] \arrow[d, "\alpha"] & B_0 \arrow[d, "\beta"] \\ A_1 \arrow[r, "f_1"'] & B_1 \end{tikzcd}$$

commutes. Thus a natural transformation is just an arrow of $\boldsymbol{C}^{\downarrow}$.

(b) Re-read Solution 1.3.10. You will find that the objects of $\boldsymbol{C}^{\nabla}$ are essentially the 'functors' from the graph $\nabla = (N, E)$, and the arrows of $\boldsymbol{C}^{\nabla}$ are essentially the 'natural transformations' between these 'functors'. The only problem is that ∇ is not a category and there are no composition requirements. Don't worry about that. Composition can be dealt with later. □

3.4.2 A presheaf on S is a contravariant functor

$$S \longrightarrow \boldsymbol{Set}$$

and an arrow between presheaves is a natural transformation between these functors. □

3.4.3 We look at the contravariant case. Let A and B be a pair of (right) R-sets viewed as functors from the 1-object category R. Since R has just one object a natural transformation will be a single function

$$A \xrightarrow{\ f\ } B$$

subject to certain conditions. For each arrow r of R ($r \in R$) the square

$$\begin{tikzcd} A \arrow[r, "f"] & B \\ A \arrow[u, "\alpha_r"] \arrow[r, "f"'] & B \arrow[u, "\beta_r"'] \end{tikzcd}$$

must commute. Here α_r and β_r are the 1-placed operations on A and B selected by r. Thus

$$f \circ \alpha_r = \beta_r \circ f$$

is the required condition. In terms of elements and actions this is

$$f(ar) = f(\alpha_r(a)) = (f \circ \alpha_r)(a) = (\beta_r \circ f)(a) = \beta_r(f(a)) = f(a)r$$

so that f is just an arrow of $\boldsymbol{Set}$-R. $\square$

3.4.4 We must show that for each $\boldsymbol{Src}$-arrow

$$A \xrightarrow{\ f\ } B$$

the induced $\boldsymbol{Trg}$-square

$$\begin{tikzcd} GA \arrow[r, "\sigma_A"] \arrow[d, "G(f)"'] & FA \arrow[d, "F(f)"] \\ GB \arrow[r, "\sigma_B"'] & Fb \end{tikzcd}$$

commutes. To do that consider the following diagram.

$$\begin{tikzcd} FA \arrow[r, "\tau_A"] \arrow[d, "F(f)"'] & GA \arrow[r, "\sigma_A"] \arrow[d, "G(f)"] & FA \arrow[d, "F(f)"] \\ FB \arrow[r, "\tau_B"'] & GB \arrow[r, "\sigma_B"'] & FB \end{tikzcd}$$

The left hand square commutes, since $\tau_\bullet$ is natural. The outer cell commutes since we are given

$$\sigma_\bullet \circ \tau_\bullet = id_\bullet$$

and hence both trips from FA to FB are equal to $F(f)$. But now

$$F(f) \circ \sigma_A \circ \tau_A = \sigma_B \circ \tau_B \circ F(f) = \sigma_B \circ G(f) \circ \tau_A$$

and hence since

$$\tau_\bullet \circ \sigma_\bullet = id_\bullet$$

we have

$$F(f) \circ \sigma_A = F(f) \circ \sigma_A \circ \tau_A \circ \sigma_A = \sigma_B \circ G(f) \circ \tau_A \circ \sigma_A = \sigma_B \circ G(f)$$

for the required result. □

3.5 Examples of natural transformations

3.5.1 Given an arrow

$$L \xrightarrow{k} K$$

we certainly have a function

$$[A, L] \xrightarrow{k \circ -} [A, K]$$

for each object A. For this to be natural we require the square

$$\begin{tikzcd} A & & {[A,L]} \arrow[r, "k \circ -"] \arrow[d, "- \circ f"'] & {[A,K]} \arrow[d, "- \circ f"] \\ B \arrow[u, "f"] & & {[B,L]} \arrow[r, "k \circ -"'] & {[B,K]} \end{tikzcd}$$

to commute for each arrow f. This is trivially satisfied. □

3.5.2 Each of F and G is a composite of contravariant functors, and hence each is a covariant functor.

We need an explicit description of the behaviour on arrows. Consider an arrow

$$A \xrightarrow{f} B$$

of $\boldsymbol{C}$. This must produce a function

$$\boldsymbol{Set}[\boldsymbol{C}[A, P], R] \xrightarrow{F(f)} \boldsymbol{Set}[\boldsymbol{C}[B, P], R]$$

mapping functions to functions. In other words, for each input function

$$\boldsymbol{C}[A, P] \xrightarrow{l} R$$

an output function

$$\boldsymbol{C}[B, P] \longrightarrow R$$

is required. Thus we set

$$F(f)(l)(b) = l(b \circ f)$$

for each *arrow*

$$B \xrightarrow{\ b\ } P$$

of $\boldsymbol{C}$.

There is a similar description of G.

For the natural transformation we require a function

$$\boldsymbol{Set}[\boldsymbol{C}[A, P], R] \xrightarrow{\ \tau_A\ } \boldsymbol{Set}[\boldsymbol{C}[A, Q], S]$$

for each object A of $\boldsymbol{C}$. In other words, for each input function

$$\boldsymbol{C}[A, P] \xrightarrow{\ l\ } R$$

an output function

$$\boldsymbol{C}[A, Q] \longrightarrow S$$

is required. Thus we set

$$\tau_A(l)(a) = (s \circ l)(p \circ a)$$

for each *arrow*

$$A \xrightarrow{\ a\ } Q$$

of $\boldsymbol{C}$. Finally, we show that the square

$$\begin{tikzcd} A & & FA \arrow[r, "\tau_A"] \arrow[d, "F(f)"'] & GA \arrow[d, "G(f)"] \\ B \arrow[u, "f"] & & FB \arrow[r, "\tau_B"'] & GB \end{tikzcd}$$

commutes for each arrow f of $\boldsymbol{C}$. In other words we require

$$\big(\tau_B \circ F(f)\big)(l) = \big(G(f) \circ \tau)A\big)(l)$$

for each member

$$\boldsymbol{C}[A, P] \xrightarrow{\ l\ } R$$

of FA. Thus we require

$$\big(\tau_B \circ F(f)\big)(l)(b) = \big(G(f) \circ \tau)A\big)(l)(b)$$

for each *arrow*

$$B \xrightarrow{\ b\ } P$$

of $\boldsymbol{C}$. But we have

$$\begin{aligned}
\big(\tau_B \circ F(f)\big)(l)(b) &= \tau_B\big(F(f)(l)\big)(b) \\
&= \big(s \circ F(f)(l)\big)(p \circ b) \\
&= s\big(F(f)(l)(p \circ b)\big) \\
&= s\big((l(p \circ b \circ f)\big) \qquad = (s \circ l)(p \circ b \circ f)
\end{aligned}$$

and

$$\big(G(f) \circ \tau_A\big)(l)(b) = G(f)\big(\tau_A(l)\big)(b) = \tau_A(l)(b \circ f) = (s \circ l)(p \circ b \circ f)$$

to give the required result. □

3.5.3 It remains to check the last bit of part (b), namely that

$$\tau_A(l) = F(l)(k)$$

holds. We track id_K round both sides of the given commuting square

$$\begin{tikzcd}
id_K \arrow[rrr, mapsto] \arrow[dddd, mapsto] & & & \tau_K(id_K) = k \arrow[ddd, mapsto] \\
& {[K,K]} \arrow[r, "\tau_K"'] \arrow[dd, "l \circ -"'] & FK \arrow[dd, "F(l)"] & \\
& & & \\
& {[K,B]} \arrow[r, "\tau_A"] & FA & F(l)(k) \\
l \arrow[rr, mapsto] & & \tau_A(l) &
\end{tikzcd}$$

for the required result. □

3.5.4 The particular case where the category $\boldsymbol{C}$ is a poset is dealt with in Example 1.4.1. □

3.5.5 This is the contravariant version of the first part of Example 3.5.2. □

3.5.6 This is the contravariant version of the second part of Example 3.5.2. □

3.5.7 For each set A we set

$$\eta^{\exists}_A(a) = \{a\} \qquad \eta^{\forall}_A(a) = \{a\}'$$

for each $a \in A$. □

3.5.8 We require $q = p \circ f$, but since each of these functions has target $\mathsf{2}$ is suffices to show they output 1 at the same inputs. For each $b \in B$ we have

$$\begin{aligned}
q(b) = 1 &\iff \chi_B(f^{l}a(X))(b) = 1 \\
&\iff b \in f^{\leftarrow}(X) \\
&\iff f(b) \in X \\
&\iff \chi_A(X)(f)(b) = 1 \qquad \iff (p \circ f)(b) = 1
\end{aligned}$$

for the required result. □

3.5.9 (a) From the diagram in Example 3.5.5 we require

$$\eta_B(f(a)) = \Pi(f)(\eta_A(a))$$

for each $a \in A$. Each side of this equality is a member of $\mathcal{P}^2 B$. Thus, for $Y \in \mathcal{P}B$, we have

$$\begin{aligned} Y \in \Pi(f)(\eta_A(a)) &\iff f^{\leftarrow}(Y) \in \eta_A(a) \\ &\iff a \in f^{\leftarrow}(Y) \\ &\iff f(a) \in Y \qquad \iff Y \in \eta_B(f(a)) \end{aligned}$$

to give the required equation.

(b) As in the question let I be the inverse power set endofunctor on $\boldsymbol{Set}$, and also let J be $\boldsymbol{Set}[-,\mathsf{2}]$, the endo-hom-functor on $\boldsymbol{Set}$. We know that the arrow behaviour of J is by composition. Thus we have

$$\begin{tikzcd} A \arrow[d, "f"] & \mathsf{J}A & q \circ f & \mathsf{J}^2 A \arrow[d, "\mathsf{J}^2(f)"] & \pi \arrow[d, mapsto] \\ B & \mathsf{J}B \arrow[u, "\mathsf{J}(f)"] & q \arrow[u, mapsto] & \mathsf{J}^2 B & \pi \circ \mathsf{J}(f) \end{tikzcd}$$

for each arrow f of $\boldsymbol{Set}$, each $q : B \longrightarrow \mathsf{2}$, and each $\pi : [A, \mathsf{2}] \longrightarrow \mathsf{2}$. In particular, we have

$$\mathsf{J}^2(f)(\pi)(q) = \pi(q \circ f)$$

for each such f, π, q.

For each set A consider the 'evaluation' function

$$A \xrightarrow{\ \phi_A\ } \mathsf{J}^2 A \qquad \text{given by} \qquad \phi_A(a)(p) = p(a)$$

for each $p : A \longrightarrow \mathsf{2}$ and $a \in A$. We show this is a natural transformation. In other words we show that the square

$$\begin{tikzcd} A \arrow[r, "\phi_A"] \arrow[d, "f"'] & \mathsf{J}^2 A \arrow[d, "\mathsf{J}^2(f)"] \\ B \arrow[r, "\phi_B"'] & \mathsf{J}^2 B \end{tikzcd} \qquad \phi_B \circ f = \mathsf{J}^2(f) \circ \phi_A$$

commutes. For each $a \in A$ and $q : B \longrightarrow \mathsf{2}$ we have

$$(\phi_B \circ f)(a)(q) = \phi_B(f(a))(q) = q(f(a)) = (q \circ f)(a)$$

and

$$\big(\mathsf{J}^2(f) \circ \phi_A\big)(a)(q) = \mathsf{J}^2(f)\big(\phi_A(a)\big)(q) = \phi_A(a)(q \circ f) = (q \circ f)(a)$$

to give the required result.

(c) On the whole the use of characters rather than subsets does lead to neater results.

The statement in part (b) of the question that 'I^2 is naturally isomorphic to J^2' is a bit glib. It is true, but not entirely obvious. It can be justified using horizontal and vertical composition of natural transformations. This is not something we can go into here, but we can give a hint of what it is about.

We know we have an inverse pair of natural transformations

$$\begin{tikzcd} \mathsf{I} \arrow[r, shift left, "\chi_\bullet"] & \mathsf{J} \arrow[l, shift left, "\xi_\bullet"] \end{tikzcd}$$

given by

$$\chi_A(X)(a) = 1 \iff a \in X \qquad a \in \xi_A(p) \iff p(a) = 1$$
$$\chi_A(X)(a) = 0 \iff a \notin X \qquad a \notin \xi_A(p) \iff p(a) = 0$$

for each set $A, X \in \mathcal{P}A$, $p : A \longrightarrow \mathbf{2}$, and $a \in A$. We modify these in several ways.

For each set A we may hit each of the arrows

$$\mathsf{I}A \xrightarrow{\chi_A} \mathsf{J}A \qquad \mathsf{I}A \xleftarrow{\xi_A} \mathsf{J}A$$

with each of the two contravariant functors I and J, and take particular instances (by replacing the set A).

$$\mathsf{I}^2A \xleftarrow{\mathsf{I}(\chi_A)} (\mathsf{I}\circ\mathsf{J})A \qquad (\mathsf{J}\circ\mathsf{I})A \xrightarrow{\mathsf{J}(\xi_A)} \mathsf{J}^2A$$
$$\mathsf{I}^2A \xrightarrow{\chi_{\mathsf{I}A}} (\mathsf{J}\circ\mathsf{I})A \qquad \mathsf{I}^2A \xleftarrow{\xi_{\mathsf{I}A}} (\mathsf{J}\circ\mathsf{I})A$$
$$(\mathsf{I}\circ\mathsf{J})A \xrightarrow{\chi_{\mathsf{J}A}} \mathsf{J}^2A \qquad (\mathsf{I}\circ\mathsf{J})A \xleftarrow{\xi_{\mathsf{J}A}} \mathsf{J}^2A$$

We can combine these in various ways. In particular, we may form

$$\begin{tikzcd} \mathsf{I}^2 \arrow[r, shift left, "\Gamma_A = \mathsf{J}(\xi_A)\circ\chi_{\mathsf{I}A}"] & \mathsf{J}^2 \arrow[l, shift left, "\Delta_A = \mathsf{I}(\chi_A)\circ\xi_{\mathsf{J}A}"] \end{tikzcd}$$

going via $(\mathsf{J}\circ\mathsf{I})A$ for Γ and $(\mathsf{I}\circ\mathsf{J})A$ for Δ. We show that these are natural transformations, and are inverses.

For an arbitrary arrow

$$A \xrightarrow{f} B$$

consider the following two commuting squares.

$$\begin{tikzcd} \mathsf{I}^2A \arrow[r, "\chi_{\mathsf{I}A}"] \arrow[d, "\mathsf{I}^2(f)"'] & (\mathsf{J}\circ\mathsf{I})A \arrow[d, "\mathsf{J}(\mathsf{I}(f))"] \\ \mathsf{I}^2B \arrow[r, "\chi_{\mathsf{I}B}"'] & (\mathsf{J}\circ\mathsf{I})B \end{tikzcd} \qquad \begin{tikzcd} \mathsf{I}A \arrow[r, "\xi_A"] \arrow[d, "\mathsf{I}(f)"'] & \mathsf{J}A \arrow[d, "\mathsf{J}(f)"] \\ \mathsf{I}B \arrow[r, "\xi_B"'] & \mathsf{J}B \end{tikzcd}$$

The right hand square is an instance of the naturality of $\xi_\bullet$ across the arrow f. The left hand square is an instance of the naturality of $\chi_\bullet$ across the arrow $\mathsf{I}(f)$.

Hitting the right hand square with J give a commuting diagram

$$\begin{tikzcd} \mathsf{I}^2A \arrow[r, "\chi_{\mathsf{I}A}"] \arrow[d, "\mathsf{I}^2(f)"'] & (\mathsf{J}\circ\mathsf{I})A \arrow[r, "\mathsf{J}(\xi_A)"] \arrow[d, "(\mathsf{J}\circ\mathsf{I})(f)"] & \mathsf{J}^2A \arrow[d, "\mathsf{J}(f)"] \\ \mathsf{I}^2B \arrow[r, "\chi_{\mathsf{I}B}"'] & (\mathsf{J}\circ\mathsf{I})B \arrow[r, "\mathsf{J}(\xi_B)"'] & \mathsf{J}^2B \end{tikzcd}$$

to show that $\Gamma_\bullet$ is natural.

A similar argument shows that $\Delta_\bullet$ is natural.

To show that

$$\mathsf{I}^2 \xrightarrow{\Delta_A \circ \Gamma_A} \mathsf{I}^2$$

is the identity consider any

$$\mathcal{X} \in \mathsf{I}^2 A = \mathcal{P}^2 A \qquad X \in \mathsf{I}A = \mathcal{P}A \qquad \text{and let} \qquad Y = \chi_A(X)$$

to use in the calculation. We have

$$\begin{aligned}
X \in \big(\Delta_A \circ \Gamma_A\big)(\mathcal{X}) &\iff X \in \big(\mathsf{I}(\chi_A) \circ \xi_{\mathsf{J}A} \circ \Gamma_A\big)(\mathcal{X}) \\
&\iff X \in \mathsf{I}(\chi_A)\Big(\big(\xi_{\mathsf{J}A} \circ \Gamma_A\big)(\mathcal{X})\Big) \\
&\iff X \in \chi_A^{\leftarrow}\Big(\big(\xi_{\mathsf{J}A} \circ \Gamma_A\big)(\mathcal{X})\Big) \\
&\iff Y = \chi_A(X) \in \big(\xi_{\mathsf{J}A} \circ \Gamma_A\big)(\mathcal{X}) \\
&\iff Y \in \xi_{\mathsf{J}_A}\big(\Gamma_A(\mathcal{X})\big) \\
&\iff \big(\Gamma_A(\mathcal{X})\big)(Y) = 1 \\
&\iff \big(\mathsf{J}(\xi_A) \circ \chi_{\mathsf{J}A}\big)(\mathcal{X})(Y) = 1 \\
&\iff \mathsf{J}(\xi_A)\big(\chi_{\mathsf{J}A}(\mathcal{X})\big)(Y) = 1 \\
&\iff \big(\chi_{\mathsf{J}A}(\mathcal{X}) \circ \xi_A\big)(Y) = 1 \\
&\iff \chi_{\mathsf{J}A}(\mathcal{X})\big(\xi_A(Y)\big) = 1 \\
&\iff \xi_A(Y) \in \mathcal{X} \iff \big(\xi_A \circ \chi_A\big)(X) \in \mathcal{X}
\end{aligned}$$

and now we remember that

$$\big(\xi_A \circ \chi_A\big)(X) = X \quad \text{to give} \quad \big(\Delta_A \circ \Gamma_A\big)(\mathcal{X}) = \mathcal{X}$$

as required.

A similar argument deals with $\Gamma_A \circ \Delta_A$.

In the same way we can show how $\eta_\bullet$ and $\phi_\bullet$ are related.

Consider the following composite.

$$A \xrightarrow{\phi_A} \mathsf{J}^2 A \xrightarrow{\Delta_A} \mathsf{I}^2 A$$

For each $a \in A$ and $X \in \mathsf{I}A$ we have

$$\begin{aligned}
X \in (\Delta_A \circ \phi_A)(a) &\iff X \in \big(\mathsf{I}(\chi_A) \circ \xi_{\mathsf{J}A} \circ \phi_A\big)(a) \\
&\iff X \in \mathsf{I}(\chi_A)\big((\xi_{\mathsf{J}A} \circ \phi_A)(a)\big) \\
&\iff \chi_A(X) \in (\xi_{\mathsf{J}A} \circ \phi_A)(a) \\
&\iff \chi_A(X) \in \xi_{\mathsf{J}A}\big(\phi_A(a)\big) \\
&\iff \phi_A(a)\big(\chi_A(X)\big) = 1 \\
&\iff \chi_A(X)(a) = 1 \iff a \in X \iff X \in \eta_A(a)
\end{aligned}$$

and hence

$$\Delta_A \circ \phi_A = \eta_A$$

to show how $\phi_\bullet$ determines $\eta_\bullet$. A similar argument gives

$$\Gamma_A \circ \eta_A = \phi_A$$

and hence $\phi_\bullet$ and $\eta_\bullet$ are equivalent. $\square$

3.5.10 (a) Actually, you don't need to write them down, you can look them up in your favourite textbook on linear algebra. However, you should count the overloading of the symbols. How many different operations does '$\times$' stand for? How many multiplication operations are indicated by juxtaposition? Don't tell anybody, but this is done to confuse the people in suits who think they know how to run things.

(b) We don't need to do them all but we should at least look at the crucial axiom on the action, namely that

$$r(\alpha+\beta) = r\alpha + r\beta$$

for each scalar r and pair α, β of characters. Recall that this means that

$$\big(r(\alpha+\beta)\big)(a) = \big(r\alpha + r\beta\big)(a)$$

for each vector a. Remembering how the operations are defined we have

$$\begin{aligned}
\big(r(\alpha+\beta)\big)(a) &= r\Big(\big((\alpha+\beta\big)(a)\Big)\\
&= r\big(\alpha(a)+\beta(a)\big)\\
&= r\big(\alpha(a)\big) + r\big(\beta(a)\big)\\
&= \big(r\alpha\big)(a) + \big(r\beta\big)(a) \quad = \big(r\alpha + r\beta\big)(a)
\end{aligned}$$

for the required result. You should make sure that you know why each step is valid. Also, you young things should realize how lucky you are. When I was a lad there was only one size of bracket, which meant that calculations like this were harder to read.

(c) Once you have a modicum of experience with categories and functors this becomes a sybilean construction. We have to convert a linear transformation between two spaces, as on the left,

$$V \xrightarrow{\ f\ } W \qquad \longmapsto \qquad W^* \xrightarrow{\ f^*\ } V^*$$

into a linear transformation between the two dual spaces, as on the right. The direction of the linear transformation must swap round.

Consider a member $\gamma \in W^*$, that is a character on W. As functions we have a composite

$$V \xrightarrow{\ f\ } W \xrightarrow{\ \gamma\ } K$$

which we check is a character on V. However, we view K as a vector space over itself, and then both f and γ are linear transformations. The composite of two linear transformations is a linear transformation, so

$$V \xrightarrow{\ f^*(\gamma) = \gamma\circ f\ } K$$

is a linear transformation, a character on V. This gives us a function

$$f^* : W^* \longrightarrow V^*$$

which we show is a linear transformation between the two dual spaces.

We require

$$f^*(\gamma+\delta) = f^*(\gamma) + f^*(\delta) \qquad f^*(r\gamma) = rf^*(\gamma)$$

for each characters γ, δ on W^* and scalar r. We verify these by evaluation at an arbitrary vector $a \in V$. Let's check the right hand equality.

We have

$$\begin{aligned}\big(f^*(r\gamma)\big)(a) &= \big((r\gamma)\circ f\big)(a)\\ &= \big(r\gamma\big)\big(f(a)\big)\\ &= r\Big(\gamma\big(f(a)\big)\Big)\\ &= r\Big(\big(\gamma\circ f\big)(a)\Big)\\ &= r\Big(f^*\big(\gamma\big)(a)\Big) \quad = \big(rf^*(\gamma)\big)(a)\end{aligned}$$

for the required result.

This shows that we certainly have a construction that converts a linear transformation into a linear transformation with the required type (source and target). But why do we have a functor? We need to show that various diagrams commute. However!

Notice that if we forget some of the carried structure we have the contravariant hom-functor induced by K.

$$\boldsymbol{Vect}_K \xrightarrow{[-,K]} \boldsymbol{Set}$$

Each diagram we must look at is a diagrams of functions (enriched in some way), so that hom-functor calculations give us the required results.

What we have here is an enriched hom-functor. These occur all over mathematics, so you have just seen a very useful trick.

Notice that if you had to do this exercise from scratch there would by spondoodles of rather trivial calculations, and you might find it hard to see the wood from the trees. Category theory helps to organize this kind of thing. It chops the dead wood and grows the trees. $\square$

3.5.11 We look at (1, 2, 3) in turn.

(1) We require

$$a^\frown(\alpha+\beta) = a^\frown(\alpha) + a^\frown(\beta) \qquad a^\frown(r\alpha) = ra^\frown(\alpha)$$

for each scalar r, each vector a, and each pair of characters α, β. Remembering the definition of $a^\frown$ these translate to

$$(\alpha+\beta)(r) = \alpha(r) + \beta(r) \qquad \big(r\alpha\big)(a) = r\big(\alpha(a)\big)$$

which are true by definition.

(2) We require

$$(a+b)^\frown = a^\frown + b^\frown \qquad (ra)^\frown = ra^\frown$$

for each scalar r and pair a, b of vectors. The two equalities are verified by evaluation at an arbitrary vector α. The conditions become

$$\alpha(a+b) = \alpha(a) + \alpha(b) \qquad \alpha(ra) = r\big(\alpha(a)\big)$$

which are true.

(3) We must show that the square commutes

$$\begin{tikzcd} V \arrow[r, "(\cdot)^\frown"] \arrow[d, "f"'] & V^{**} \arrow[d, "f^{**}"] \\ W \arrow[r, "(\cdot)^\frown"'] & W^{**} \end{tikzcd}$$

where f is an arbitrary linear transformation. To do that we take an arbitrary vector $a \in V$ and track it both ways to the second dual W^{**}.

By definition we have

$$f^{**}(\mathfrak{a}) = \mathfrak{a} \circ f^*$$

for each $\mathfrak{a} \in V^{**}$. (We are beginning to run out of different types of letters, but that won't last long.)

Consider any $a \in V$. Then going via W we obtain

$$f(a)^\frown \in W^{**} \quad \text{so that} \quad f(a)^\frown(\gamma) = \gamma(f(a)) = (\gamma \circ f)(a)$$

for each $\gamma \in W^*$.

Going the other way we have

$$f^{**}(a^\frown) = a^\frown \circ f^*$$

so that

$$f^{**}(a^\frown)(\gamma) = \big(a^\frown \circ f^*\big)(\gamma) = a^\frown\big(f^*(\gamma)\big) = a^\frown\big(\gamma \circ f\big) = \big(\gamma \circ f\big)(a)$$

to give the required result. □

3.5.12 By Exercises 3.3.7 we have a bijection

$$\mathcal{O}S \xrightarrow{\chi_S} \Xi S$$

for each space S. It suffices to show that χ_S is natural for variation of S. To this end consider the square

$$\begin{tikzcd} S & \mathcal{O}S \arrow[r, "\chi_S"] \arrow[d, "\phi^\leftarrow"'] & \Xi S \arrow[d, "- \circ \phi"] \\ T \arrow[u, "\phi"] & \mathcal{O}T \arrow[r, "\chi_T"'] & \Xi T \end{tikzcd}$$

induced by a continuous map ϕ, as on the left. We must show that the square commutes. Thus we required

$$\chi_T(\phi^\leftarrow(U)) = \chi_S(U) \circ \phi$$

for each $U \in \mathcal{O}S$. These two functions can return only 0 and 1 as a value. For each $s \in S$ we have

$$\chi_T(\phi^\leftarrow(U))(s) = 1 \iff s \in \pi^\leftarrow(U) \iff \phi(s) \in U \iff \chi_S(\phi(U)) = 1$$

which gives the required equality. □

3.5.13 Let's consider the more complicated version, that dealt with by Exercise 3.3.8.

For a category $\boldsymbol{C}$ we have two functors

$$\begin{tikzcd} \boldsymbol{C}\times\boldsymbol{C} \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & \boldsymbol{C} \end{tikzcd}$$

where

$$F(A_1, A_2) = A_1 \times A_2 \qquad G(A_1, A_2) = A_1$$

are the two object assignments. (One thing you should be wary of here is the two different uses of '$\times$'. The first use gives the cartesian product of the category $\boldsymbol{C}$ with itself, and the second gives the internal product object in $\boldsymbol{C}$.)

For each pair (A_1, A_2) of objects we have an arrow

$$F(A_1, A_2) \xrightarrow{p_{(A_1,A_2)}} G(A_1, A_2)$$

namely the projection arrow. We must show this is natural for variation of (A_1, A_2).

Consider an arrow of $\boldsymbol{C} \times \boldsymbol{C}$, that is a pair of arrows of $\boldsymbol{C}$ as indicated on the left of the following diagram.

$$\begin{tikzcd} A_1 \arrow[d, "f_1"] & A_2 \arrow[d, "f_2"] & & F(A_1,A_2) \arrow[r, "p_{(A_1,A_2)}"] \arrow[d, "f_1\times f_2"'] & G(A_1,A_2) \arrow[d, "f_1"] \\ B_1 & B_2 & & F(B_1,B_2) \arrow[r, "p_{(B_1,B_2)}"'] & G(B_1,B_2) \end{tikzcd}$$

We must show that the square on the right commutes. But this is just one of the commuting squares that determines the projection, as in Solution 3.3.8 (in a slightly different notation). □

3.5.14 We must first set up ϕ_A for an arbitrary object A of $\boldsymbol{C}$. To do that consider the diagram on the left.

$$\begin{tikzcd} R \arrow[r, "\phi"] & S & & R \arrow[r, "\phi"] & S \\ A\times R \arrow[u, "r_A"] \arrow[d, "p_A"'] & A\times S \arrow[u, "s_A"'] \arrow[d, "q_A"] & & A\times R \arrow[u, "r_A"] \arrow[d, "p_A"'] \arrow[r, "\phi_A"] \arrow[ur, phantom, "(1)"] \arrow[dr, phantom, "(2)"] & A\times S \arrow[u, "s_A"'] \arrow[d, "q_A"] \\ A \arrow[r, "id_A"'] & A & & A \arrow[r, "id_A"'] & A \end{tikzcd}$$

This consists of two product wedges with the associated projections r_A, p_A, s_A, q_A. The generating arrow ϕ has also been inserted. Using the product property of $A \times S$ we see there is a unique arrow ϕ_A to produce a pair of commuting squares as on the right. This is just $id_A \times \phi$ in product notation. We have labelled the two squares for later use.

We must show that $\phi_\bullet$ is natural for variation of the object. Let's set up that problem.

Consider an arbitrary arrow f as on the left. We must show that the square on the right commutes.

$$
\begin{tikzcd}
A \arrow[d, "f"'] \\
B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \times R \arrow[r, "\phi_A"] \arrow[d, "f \times id_R"'] \arrow[dr, phantom, "(?)"] & A \times S \arrow[d, "f \times id_S"] \\
B \times R \arrow[r, "\phi_B"'] & B \times S
\end{tikzcd}
$$

How are we going to do this?

Consider the diagram on the left.

$$
\begin{tikzcd}
R \arrow[r, "\phi"] & S \\
A \times R \arrow[u, "r_A"] \arrow[d, "p_A"'] & B \times S \arrow[u, "s_B"'] \arrow[d, "q_B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "\phi"] & S \\
A \times R \arrow[u, "r_A"] \arrow[d, "p_A"'] \arrow[r, "\psi"] \arrow[ur, phantom, "(rs)"] \arrow[dr, phantom, "(pq)"] & B \times S \arrow[u, "s_B"'] \arrow[d, "q_B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

This is not the same as the first diagram. We have now varied the object A along the arrow f. However, in the same way the product property of $B \times S$ ensures there is a unique arrow ψ that makes both squares commute, that is

$$
\begin{aligned}
&(rs) \quad s_B \circ \psi = \phi \circ r_A \\
&(pq) \quad q_B \circ \psi = f \circ p_A
\end{aligned}
$$

for some unique arrow ψ. In the square $(?)$ we show that both trips from $A \times R$ to $B \times S$

$$(f \times id_S) \circ \phi_A \qquad \phi_B \times (f \times id_R)$$

satisfy (rs) and (pq), and hence must be equal.

We need a property of the product construction, namely that projections are natural.

$$
\begin{tikzcd}
R \arrow[r, "id_R"] & R \\
A \times R \arrow[u, "r_A"] \arrow[d, "p_A"'] \arrow[r, "f \times id_R"] \arrow[ur, phantom, "(3)"] \arrow[dr, phantom, "(4)"] & B \times R \arrow[u, "r_B"'] \arrow[d, "p_B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
S \arrow[r, "id_S"] & S \\
A \times R \arrow[u, "sa_A"] \arrow[d, "q_A"'] \arrow[r, "f \times id_S"] \arrow[ur, phantom, "(5)"] \arrow[dr, phantom, "(6)"] & B \times S \arrow[u, "s_B"'] \arrow[d, "q_B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

Thus all four of these squares commute.

We are now ready to do the several small calculations.

With

$$\psi = (f \times id_S) \circ \phi_A$$

a use of $(6, 2)$ and then a use of $(5, 1)$ gives

$$\begin{array}{lllll}
(rs) & s_B \circ \psi = s_B \circ (f \times id_S) \circ \phi_A = & s_A \circ \phi_A & = \phi \circ r_A \\
(pq) & q_B \circ \psi = q_B \circ (f \times id_S) \circ \phi_A = & f \circ a_A \circ \phi_A & = f \circ p_A
\end{array}$$

to show that this ψ satisfies the two required conditions.

With

$$\psi = \phi_B \circ (f \times id_R)$$

a use of $(1, 3)$ and then a use of $(2, 4)$, with (3) and (2) in the B version, gives

$$\begin{array}{lllll}
(rs) & s_B \circ \psi = & s_B \circ \phi_B \circ (f \times id_R) = & \phi \circ r_B \circ (f \times id_R) & = \phi \circ r_A \\
(pq) & q_B \circ \psi = & q_B \circ \phi_B \circ (f \times id_R) = & p_B \circ (f \times id_R) & = f \circ p_A
\end{array}$$

to show that this ψ satisfies the two required conditions.

This completes the proof.

This and the next solution are a nice illustration of why reading and writing proofs in category theory can be a bit tricky. Often many small diagrams have to be looked at, and there is a tendency to combine these into one big diagram, and so make it incomprehensible. □

3.5.15 If we fix two of the three inputs then each L and each R is a composite of various known functors. However, let's see if we can make sense of the 3-placed version.

Let

$$\begin{tikzcd}
A_0 \arrow[d, "\alpha"] & B_0 \arrow[d, "\beta"] & C_0 \arrow[d, "\gamma"] \\
A_1 & B_1 & C_1
\end{tikzcd}$$

be a triple of arrows in $\boldsymbol{C}$. What are the resulting arrows

$$\begin{tikzcd}
L(A_0, B_0, C_0) \arrow[d, "L(\alpha, \beta, \gamma)"] & R(A_0, B_0, C_0) \arrow[d, "R(\alpha, \beta, \gamma)"] \\
L(A_1, B_1, C_1) & R(A_1, B_1, C_1)
\end{tikzcd}$$

in $\boldsymbol{C}$?

We look at L first. Consider the cells

$$\begin{tikzcd}
& A_0 \times C_0 \arrow[dl] \arrow[dr] & & & B_0 \times C_0 \arrow[dl] \arrow[dr] & \\
A_0 \arrow[d, "\alpha"] & & C_0 \arrow[d, "\gamma"] & B_0 \arrow[d, "\beta"] & & C_0 \arrow[d, "\gamma"] \\
A_1 & & C_1 & B_1 & & C_1 \\
& A_1 \times C_1 \arrow[ul] \arrow[ur] & & & B_1 \times C_1 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

where the unnamed arrows are the projections. The product property provides two unique arrows

$$
\begin{tikzcd}
 & A_0 \times C_0 \arrow[dl] \arrow[dr] \arrow[ddd, "\alpha \times \gamma"] & \\
A_0 \arrow[d, "\alpha"] & & C_0 \arrow[d, "\gamma"] \\
A_1 & & C_1 \\
 & A_1 \times C_1 \arrow[ul] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & B_0 \times C_0 \arrow[dl] \arrow[dr] \arrow[ddd, "\beta \times \gamma"] & \\
B_0 \arrow[d, "\beta"] & & C_0 \arrow[d, "\gamma"] \\
B_1 & & C_1 \\
 & B_1 \times C_1 \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

to makes the diagrams commute. This is just the functorial property of the binary product.

Consider the left cell where the unnamed arrows are the insertions.

$$
\begin{tikzcd}
 & L(A_0, B_0, C_0) & \\
A_0 \times C_0 \arrow[ur] \arrow[d, "\alpha \times \gamma"] & & B_0 \times C_0 \arrow[ul] \arrow[d, "\beta \times \gamma"] \\
A_1 \times C_1 \arrow[dr] & & B_1 \times C_1 \arrow[dl] \\
 & L(A_1, B_1, C_1) &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & L(A_0, B_0, C_0) \arrow[ddd, "L(\alpha, \beta, \gamma)"] & \\
A_0 \times C_0 \arrow[ur] \arrow[d, "\alpha \times \gamma"] & & B_0 \times C_0 \arrow[ul] \arrow[d, "\beta \times \gamma"] \\
A_1 \times C_1 \arrow[dr] & & B_1 \times C_1 \arrow[dl] \\
 & L(A_1, B_1, C_1) &
\end{tikzcd}
$$

The coproduct property provides a unique arrow to make the right diagram commutes.

Here is the full diagram.

$$
\begin{tikzcd}
 & & & L(A_0, B_0, C_0) \arrow[ddddd, "L(\alpha, \beta, \gamma)"] & & & \\
 & A_0 \times C_0 \arrow[urr] \arrow[dl] \arrow[dr] \arrow[ddd, "\alpha \times \gamma"] & & & & B_0 \times C_0 \arrow[ull] \arrow[dl] \arrow[dr] \arrow[ddd, "\beta \times \gamma"] & \\
A_0 \arrow[d, "\alpha"] & & C_0 \arrow[d, "\gamma"] & & B_0 \arrow[d, "\beta"] & & C_0 \arrow[d, "\gamma"] \\
A_1 & & C_1 & & B_1 & & C_1 \\
 & A_1 \times C_1 \arrow[ul] \arrow[ur] \arrow[drr] & & & & B_1 \times C_1 \arrow[ul] \arrow[ur] \arrow[dll] & \\
 & & & L(A_1, B_1, C_1) & & &
\end{tikzcd}
$$

This shows how the arrow $L(\alpha, \beta, \gamma)$ is obtained. The functorial property is induced by the uniqueness of $\alpha \times \beta, \beta \times \gamma$, and $L(\alpha, \beta, \gamma)$.

The behaviour of R on arrows is obtained in a similar way. Starting from three arrows α, β, γ, as above, we use the cell on the left to obtain

$$
\begin{tikzcd}
& A_0 + B_0 & \\
A_0 \arrow[ur] \arrow[d, "\alpha"] & & B_0 \arrow[ul] \arrow[d, "\beta"] \\
A_1 \arrow[dr] & & B_1 \arrow[dl] \\
& A_1 + B_1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A_0 + B_0 \arrow[ddd, "\alpha + \beta"] & \\
A_0 \arrow[ur] \arrow[d, "\alpha"] & & B_0 \arrow[ul] \arrow[d, "\beta"] \\
A_1 \arrow[dr] & & B_1 \arrow[dl] \\
& A_1 + B_1 &
\end{tikzcd}
$$

the unique arrow $\alpha + \beta$ and commuting diagram on the right. Here the unnamed arrows are the insertions. We now introduce the arrow γ to obtain a unique arrow $R(\alpha, \beta, \gamma)$ and commuting diagram where the new

$$
\begin{tikzcd}
& & & R(A_0, B_0, C_0) \arrow[dll] \arrow[dr] \arrow[ddddd, "R(\alpha,\beta,\gamma)"] & \\
& A_0 + B_0 \arrow[ddd, "\alpha + \beta"] & & & C_0 \arrow[ddd, "\gamma"] \\
A_0 \arrow[ur] \arrow[d, "\alpha"] & & B_0 \arrow[ul] \arrow[d, "\beta"] & & \\
A_1 \arrow[dr] & & B_1 \arrow[dl] & & \\
& A_1 + B_1 & & & C_1 \\
& & & R(A_1, B_1, C_1) \arrow[ull] \arrow[ur] &
\end{tikzcd}
$$

unnamed arrows are projections. The various uniquenesses ensure that R passes across composition in the appropriate manner to be a functor.

For the next part let

$$L = L(A, B, C) \qquad R = R(A, B, C)$$

for some arbitrary objects A, B, C. We produce an arrow

$$L \longrightarrow R$$

which, in due course, we show is natural for variation of the three objects.

So far we have managed without naming the various projections and insertion, but now we have to. Thus let

$$
\begin{array}{ll}
A\times C \xrightarrow{\ l\ } A \xrightarrow{\ u\ } A+B & A\times C \xrightarrow{\ r\ } C \\
B\times C \xrightarrow{\ m\ } B \xrightarrow{\ v\ } A+B & B\times C \xrightarrow{\ s\ } C \\
A\times C \xrightarrow{\ i\ } L & R \xrightarrow{\ p\ } A+B \\
B\times C \xrightarrow{\ j\ } L & R \xrightarrow{\ q\ } C
\end{array}
$$

be these various arrows.

The

coproduct property of L product property of R

produce unique arrows a, b, c, d such that

$$
\begin{tikzcd}
A\times C \arrow[d, "i"'] \arrow[dr, "u\circ l"] & \\
L \arrow[r, "a"] & A+B \\
B\times C \arrow[u, "j"] \arrow[ur, "v\circ m"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A+B \\
A\times C \arrow[ur, "u\circ l"] \arrow[r, "c"] \arrow[dr, "r"'] & R \arrow[u, "p"'] \arrow[d, "q"] \\
& C
\end{tikzcd}
$$

$$
\begin{tikzcd}
A\times C \arrow[d, "i"'] \arrow[dr, "r"] & \\
L \arrow[r, "b"] & C \\
B\times C \arrow[u, "j"] \arrow[ur, "s"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A+B \\
B\times C \arrow[ur, "v\circ m"] \arrow[r, "d"] \arrow[dr, "s"'] & R \arrow[u, "p"'] \arrow[d, "q"] \\
& C
\end{tikzcd}
$$

commute. With these the

product property of R coproduct property of L

produce unique arrows μ, ν such that

$$
\begin{tikzcd}
& A+B \\
L \arrow[ur, "a"] \arrow[r, "\mu"] \arrow[dr, "b"'] & R \arrow[u, "p"'] \arrow[d, "q"] \\
& C
\end{tikzcd}
\qquad
\begin{tikzcd}
A\times C \arrow[d, "i"'] \arrow[dr, "c"] & \\
L \arrow[r, "\nu"] & R \\
B\times C \arrow[u, "j"] \arrow[ur, "d"'] &
\end{tikzcd}
$$

commute. We first show that $\mu = \nu$.

By the characterizing properties of μ of ν it suffices to show that either

$$
\begin{array}{llcll}
(\mu a) & p\circ\nu = a & & (\nu c) & \mu\circ i = c \\
 & & \text{or} & & \\
(\mu b) & q\circ\nu = b & & (\nu d) & \mu\circ j = d
\end{array}
$$

for then

$$\nu = \mu \quad \text{or} \quad \mu = \nu$$

respectively. For these, using characterizing properties of a and b or c and d it suffices to show

$$\begin{array}{llll} (\mu ai) & p \circ \nu \circ i = u \circ l & (\nu cp) & p \circ \mu \circ i = u \circ l \\ (\mu aj) & p \circ \nu \circ j = v \circ m & (\nu cq) & q \circ \mu \circ i = r \end{array}$$

or

$$\begin{array}{llll} (\mu bi) & q \circ \nu \circ i = r & (\nu dp) & p \circ \mu \circ j = v \circ m \\ (\mu bj) & q \circ \nu \circ j = s & (\nu dq) & q \circ \mu \circ j = s \end{array}$$

respectively. These follows by the previous six diagrams. For instance

$$p \circ \nu \circ i = p \circ c = u \circ l$$

gives (μai).

Let us write μ for this arrow. It remains to show that μ is natural for variation of A, B, C.

To do that consider three arrows α, β, γ, as above. Let

$$\lambda = L(\alpha, \beta, \gamma) \qquad \rho = R(\alpha, \beta, \gamma)$$

so that we must show that the following square commutes.

$$\begin{tikzcd} L(A_0, B_0, C_0) \arrow[r, "\mu_0"] \arrow[d, "\lambda"'] & R(A_0, B_0, C_0) \arrow[d, "\rho"] \\ L(A_1, B_1, C_1) \arrow[r, "\mu_1"'] & R(A_1, B_1, C_1) \end{tikzcd}$$

Here μ_0 and μ_1 are the two version of μ for the triple of objects indicated by the index. We also use the various projections and insertions for the two triples with a indexed version of the notation above.

To show

$$\mu_1 \circ \lambda = \rho \circ \mu_0$$

we invoke the coproduct property of $L(0)$ to observe that the pair of equalities

$$\begin{aligned} \mu_1 \circ \lambda \circ i_0 &= \rho \circ \mu_0 \circ i_0 \\ \mu_1 \circ \lambda \circ j_0 &= \rho \circ \mu_0 \circ j_0 \end{aligned}$$

will suffice. To prove these we invoke the product property of $R(1)$ to observe that the four equalities will suffice.

$$\begin{aligned} p_1 \circ \mu_1 \circ \lambda \circ i_0 &= p_1 \circ \rho \circ \mu_0 \circ i_0 \\ q_1 \circ \mu_1 \circ \lambda \circ i_0 &= q_1 \circ \rho \circ \mu_0 \circ i_0 \\ p_1 \circ \mu_1 \circ \lambda \circ j_0 &= p_1 \circ \rho \circ \mu_0 \circ j_0 \\ q_1 \circ \mu_1 \circ \lambda \circ j_0 &= q_1 \circ \rho \circ \mu_0 \circ j_0 \end{aligned}$$

All four of these are proved in the same way. Let's look at the first.

Using various commuting cells and remembering that $\mu_1 = \nu_1$ we have

$$
\begin{array}{ll}
\quad p_1 \circ \mu_1 \circ \lambda \circ i_0 & \quad p_1 \circ \rho \circ \mu_0 \circ i_0 \\
= p_1 \circ \mu_1 \circ i_1 \circ (\alpha \times \gamma) & = (\alpha + \beta) \circ p_0 \circ \mu_0 \circ i_0 \\
= p_1 \circ c_1 \circ (\alpha \times \gamma) & = (\alpha + \beta) \circ a_0 \circ i_0 \\
= u_1 \circ l_1 \circ (\alpha \times \gamma) & = (\alpha + \beta) \circ u_0 \circ l_0
\end{array}
$$

and hence it suffices to show that the diagram

$$
\begin{tikzcd}
A_0 \times C_0 \arrow[r, "\alpha \times \gamma"] \arrow[d, "l_0"'] & A_1 \times C_1 \arrow[r, "l_1"] & A_1 \arrow[d, "u_1"] \\
A_0 \arrow[r, "u_0"'] & A_0 + B_0 \arrow[r, "\alpha + \beta"'] & A_1 + B_1
\end{tikzcd}
$$

commutes. To do this simply observe that the arrow α, as an upwards diagonal, makes the two resulting cells commute. $\square$

3.5.16 If you are a bit confused it's probably because you have forgotten the forgetful functor. Let $\boldsymbol{Sgp}$ and $\boldsymbol{Mon}$ be the the categories of semigroups and monoids. We are concerned with two functors

$$
\begin{tikzcd}
\boldsymbol{Sgp} \arrow[r, shift left, "F"] & \boldsymbol{Mon} \arrow[l, shift left, "¿"]
\end{tikzcd}
$$

where $¿$ is the given forgetful functor and F is the functor we are trying to produce. (Technically, F is a left adjoint to $¿$, but that's for later.) Let's insert $¿$ where it should appear.

(c) For each $\boldsymbol{Sgp}$ arrow

$$A \xrightarrow{f} ¿B$$

where B is monoid, there is a commuting triangle

$$
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "\iota_A"'] & & ¿B & B \\
& (¿ \circ F)A \arrow[ur, "¿(f^\sharp)"'] & & FA \arrow[u, "f^\sharp"']
\end{tikzcd}
$$

for some unique $\boldsymbol{Mon}$ arrow $f^\sharp$, as indicated. There is only one possibility for $f^\sharp$.

$$f^\sharp{}_{|A} = f \qquad f^\sharp(\omega) = \text{unit of } B$$

The rest is now standard category theory where semigroups and monoids need not be mentioned.

(d) For each $\boldsymbol{Sgp}$ arrow

$$A \xrightarrow{f} B$$

there is a unique $\boldsymbol{Mon}$ arrow

$$FA \xrightarrow{F(f)} FB$$

such that

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[d, "\iota_A"'] & & B \arrow[d, "\iota_B"] \\
(¿ \circ F)A \arrow[rr, "(¿ \circ F)(f)"'] & & (¿ \circ F)B
\end{tikzcd}$$

commutes. This follows from part (c) by setting

$$F(f) = (\iota_B \circ f)^\sharp$$

for the given arrow f. By considering a composite

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in $\boldsymbol{Sgp}$ with the induced commuting squares and by remembering the uniqueness of $F(\cdot)$, we see that

$$F(g) \circ F(f) = F(g \circ f)$$

which more or less shows that F is a functor.

(e) The commuting square above shows the naturality of ι.

(f) If A already has a unit then this is forgotten and a new unit is adjoined. The two are *not* coalesced. □

3.5.17 (a) Concatenation is associative, but not commutative. The unit is the empty word.

(b) The functor

$$\boldsymbol{Set} \xrightarrow{F} \boldsymbol{Mon}$$

goes from the category of sets to the category of monoids.

We need to describe the action

$$A \xrightarrow{f} B \qquad \longmapsto \qquad FA \xrightarrow{F(f)} FB$$

on arrows. Given a $\boldsymbol{Set}$ arrow f, a function between sets as above, let

$$F(f) : FA \longrightarrow GB$$

be the function given by

$$F(f)(\mathsf{a}) = [f(a_1), \ldots, f(a_l)]$$

for each list

$$\mathsf{a} = [a_1, \ldots, a_l]$$

in FA. Almost trivially this is a monoid morphism, and the required functorial properties are just as easy. Thus we do have a functor, as above.

(c) We have a forgetful functor

$$\boldsymbol{Set} \xleftarrow[¿]{} \boldsymbol{Mon}$$

which sends each monoid to its carrying set. We show that the insertion

$$\begin{array}{rcl} A & \xrightarrow{\iota_A} & (\text{¿} \circ F)A \\ a & \longmapsto & [a] \end{array}$$

is natural for variation on A. In other words, for each function f, as on the left,

$$\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\iota_A"] \arrow[d, "f"'] & (\text{¿} \circ F)A \arrow[d, "(\text{¿} \circ F)(f)"] \\ B \arrow[r, "\iota_B"'] & (\text{¿} \circ F)B \end{tikzcd}$$

the $\boldsymbol{Set}$ square on the right commutes. This is a trivial calculation. Both trips round the square send each element $a \in A$ to $[f(a)] \in (\text{¿} \circ F)B$.

(d) We show that for each function

$$A \xrightarrow{f} B$$

from a set to a monoid, there is a unique monoid morphism

$$FA \xrightarrow{f^\sharp} B$$

such that the triangle commutes.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\iota_A"'] & & \text{¿}B \\ & (\text{¿} \circ F)A \arrow[ur, "\text{¿}(f^\sharp)"'] & \end{tikzcd} \quad \begin{tikzcd} & B \\ FA \arrow[ur, "f^\sharp"] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\iota_A"'] & & B \\ & FA \arrow[ur, "f^\sharp"'] & \end{tikzcd} \quad \begin{tikzcd} & B \\ FA \arrow[ur, "f^\sharp"] & \end{tikzcd}$$

On the left we have the official version, and on the right we have the way it is usually written with the forgetful functor forgotten.

Given a function f we have to show two things: there is at most one fill-in morphism $f^\sharp$, and there is at least one fill-in morphism $f^\sharp$. Almost always with this kind of problem these two parts are dealt with separately. Here is a useful way to handle the first part.

We show that ι_A is 'as epic as it can be'. We show that for each parallel pair of monoid morphisms

$$\begin{tikzcd} FA \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & C \end{tikzcd}$$

we have

$$g \circ \iota_A = h \circ \iota_A \Longrightarrow g = h$$

and hence there is at most one fill-in morphism.

Consider any element $\mathsf{a} \in FA$. With

$$\mathsf{a} = [a_1, \ldots, a_l]$$

we have

$$\mathsf{a} = \iota_A(a_1) \frown \cdots \frown \iota_A(a_l)$$

where $\cdot \frown \cdot$ is the operation on FA, that is concatenation. Consider any pair of morphism g, h, as above. We have

$$\begin{aligned} g(\mathsf{a}) &= g\big(\iota_A(a_1) \frown \cdots \frown \iota_A(a_l)\big) = (g \circ \iota_A)(a_1) \star \cdots \star (g \circ \iota_A)(a_l) \\ h(\mathsf{a}) &= h\big(\iota_A(a_1) \frown \cdots \frown \iota_A(a_l)\big) = (h \circ \iota_A)(a_1) \star \cdots \star (h \circ \iota_A)(a_l) \end{aligned}$$

where $\star$ is the operation of C. Thus if

$$g \circ \iota_A = h \circ \iota_A \quad \text{then} \quad g(\mathsf{a}) = h(\mathsf{a})$$

and so $g = h$.

It remains to show that there is at least one morphism $f^\sharp$ that make the triangle commute. To do that we simple set

$$f^\sharp(\mathsf{a}) = f(a_1) * \cdots * f(a_l)$$

for each $\mathsf{a} \in FA$ (as above) where $*$ is the operation on B.

(e) If A is already a monoid then this structure is forgotten and a much bigger monoid is produced. Even when A is the 1-element monoid, the monoid FA is infinite. □

3.5.18 (a) Solution 3.3.13 show that we have two functors

$$\begin{tikzcd} \boldsymbol{Grp} \arrow[r, shift left, "G"] \arrow[r, shift right, "F"'] & \boldsymbol{AGrp} \end{tikzcd}$$

given by

$$GA = \delta A \qquad FA = A/\delta A$$

for each group A. The diagrams in that solution show that ι and η are natural.

(b) When B is abelian the subgroup δB is trivial. Thus the construction of $f^\sharp$ is a particular case of the construction of f/δ given in the latter part of Solution 3.3.13. □

3.5.19 As in Solution 3.3.18, the construction of the arrow assignment ensures that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta_A"'] & B \arrow[d, "\eta_B"] \\ (\text{¿} \circ F)A \arrow[r, "(\text{¿} \circ F)A"'] & (\text{¿} \circ F)B \end{tikzcd}$$

commutes for each arrow of $\boldsymbol{Src}$.

$$A \xrightarrow{f} B$$

This show that η is a natural transformation from the identity endo-functor on $\boldsymbol{Src}$ to $\text{¿} \circ F$. □

3.5.20 Consider three objects of $\boldsymbol{C}^{\nabla}$ and two arrows which ought to be composible.

$$F \xrightarrow{\ \sigma\ } G \xrightarrow{\ \tau\ } H$$

Thus we have three functors and two natural transformations. We require an appropriate composite

$$F \xrightarrow{\ \tau\circ\sigma\ } H$$

of the two transformations.

For each object i of ∇ we have a composible pair of arrows

$$Fi \xrightarrow{\ \sigma_i\ } Gi \xrightarrow{\ \tau_i\ } Hi$$

of arrows of $\boldsymbol{C}$. We set

$$(\tau\circ\sigma)_i = \tau_i\circ\sigma_i$$

to obtain a ∇-indexed family of arrows of $\boldsymbol{C}$. We show this family is natural for variation of i.

Consider any arrow e of ∇ and suppose this starts at i and finishes at j. We know that the two squares on the left do commute.

$$\begin{tikzcd}
Fi \arrow[r, "\sigma_i"] \arrow[d, "F(e)"'] & Gi \arrow[r, "\tau_i"] \arrow[d, "G(e)"] & Hi \arrow[d, "H(e)"] \\
Fj \arrow[r, "\sigma_j"'] & Gj \arrow[r, "\tau_j"'] & Hj
\end{tikzcd}
\qquad
\begin{tikzcd}
Fi \arrow[r, "(\tau\circ\sigma)_i"] \arrow[d, "F(e)"'] & Hi \arrow[d, "H(e)"] \\
Fj \arrow[r, "(\tau\circ\sigma)_j"'] & Hj
\end{tikzcd}$$

Hence so does the square on the right, to show the required naturality.

A similar argument shows that this composition is associative. □

Warning: Sometimes the symbol '$\circ$' is not used for the composition of natural transformations described in the previous solution, but it is used for the composition $\star$ described in the next solution.

3.5.21 The naturality of ρ ensures that the $\boldsymbol{B}$-square

$$\begin{tikzcd}
B_0 \arrow[d, "g"'] \\
B_1
\end{tikzcd}
\qquad
\begin{tikzcd}
KB_0 \arrow[r, "\rho_0"] \arrow[d, "K(g)"'] & LB_0 \arrow[d, "L(g)"] \\
KB_1 \arrow[r, "\rho_1"'] & LB_1
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \arrow[d, "\lambda_A"] \\
GA
\end{tikzcd}$$

commutes for each $\boldsymbol{B}$-arrow g, as on the left. Each object A of $\boldsymbol{A}$ gives us an arrow λ_A of $\boldsymbol{B}$, as on the right. Taking this for g gives the required commuting $\boldsymbol{C}$-square. □

3.5.22 For an arbitrary $\boldsymbol{A}$-object A consider the following diagram.

$$\begin{tikzcd}
(K\circ F)A \arrow[r, "\rho_{FA}"] \arrow[d, "K(\lambda_A)"'] & (L\circ F)A \arrow[r, "\sigma_{FA}"] \arrow[d, "L(\lambda_A)"] & (M\circ F)A \arrow[d, "M(\lambda_A)"] \\
(K\circ G)A \arrow[r, "\rho_{GA}"] \arrow[d, "K(\mu_A)"'] & (L\circ G)A \arrow[r, "\sigma_{GA}"] \arrow[d, "L(\mu_A)"] & (M\circ G)A \arrow[d, "M(\mu_A)"] \\
(K\circ H)A \arrow[r, "\rho_{HA}"'] & (L\circ H)A \arrow[r, "\sigma_{HA}"'] & (M\circ H)A
\end{tikzcd}$$

Each of the four small squares commutes. This is four instances of the result of Exercise 3.5.21. The diagonals of the top let and bottom right squares are

$$(\rho\star\lambda)_A \qquad (\sigma\star\mu)_A$$

respectively, and hence

$$\big((\sigma\star\mu)\circ(\rho\star\lambda)\big)_A$$

is the composite diagonal.

The outside square commutes, and this is just

$$\begin{tikzcd}
(K\circ F)A \arrow[r, "(\sigma\circ\rho)_{FA}"] \arrow[d, "K\left((\mu\circ\lambda)_A\right)"'] & (M\circ F)A \arrow[d, "M\left((\mu\circ\lambda)_A\right)"] \\
(K\circ H)A \arrow[r, "(\sigma\circ\rho)_{HA}"'] & (M\circ H)A
\end{tikzcd}$$

using the construction of the vertical composition. The diagonal of this square is

$$\big((\sigma\circ\rho)\star(\mu\circ\lambda)\big)_A$$

by the definition of horizontal composition.

Comparing the two descriptions of the full diagonal gives the required result. □

4
Limits and colimits in general

4.1 Template and diagram – a first pass

4.1.1 Sticking paths end to end is associative, so composition in $\boldsymbol{Pth}(\nabla)$ is associative. We must ensure that each object (node of ∇) has an identity arrow on $\boldsymbol{Pth}(\nabla)$. Each node has an associated path of length zero, and sticking this path on the end of some other path doesn't change that second path. The identity arrows are the paths of length zero. $\square$

4.1.2 (a) Let us label the four edges as follows.

$$\begin{tikzcd} & 1 \arrow[dr, "{(3,1)}"] & \\ 0 \arrow[ur, "{(1,0)}"] \arrow[dr, "{(2,0)}"'] & & 3 \\ & 2 \arrow[ur, "{(3,2)}"'] & \end{tikzcd}$$

There are just 10 possible paths.

Paths of length		Number of such paths
0	The four nodes $0, 1, 2, 3$	4
1	The four edges $(1,0), (2,0), (3,1), (3,2)$	4
2	The two formal composites $(3,1)\circ(1,0) \quad (3,2)\circ(2,1)$	2

This graph generates a category of four objects and 10 arrows. This category is not a poset since there are two distinct arrows

$$(3,1)\circ(1,0) \qquad (3,2)\circ(2,1)$$

from 0 to 3.

(b) The graph is

$$\begin{tikzcd} & 1 \arrow[dr, "{(2,1)}"] & \\ 0 \arrow[ur, "{(1,0)}"] \arrow[dr, "{(3,0)}"'] & & 2 \arrow[dl, "{(3,2)}"] \\ & 3 & \end{tikzcd}$$

to give a category of four nodes and 11 arrows.

Paths of length		Number of
0	The four nodes $0, 1, 2, 3$	4
1	The four edges $(1,0), (2,1), (3,2), (0,3)$	4
2	The four formal composites $(2,1) \circ (1,0) \quad (3,2) \circ (2,1)$ $(0,3) \circ (3,2) \quad (1,0) \circ (0,3)$	4
3	The formal composites $(3,2) \circ (2,1) \circ (1,0)$ $(0,3) \circ (3,2) \circ (2,1)$ $(1,0) \circ (0,3) \circ (3,2)$ $(2,1) \circ (1,0) \circ (0,3)$	4
4	The formal composites $(0,3) \circ (3,2) \circ (2,1) \circ (1,0)$ $(1,0) \circ (0,3) \circ (3,2) \circ (2,1)$ $(2,1) \circ (1,0) \circ (0,3) \circ (3,2)$ $(3,2) \circ (2,1) \circ (1,0) \circ (0,3)$	4
5	The formal composites $(1,0 \circ (0,3) \circ (3,2) \circ (2,1) \circ (1,0)$ $(2,1) \circ (1,0) \circ (0,3) \circ (3,2) \circ (2,1)$ $\vdots$	∞

Table 4.1: Paths for graph (c)

Paths of length		Number of such paths
0	The four nodes $0, 1, 2, 3$	4
1	The four edges $(1,0), (2,1), (3,2), (3,0)$	4
2	The two formal composites $(2,1) \circ (1,0) \quad (3,2) \circ (2,1)$	2
3	The formal composite $(3,2) \circ (2,1) \circ (1,0)$	1

(c) The labelled graph is

$$\begin{tikzcd}
& 1 \arrow[dr, "{(2,1)}"] & \\
0 \arrow[ur, "{(1,0)}"] & & 2 \arrow[dl, "{(3,2)}"] \\
& 3 \arrow[ul, "{(0,3)}"] &
\end{tikzcd}$$

and this generates a category of four objects and infinitely many arrows, as indicated in Table 4.1. We can cycle round the edges for ever. □

4.1.3 No! The generated category is bigger. Consider the category

$$\begin{tikzcd} a0 \arrow[rr, "{(2,0)}"] \arrow[dr, "{(1,0)}"'] & & 2 \\ & 1 \arrow[ur, "{(2,1)}"'] & \end{tikzcd}$$

of three objects, three identity arrows (not shown), and three other arrows where

$$(2,1) \circ (1,0) = (2,0)$$

in the category. There are (at least) two paths from 0 to 2, namely

$$(2,0) \quad \text{and} \quad (1,0) \text{ followed by } (2,1)$$

and these are *not* the same path.

The parent category is a quotient of the generated path category. □

4.1.4 Let ∇ be the graph with nodes i and edges e. A ∇-diagram in the category $\boldsymbol{C}$ consists of

$$\begin{array}{cc} \text{objects} & \text{arrows} \\ A(i) & A(e) \end{array}$$

indexed by the

$$\begin{array}{cc} \text{nodes} & \text{edges} \end{array}$$

respectively. For each edge

$$i \xrightarrow{\quad e \quad} j \quad \text{we require an arrow} \quad A(i) \xrightarrow{A(e)} A(j)$$

but there are no requirements that certain triangles must commute.

Consider a functor $\boldsymbol{Pth}(\nabla) \longrightarrow \boldsymbol{C}$ from the path category. This gives a family of objects of $\boldsymbol{C}$

$$A(i)$$

indexed by the objects of $\boldsymbol{Pth}(\nabla)$, the nodes of ∇. It also gives an arrow of $\boldsymbol{C}$

$$A(\pi)$$

for each arrow of $\boldsymbol{Pth}(\nabla)$, each path of through ∇. In particular, we have an arrow in $\boldsymbol{C}$

$$A(e)$$

for each edge of ∇, each path in $\boldsymbol{Pth}(\nabla)$ of length 1. Thus there is a ∇-diagram embedded in the functor.

Each path π has a unique decomposition

$$i(0) \xrightarrow{e(1)} i(1) \xrightarrow{e(2)} i(2) \longrightarrow \cdots\cdots \xrightarrow{e(l)} i(l)$$

as a sequence of edges through ∇. The functorial properties ensure that

$$A(\pi) = A(e(l)) \circ \cdots \circ A(e(1))$$

to show the functor is uniquely determined by the embedded ∇-diagram.

Observe that almost the same proof shows that each ∇-diagram extends to a functor. □

4.2 Functor categories

4.2.1 This is just the same as Exercise 3.5.20. ☐

4.2.2 We must show that each arrow induces a natural transformation

$$X \xrightarrow{\ f\ } Y \qquad \longmapsto \qquad \Delta X \xrightarrow{\ \Delta(f)_\bullet\ } \Delta Y$$

such that

$$\begin{tikzcd}
(\Delta X)(i) \arrow[r, "\Delta(f)_i"] \arrow[d, "(\Delta X)(e)"'] & (\Delta Y)(i) \arrow[d, "(\Delta Y)(e)"] & i \arrow[d, dash, "e"] \\
(\Delta X)(j) \arrow[r, "\Delta(f)_j"'] & (\Delta Y)(j) & j
\end{tikzcd}$$

commutes for each edge e of ∇, as on the right. When we insert the values of ΔX and ΔY we see that a commuting square

$$\begin{tikzcd}
X \arrow[r, "\Delta(f)_i"] \arrow[d, "id_X"'] & Y \arrow[d, "id_Y"] \\
X \arrow[r, "\Delta(f)_j"'] & Y
\end{tikzcd}$$

is required. Thus we set

$$\Delta(f)_i = f$$

for each node i. ☐

4.2.3 We do both solutions in parallel.

A typical arrow in $\boldsymbol{C}^\nabla$

$$\Delta X \longrightarrow A \qquad\qquad A \longrightarrow \Delta X$$

is a natural transformation, a family of arrows of $\boldsymbol{C}$

$$(\Delta X)(i) \xrightarrow{\ \xi(i)\ } A(i) \qquad\qquad A(i) \xrightarrow{\ \xi(i)\ } (\Delta X)(i)$$

indexed by the nodes and such that

$$\begin{tikzcd}
(\Delta X)(i) \arrow[r, "\xi(i)"] \arrow[d, "(\Delta X)(e)"'] & A(i) \arrow[d, "A(e)"] & i \arrow[d, dash, "e"] & A(i) \arrow[r, "\xi(i)"] \arrow[d, "A(e)"'] & (\Delta X)(i) \arrow[d, "(\Delta X)(e)"] \\
(\Delta X)(j) \arrow[r, "\xi(j)"'] & A(j) & j & A(j) \arrow[r, "\xi(j)"'] & (\Delta X)(j)
\end{tikzcd}$$

commutes for each edge e. When we insert the values of ΔX we see that we require a commuting triangle

$$\begin{tikzcd}
 & A(i) \arrow[dd, "A(e)"] & i \arrow[dd, dash, "e"] & A(i) \arrow[dd, "A(e)"'] \arrow[dr, "\xi(i)"] & \\
X \arrow[ur, "\xi(i)"] \arrow[dr, "\xi(i)"'] & & & & X \\
 & A(j) & j & A(j) \arrow[ur, "\xi(i)"'] &
\end{tikzcd}$$

for each edge e. Such an arrow is just a left/right solution for the diagram A. ☐

4.3 Problem and solution

4.3.1 Consider any ∇-diagram and the corresponding $\boldsymbol{Pth}(\nabla)$-diagram. These have the same family

$$A(i)$$

of objects indexed by the nodes i of ∇. Each edge

$$i \xrightarrow{\ e\ } j \qquad \text{of } \nabla \text{ gives an arrow} \qquad A(i) \xrightarrow{A(e)} A(j)$$

of the ∇-diagram, and this is also an arrow of the $\boldsymbol{Pth}(\nabla)$-diagram. However, there are more arrows in the $\boldsymbol{Pth}(\nabla)$-diagram. Each path

$$i \xrightarrow{\ \pi\ } j \qquad \text{in } \boldsymbol{Pth}(\nabla) \text{ gives an arrow} \qquad A(i) \xrightarrow{A(\pi)} A(j)$$

in the $\boldsymbol{Pth}(\nabla)$-diagram.

In the ∇-diagram there are no requirements that certain triangles commute (for there is no notion of composition in the graph ∇).

In the $\boldsymbol{Pth}(\nabla)$-diagram each composite path requires that certain triangles commute. For example let π be the composite path

$$i(0) \xrightarrow{e(1)} i(1) \xrightarrow{e(2)} i(2) \longrightarrow \cdots\cdots \xrightarrow{e(l)} i(l)$$

of l edges. Then then the two arrows

$$A(i(0)) \xrightarrow{A(e(1))} A(i(1)) \xrightarrow{A(e(2))} A(i(2)) \longrightarrow \cdots\cdots \xrightarrow{A(e(l))} A(i(l))$$

$$A(i(0)) \xrightarrow[A(\pi)]{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad} A(e(l))$$

must agree. The ∇-diagram completely determines the $\boldsymbol{Pth}(\nabla)$-diagram. (You should also think of how the conditions on identity arrows are handled in the $\boldsymbol{Pth}(\nabla)$-diagram.)

At this point we have to decide whether we look at left solutions or right solutions. The two discussions are entirely symmetrical, so let's look at left solutions.

Consider a solution of each diagram with the same apex X. Each is a family of arrows

$$X \xrightarrow{\xi(i)} A(i)$$

indexed by the nodes of ∇. There are certain commuting conditions. All the triangles

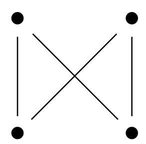
$$\begin{array}{cc}
\nabla & \boldsymbol{Pth}(\nabla) \\
\begin{tikzcd}
 & A(i) \arrow[dd, "A(e)"] & i \arrow[dd, dash, "e"] \\
X \arrow[ur, "\xi(i)"] \arrow[dr, "\xi(i)"'] & & \\
 & A(j) & j
\end{tikzcd}
&
\begin{tikzcd}
 & A(i) \arrow[dd, "A(\pi)"] & i \arrow[dd, "\pi"] \\
X \arrow[ur, "\xi(i)"] \arrow[dr, "\xi(i)"'] & & \\
 & A(j) &
\end{tikzcd}
\end{array}$$

for each edge e of ∇ and each path π of $\boldsymbol{Pth}(\nabla)$. □

4.3.2 See Solution 4.2.3. □

4.4 Universal solution

4.4.1 Consider the 4-element pre-set



(which is not a poset). The set of the two upper elements has two distinct limits (infima), namely each of the two lower elements. $\square$

4.4.2 Consider any set X which is the limit or the colimit of some diagram in $\boldsymbol{Set}$. This means that X is suitably furnished to be the limit or colimit. Now take any set Y of the same size together with a bijection between X and Y. This bijection is an isomorphism in $\boldsymbol{Set}$. By furnishing Y in the obvious way Y becomes a limit of a colimit of the diagram.

In other categories we can use the same idea. We simply take an isomorphic copy of the limit of colimit. $\square$

4.4.3 Let

$$A(i) \xrightarrow{\sigma(i)} S$$

be a colimit of a diagram indexed by the nodes i of a template.

If

$$S \underset{\psi}{\overset{\theta}{\rightrightarrows}} X$$

is a parallel pair of arrows with

$$\theta \circ \sigma(i) = \psi \circ \sigma(i)$$

for each node i, then $\theta = \psi$.

If

$$S \xrightarrow{\epsilon} S$$

is an endo-arrow of S such that

$$\epsilon \circ \sigma(i) = \sigma(i)$$

for each node i, then $\epsilon = id_S$.

If

$$A(i) \xrightarrow{\tau(i)} T$$

is also a limit of the diagram then there is a unique arrow

$$S \xrightarrow{\tau} T$$

such that

$$\tau(i) = \tau(i) \circ \sigma(i)$$

for each node i. Furthermore, τ is an isomorphism.

These are prove simply by reversing the arrows of the limit proofs. $\square$

4.5 A geometric limit and colimit

4.5.1 Consider any left solution of the diagram.

$$A \xrightarrow{f_m} \mathbb{Z}$$

This is a $\mathbb{Z}$-indexed family of arrows, as indicated, such that

$$\begin{tikzcd} & \mathbb{Z} \arrow[r, "d"] & \mathbb{Z} \\ A \arrow[ur, "f_m"] \arrow[urr, "f_{m+1}"'] & & \end{tikzcd}$$

commutes for each $m \in \mathbb{Z}$. Thus

$$f_{m+1}(a) = 2f_m(a)$$

for each $a \in A$ and $m \in \mathbb{Z}$. By a trivial induction this gives

$$f_{m+r}(a) = 2^r f_m(a)$$

for each $a \in A, m \in \mathbb{Z}, r \in \mathbb{N}$. Each value $f_m(a)$ of f_m is in $\mathbb{Z}$. The above shows that the value is divisible by 2^r for arbitrarily large $r \in \mathbb{N}$. Thus

$$f_m(a) = 0$$

for each $a \in A$ and $m \in \mathbb{Z}$. Thus each left solution has a simple structure.

Consider the singleton

$$L = \{*\}$$

furnished with the constant functions

$$\begin{array}{rcl} L & \xrightarrow{\lambda_m} & \mathbb{Z} \\ * & \longmapsto & 0 \end{array}$$

for each $m \in \mathbb{Z}$. Since

$$\lambda_{m+1}(*) = 0 = 2 \times 0 = 2\lambda_m(*)$$

this certainly gives a left solution of the diagram.

To show that this is the limit consider any left solution, as above. We require a unique function

$$A \xrightarrow{h} L$$

such that

$$f_m = \lambda_m \circ h$$

for each $m \in \mathbb{Z}$. In fact, there is only one possible function h of the indicated type, namely that given by

$$h(a) = *$$

for each $a \in A$. But now

$$(\lambda_m \circ h)(a) = \lambda_m(*) = 0 = f_m(a)$$

to show that h is the required mediator.

Consider any right solution of the diagram.

$$\mathbb{Z} \xrightarrow{f_m} A$$

This is a $\mathbb{Z}$-indexed family of arrows, as indicated, such that

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "d"] \arrow[drr, "f_m"'] & \mathbb{Z} \arrow[dr, "f_{m+1}"] & \\ & & A \end{tikzcd}$$

commutes for each $m \in \mathbb{Z}$. Thus

$$f_m(z) = f_{m+1}(2z)$$

for all $m, z \in \mathbb{Z}$. By a trivial induction this gives

$$f_m(z) = f_{m+r}(2^r z)$$

for all $m, z \in \mathbb{Z}$ and $r \in \mathbb{N}$. For later we need a refined version of this. We require

$$(\star) \quad 2^{-m}x = 2^{-n}y \Longrightarrow f_m(x) = f_n(y)$$

for all $m, n, x, y \in \mathbb{Z}$. To prove this suppose

$$2^{-m}x = 2^{-n}y \quad \text{so that} \quad 2^n x = 2^m y$$

holds. By symmetry we may suppose $m \leq n$, so that $n = m + r$ for some $r \in \mathbb{N}$. Thus

$$y = 2^r x \quad \text{and hence} \quad f_m(x) = f_{m+r}(2^r x) = f_n(y)$$

as required.

The dyadic rationals $\mathbb{D}$ consists of those rationals of the form

$$2^{-m}x$$

for $m, x \in \mathbb{Z}$. Of course, this representation is not unique (which is why we proved $(\star)$).

For each $m \in \mathbb{Z}$ consider the function

$$\mathbb{Z} \xrightarrow{\rho_m} \mathbb{D} \qquad \text{given by} \qquad \rho_m(x) = 2^{-m}x$$

for each $x \in \mathbb{Z}$. This gives a right solution since

$$\rho_{m+1}(2x) = 2^{-m-1} \times 2x = 2^{-m}x = \rho_m(x)$$

for each $m, x \in \mathbb{Z}$. We show that this is the colimit of the diagram.

Consider any right solution, as above. We require a unique function

$$\mathbb{D} \xrightarrow{h} A$$

such that

$$f_m = h \circ \rho_m$$

for each $m \in \mathbb{N}$. If there is such a function h then it must satisfy

$$h(2^{-m}x) = (h \circ \rho_m)(x) = f_m(x)$$

for each $m, x \in \mathbb{Z}$. Thus there is only one possible function h.

Consider and $d \in \mathbb{D}$. We may have

$$2^{-m}x = d = 2^{-n}y$$

for $m, n, x, y \in \mathbb{Z}$. The result $(\star)$ gives

$$f_m(x) = f_n(y)$$

and hence we may set

$$h(2^{-m}x) = f_m(x)$$

to obtain a well-defined function of the required type. Finally, for each $m, x \in \mathbb{Z}$, we have

$$(h \circ \rho_m)(x) = h(2^{-m}x) = f_m(x)$$

so that h does the required mediating job. □

4.5.2 Consider first any possible left solution of the problem. This is a poset X together with a monotone map

$$X \xrightarrow{\alpha(i)} A(i)$$

for each node i. Of course, these various maps must combine in the appropriate fashion. Thus for each pair $i-1, i$ of consecutive nodes the composite

$$X \xrightarrow{\alpha(i-1)} A(i-1) \xrightarrow{(i,i-1)} A(i)$$

must be the map $\alpha(i)$. Thus

$$\alpha(i)(x) = (i, i-1)\big(\alpha(i-1)(x)\big) = \star$$

for each $x \in X$. This shows that each left solution is a poset X together with the family

$$\begin{array}{ccc} X & \xrightarrow{\alpha(i)} & A(i) \\ x & \longmapsto & \star \end{array}$$

of constant functions. In particular, the limit of the diagram is the 1-element poset together with the maps that pick out $\star$ at each node.

Next consider any possible right solution of the problem. This is a poset X together with a monotone map

$$A(i) \xrightarrow{\alpha(i)} X$$

for each node i. These various maps must combine in the appropriate fashion. Thus for each pair $i, i+1$ of consecutive nodes the composite

$$A(i) \xrightarrow{(i+1,i)} A(i+1) \xrightarrow{\alpha(i+1)} X$$

must be the map $\alpha(i)$. Thus

$$\alpha(i)(a) = (i+1, i)\big(\alpha(i+1)(a)\big) = \star$$

for each $a \in A(i)$. This shows that for each right solution X each map $\alpha(i)$ is constant with value $\star$. In particular, the colimit is the singleton poset. □

4.6 How to calculate certain limits

4.6.1 Limits in $\boldsymbol{Set}$

4.6.1 Let A be the set of threads. We furnish A with a distinguished subset R to obtain an object (A, R) of $\boldsymbol{SetD}$. We let

$$a \in R \Longleftrightarrow (\forall i \in \mathbb{I})[a(i) \in R(i)]$$

for each thread a. We need to check various conditions.

For an arbitrary index i consider the connecting function

$$A \xrightarrow{\alpha(i)} A(i)$$

in $\boldsymbol{Set}$. We check that this function is an arrow

$$(A, R) \xrightarrow{\alpha(i)} (A(i), R(i))$$

of $\boldsymbol{SetD}$. Remembering the definition of $\alpha(i)$, for each $a \in A$ we have

$$a \in R \Longrightarrow \alpha(i)(a) = a(i) \in R(i)$$

to show that $\alpha(i)$ is an arrow.

Next we observe that we have a solution of the diagram in $\boldsymbol{SetD}$. This requires that certain triangles in $\boldsymbol{SetD}$ commute. But we know that these triangles commute in $\boldsymbol{Set}$, so there is nothing to prove.

We check that this solution is a universal solution in $\boldsymbol{SetD}$.

Consider any solution in $\boldsymbol{SetD}$

$$(X, W) \xrightarrow{\xi(i)} (A(i), R(i))$$

a $\boldsymbol{SetD}$-object (X, W) and an $\mathbb{I}$-indexed family of $\boldsymbol{SetD}$-arrows $\xi(i)$. We are given that certain triangles in $\boldsymbol{SetD}$ commute. We must show that there is a unique mediator in $\boldsymbol{SetD}$.

$$(X, W) \xrightarrow{\mu} (A, R)$$

The trick is to forget the furnishings for a moment and drop down to $\boldsymbol{Set}$. We have a ∇-diagram in $\boldsymbol{Set}$, a solution of the diagram based on X, and a universal solution based on A. Thus if the $\boldsymbol{SetD}$-situation has a mediator, then it can only be the $\boldsymbol{Set}$-mediator, given by

$$\mu(x)(i) = \xi(i)(x)$$

for each $x \in X$ and $i \in \mathbb{I}$. It suffices to show that μ is a $\boldsymbol{SetD}$-arrow.

Remembering that each $\xi(i)$ is a $\boldsymbol{SetD}$-arrow, for each $x \in X$ we have

$$x \in W \Longrightarrow (\forall i \in \mathbb{I})[\mu(x)(i) = \xi(i)(x) \in R(i)] \Longrightarrow \mu(x) \in R$$

to give the required result. $\square$

4.6.2 Limits in $\boldsymbol{Pos}$

4.6.2 (a) An arrow

$$(A, \sim) \xrightarrow{f} (B, \approx)$$

is a function f from A to B for which

$$a_1 \sim a_2 \Longrightarrow f(a_1) \approx f(a_2)$$

for all $a_1, a_2 \in A$. Note this is only an implication, not an equivalence.

(b) Let A be the set of threads. We furnish A with an equivalence relation $\sim$ to obtain an object $(A, \sim)$ of $\boldsymbol{Eqv}$. We let

$$a \sim b \Longleftrightarrow (\forall i \in \mathbb{I})[a(i) \sim_i b(i)]$$

for each pair of threads a, b. Trivially, this is reflexive and symmetric, and a few moment's thought shows that it is transitive. Thus we do have an equivalence relation.

We need to check various conditions.

For an arbitrary index i consider the connecting function

$$A \xrightarrow{\alpha(i)} A(i)$$

in $\boldsymbol{Set}$. We check that this function is an arrow

$$(A, \sim) \xrightarrow{\alpha(i)} (A(i), \sim_i)$$

of $\boldsymbol{Eqv}$. In other words that

$$a \sim b \Longrightarrow \alpha(i)(a) \sim_i \alpha(i)(b)$$

for all $a, b \in A$. But

$$\alpha(i)(a) = a(i) \qquad \alpha(i)(b) = b(i)$$

so this is an immediate consequence of the definition of $\sim$.

Next we observe that we have a solution of the diagram in $\boldsymbol{Eqv}$. This requires that certain triangles in $\boldsymbol{Eqv}$ commute. But we know that these triangles commute in $\boldsymbol{Set}$, so there is nothing to prove.

We check that this solution is a universal solution in $\boldsymbol{Eqv}$. Consider any solution

$$(X, \approx) \xrightarrow{\xi(i)} (A(i), \sim_i)$$

in $\boldsymbol{Eqv}$. This is a $\boldsymbol{Eqv}$-object $(X, \approx)$ and an $\mathbb{I}$-indexed family of $\boldsymbol{Eqv}$-arrows $\xi(i)$. We are given that certain triangles in $\boldsymbol{Eqv}$ commute. We must show that there is a unique mediator

$$(X, \approx) \xrightarrow{\mu} (A, \sim)$$

in $\boldsymbol{Eqv}$.

The trick is to forget the furnishings for a moment and drop down to $\boldsymbol{Set}$. We have a ∇-diagram in $\boldsymbol{Set}$, a solution of the diagram based on X, and a universal solution based on A. Thus if the $\boldsymbol{Eqv}$-situation has a mediator, then it can only be the $\boldsymbol{Set}$-mediator, given by

$$\mu(x)(i) = \xi(i)(x)$$

for each $x \in X$ and $i \in \mathbb{I}$. In other words, it suffices to show that this function μ is a $\boldsymbol{Eqv}$-arrow, that is

$$x \approx y \Longrightarrow \mu(x) \sim \mu(y)$$

for each $x, y \in X$. For each $x, y \in X$ we have

$$\begin{aligned} x \approx y &\Longrightarrow (\forall i \in \mathbb{I})[\xi(i)(x) \sim_i \xi(i)(y)] \\ &\Longrightarrow (\forall i \in \mathbb{I})[\mu(x)(i) \sim_i \mu(y)(i)] \;\Longrightarrow \mu(x) \sim \mu(y) \end{aligned}$$

to give the required result. □

4.6.3 Let A be the set of threads. We must first furnish A with an R-action.

Consider any $a \in A$ and $r \in R$. For each node i set

$$(ar)(i) = a(i)r$$

to produce a function $ar : \mathbb{I} \longrightarrow \bigcup \mathsf{A}$. Almost trivially this is a choice function. We must show that it is a thread. To this end consider the R-morphism

$$A(i) \xrightarrow{a(e)} A(j)$$

given by an edge e. Then, since a is a thread, we have

$$A(e)\big((ar)(i)\big) = A(e)\big(a(i)r\big) = A(e)\big(a(i)\big)r = a(j)r = (ar)(j)$$

to show that ar is a thread.

We now require

$$(ar)s = a(rs) \qquad a1 = a$$

for arbitrary $r, s \in R$. For each node i, working in the R-set $A(i)$, we have

$$\big((ar)s\big)(i) = \big((ar)(i)\big)s = \big(a(i)r\big)s = a(i)(rs) = \big(a(rs)\big)(i)$$

to give the left hand requirement. The right hand requirement is easier.

Next we must show that each evaluation function

$$A \xrightarrow{\alpha(i)} A(i)$$

is an R-morphism. But for each $a \in A$ and $r \in R$ we have

$$\alpha(i)\big(ar\big) = \big(ar\big)(i) = a(i)r = \alpha(i)(a)r$$

to give the required result.

This shows that we do have a solution of the ∇-diagram in $\boldsymbol{Set}$-R. It remains to show that it is universal.

Consider any solution X in $\boldsymbol{Set}$-R. Thus X is an R-set and each function

$$X \xrightarrow{\xi(i)} A(i)$$

is an R-morphism. By passing down to $\boldsymbol{Set}$ we know there is a unique *function*

$$X \xrightarrow{\mu} A$$

for which the required triangles commute. It suffices to show that this function is an R-morphism, that is

$$\mu(xr) = \mu(x)r$$

for each $x \in X$ and $r \in R$. To do that we show that these two functions agree at an arbitrary node i.

We know that

$$\mu(x)(i) = \xi(i)(x)$$

for each $x \in X$ and node i. Thus, for arbitrary $r \in R$, we have

$$\mu(xr)(i) = \xi(i)(xr) = \big(\xi(i)(x)\big)r = \big(\mu(x)(i)\big)r = \big(\mu(x)r\big)i$$

for the required result. $\square$

4.6.3 Limits in $\boldsymbol{Mon}$

4.6.4 (a) Each commutative monoid is a monoid with an extra property. The crucial observation is that any $\boldsymbol{Mon}$-arrow between commutative monoids is automatically a $\boldsymbol{CMon}$-arrow. (Technically this says that $\boldsymbol{CMon}$ is a full subcategory of $\boldsymbol{Mon}$.)

For a template ∇ consider a ∇-diagram

$$\mathsf{A} = \big(A(i) \mid i \in \mathbb{I}\big) \qquad \mathcal{A} = \big(A(e) \mid e \in \mathbb{E}\big)$$

in $\boldsymbol{CMon}$. Thus each $A(i)$ is a commutative monoid, and each $A(e)$ is a monoid morphism. By forgetting the commutative property we have a diagram in $\boldsymbol{Mon}$. We know this has a limit

$$A \xrightarrow{\alpha(i)} A(i)$$

carried by the set of threads. We show that A is commutative, and then it is automatically a limit in $\boldsymbol{CMon}$.

The operation $\star$ on A is given by

$$(a \star b)(i) = a(i)b(i)$$

for each $a, b \in A$ and $i \in \mathbb{I}$. Since $A(i)$ is commutative we have

$$(a \star b)(i) = a(i)b(i) = b(i)a(i) = (b \star a)(i)$$

which, since i is arbitrary, gives

$$a \star b = b \star a$$

to show that $\star$ is commutative.

(b) Each group is a monoid with some extra structure. The crucial observation, which takes a few moment's thought to justify, is that any $\boldsymbol{Mon}$-arrow between groups is automatically a $\boldsymbol{Grp}$-arrow. (Technically this says that $\boldsymbol{Grp}$ is a full subcategory of $\boldsymbol{Mon}$.)

For a template ∇ consider a ∇-diagram

$$\mathsf{A} = \big(A(i) \,|\, i \in \mathbb{I}\big) \qquad \mathcal{A} = \big(A(e) \,|\, e \in \mathbb{E}\big)$$

in $\boldsymbol{Grp}$. Thus each $A(i)$ is a group, and each $A(e)$ is a group. By forgetting the existence of inverses we have a diagram in $\boldsymbol{Mon}$. We know this has a limit

$$A \xrightarrow{\alpha(i)} A(i)$$

carried by the set of threads. We show that A is a group, and then it is automatically a limit in $\boldsymbol{Grp}$.

Consider any element $a \in A$. We produce an inverse of a in A, a thread $b \in A$ such that

$$a \star b = b \star a \quad \text{that is} \quad (a \star b)(i) = (b \star a)(i)$$

for each index i. For each such index i we have

$$a(i)b(i) = 1(i) = b(i)a(i)$$

for a unique element $b(i) \in A(i)$. In other words, $b(i)$ is the unique inverse of $a(i)$ in $A(i)$. This certainly gives us a choice function

$$b(\cdot) : \mathbb{I} \longrightarrow \bigcup \mathsf{A}$$

but we need to show that b is a thread, that is with

$$A(e)\big(b(i)\big) = b(j) \qquad \text{for each edge} \qquad i \xrightarrow{e} j$$

of ∇. We remember that a is a thread and $A(e)$ is a group morphism. Thus

$$\begin{aligned} a(j)\big(A(e)(b(i))\big) &= \big(A(e)(a(i))\big)\big(A(e)(b(i))\big) \\ &= A(e)\big(a(i)b(i)\big) = A(e)(1(i)) = 1(j) \end{aligned}$$

with

$$\big(A(e)(b(i))\big)a(j) = 1(j)$$

by a similar argument. But $a(j)$ has a unique inverse in $A(j)$, namely $b(j)$, and hence

$$\big(A(e)(b(i))\big) = b(j)$$

as required.

This shows that $b \in A$, and for each index i we have

$$(a \star b)(i) = a(i)b(i) = 1(i) = b(i)a(i) = (b \star a)(i)$$

and hence b is the inverse of a in A.

This shows that A is a group. The required arrow-theoretic properties to show that A is the limit of the diagram are immediate, since they hold in $\boldsymbol{Mon}$.

(c) Each ring is a set which carries both a monoid structure, the multiplication, and a group structure, the addition. Furthermore these two structures interact to satisfy the associative laws.

For a template ∇ consider a ∇-diagram

$$\mathsf{A} = \big(A(i) \,|\, i \in \mathbb{I}\big) \qquad \mathcal{A} = \big(A(e) \,|\, e \in \mathbb{E}\big)$$

in $\boldsymbol{Rng}$. Each $A(i)$ is a unital ring, and each $A(e)$ is a ring morphism. By forgetting all the structure, or the additive structure, or the multiplicative structure we obtain diagrams in $\boldsymbol{Set}$, $\boldsymbol{Mon}$, $\boldsymbol{Grp}$, respectively. Each of these diagrams has a limit in its parent category. More importantly, each of the limit structures is carried by the set of threads through the diagram. This set of threads does *not* depend on the parent category.

Let

$$A \xrightarrow{\alpha(i)} A(i)$$

be the limit of the $\boldsymbol{Set}$ diagram. Thus A is the set of threads and the α are the evaluation functions. We show that this can be furnished to give a limit in $\boldsymbol{Rng}$.

As in parts (a,b) the furnishings on A are pointwise. Thus we furnish A with an addition and a zero and with a multiplication and a one. As in part (b) the addition furnishes A as a commutative group, and is in part (a) the multiplication furnishes A as a monoid. We must check that the associative laws hold. We require

$$a(b+c) = ab + ac \qquad (a+b)c = ac + bc$$

for threads a, b, c. To check these we evaluate at an arbitrary index i and so pass down to $A(i)$, where the corresponding equality does hold.

This furnishes A as a ring, and two calculations, as in parts (a,b) show that each evaluation function α is a ring morphism. Furthermore, all the required triangles commute, so we have produces a solution of the diagram in $\boldsymbol{Rng}$.

We show that this solution is universal, and so is a limit in $\boldsymbol{Rng}$. To do that consider any other solution

$$X \xrightarrow{\xi(i)} A(i)$$

in $\boldsymbol{Rng}$ indexed by the nodes i of ∇. By forgetting the carried structure this gives a solution of the diagram in $\boldsymbol{Set}$, and hence there is a unique function

$$X \xrightarrow{\mu} A$$

such that

$$\xi(i) = \alpha(i) \circ \mu$$

for each index i. It suffices to show that this function μ is a ring morphism. This is done as in the $\boldsymbol{Mon}$ case. $\square$

4.6.5 Given a ∇-diagram in $\boldsymbol{Pom}$ in the usual notation, let A be the set of threads. This set A can be furnished as a poset and a monoid by

$$x \leq a \Longleftrightarrow (\forall i)[x(i) \leq a(i)] \qquad (ab)(i) = a(i)b(i)$$

for all $a, b, x \in A$ and index i. (We can now drop the use of $\star$ for the operation on A.) We show this is a pom.

Consider $a, b, x, y \in A$ with $x \leq a$ and $y \leq b$. Then

$$(\forall i)[(xy)(i) = x(i)y(i) \leq a(i)b(i) = (ab)(i)]$$

to verify the required comparison property. By now you should find the required arrow-theoretic properties routine. $\square$

4.6.4 Limits in $\boldsymbol{Top}$

4.6.6 In this case a thread is just a choice function for the $\mathbb{I}$-indedex family A. The topology as described in the subsection is just the standard product topology on $\prod$A. $\square$

4.7 Confluent colimits in $\boldsymbol{Set}$

4.7.1 By dropping down to $\boldsymbol{Set}$ we know that the corresponding $\mathbb{I}$-diagram has a colimit L. We show that this set L can be furnished to form a colimit in $\boldsymbol{Set}$-R.

We are given that each function

$$\begin{array}{ccc} A(i) & \xrightarrow{A(j,i)} & A(j) \\ x & \longmapsto & x|j \end{array}$$

is an R-morphism, that is

$$(xr)|j = (x|j)r$$

for each $x \in A(i)$ and $r \in R$.

We know that L is the set of blocks $[x, i]$ of a certain equivalence relation $\sim$ on $\amalg A$. We must produce an R-action on L. Given a block $[x, i]$ in L we wish to set

$$[x, i]r = [xr, i]$$

for each $r \in R$. We must show that this is well defined.

Consider two representatives

$$[x, i] \qquad [y, j]$$

of the same block. Thus

$$x|k = y|k$$

for some node $i, j \leq k$. Since each $A(k, \cdot)$ is an R-morphism we have

$$(xr)|k = (x|k)r = (y|k)r = (yr)|k$$

to show that

$$[xr, i] = [yr, j]$$

and hence the action is well-defined.

We need to verify the action axioms, but that is straight forward.

This furnishes L as an R-set. Also, for each node i and $x \in A(i)$, by definition, we have

$$[xr, i] = [x, i]r$$

so that the connection function

$$\begin{array}{ccc} A(i) & \longrightarrow & L \\ x & \longmapsto & [x,i] \end{array}$$

is an R-morphism.

This shows that we have a right solution to the diagram in $\boldsymbol{Set}$-R. We must show it is the universal right solution. To this end consider any right solution

$$A(i) \xrightarrow{\alpha_i} M$$

of the diagram. Here M is an R-set and each α_i is an R-morphism. Dropping down to $\boldsymbol{Set}$ we know there is a unique function such that

$$\begin{tikzcd} A(i) \arrow[rr, "\alpha_i"] \arrow[dr] & & M \\ & L \arrow[ur, "\mu"'] & \end{tikzcd}$$

commutes. It suffices to show that μ is an R-morphism, that is

$$\mu\big([x, i]r\big) = \mu\big([x, i]\big)r$$

for each node i, element $x \in A(i)$, and $r \in R$. But, since

$$\mu\big([x, i]\big) = \alpha_i(x)$$

we have

$$\mu\big([x, i]r\big) = \mu\big([xr, i]\big) = \alpha_i(xr) = \alpha_i(x)r = \mu\big([x, i]\big)r$$

as required. □

4.7.2 Dropping down to $\boldsymbol{Set}$ the corresponding $\mathbb{I}$-diagram has a colimit L. We show that this set L can be furnished to form a colimit in $\boldsymbol{Mon}$.

We are given that each function

$$\begin{array}{ccc} A(i) & \xrightarrow{A(j,i)} & A(j) \\ x & \longmapsto & x|j \end{array}$$

is a monoid morphism, that is

$$(xy)|j = (x|j)(y|j)$$

for each $x, y \in A(i)$.

We know L is the set of blocks $[x, i]$ of a equivalence relation $\sim$ on $\amalg A$. We must produce a binary operation on L. Given a pair of blocks

$$[x, i] \qquad [y, j]$$

in L we wish to set

$$[x, i] \cdot [y, j] = [(x|k)(y|k), k]$$

where k is any node with $i, j \leq k$. We must show that this is well defined.

Consider

$$[x_1, i_1] = [x_2, i_2] \qquad [y_1, j_1] = [y_2, i_2] \qquad \text{that is} \qquad x_1|i = x_2|i \qquad y_1|j = y_2|j$$

for some nodes

$$i_1, i_2 \le i \qquad j_1, j_2 \le j$$

from $\mathbb{I}$. We must show that

$$[(x_1|k_1)(y_1|k_1), k_1] = [(x_2|k_2)(y_2|k_2), k_2]$$

where k_1, k_2 are any nodes with

$$i_1, j_1 \le k_1 \qquad i_2, j_2 \le k_2$$

respectively. Given such a pair k_1, k_2 of nodes we must show that

$$\big((x_1|k_1)(y_1|k_1)\big)|l = \big((x_2|k_2)(y_2|k_2)\big)|l$$

for at least one node l with $k_1, k_2 \le l$. Consider any node l with $i, j, k_1, k_2 \le l$. We have

$$\begin{aligned}\big((x_1|k_1)(y_1|k_1)\big)|l &= \big((x_1|k_1)|l\big)\big((y_1|k_1)|l\big)\\ &= (x_1|l)(y_1|l) = \big((x_1|i)|l\big)\big((y_1|j)|l\big)\end{aligned}$$

with

$$\big((x_2|k_2)(y_2|k_2)\big)|l = \big((x_2|i)|l\big)\big((y_2|j)|l\big)$$

by a similar calculation. The relationship between the 1- and the 2-components now gives the required result.

This furnishes L with a binary operation. A further calculation of this kind shows that this operation in associative. And then another small calculation shows that L is a monoid.

Next we must show that for each node i the connecting function

$$\begin{array}{ccc} A(i) & \longrightarrow & L \\ x & \longmapsto & [x, i] \end{array}$$

is a monoid morphism, that is

$$[x, i] \cdot [y, i] = [xy, i]$$

for each $x, y \in A(i)$. Since both x, y are in $A(i)$ this is immediate from the definition of the operation on L.

This shows that we have a right solution to the diagram in $\boldsymbol{Mon}$. We must show it is the universal right solution. To this end consider any right solution

$$A(i) \xrightarrow{\alpha_i} M$$

of the diagram. Here M is a monoid and each α_i is a monoid morphism. Dropping down to $\boldsymbol{Set}$ we know there is a unique function such that

$$\begin{tikzcd} A(i) \arrow[rr, "\alpha_i"] \arrow[dr] & & M \\ & L \arrow[ur, "\mu"'] & \end{tikzcd}$$

commutes. It suffices to show that this function μ is a monoid morphism.

We know that

$$\mu\big([x,i]\big) = \alpha_i(x)$$

for each node i and $x \in A(i)$. We require

$$\mu\big([x,i]\cdot[y,j]\big) = \mu\big([x,i]\big)\mu\big([y,j]\big)$$

for each pair i,j of nodes and elements $x \in A(i), y \in A(j)$. Consider any node k with $i,j \le k$. We have

$$\begin{aligned}
\mu\big([x,i]\cdot[y,j]\big) &= \mu\big((x|k,k)(y|k,k)\big)\\
&= \mu\big((x|k)(y|k),k)\big)\\
&= \alpha_k\big((x|k)(y|k)\big) = \alpha_k(x|k)\alpha(y|k)
\end{aligned}$$

and

$$\mu\big([x,i]\big)\mu\big([y,j]\big) = \alpha_i(x)\alpha_j(y)$$

so the required result follows by the given commuting diagrams that the $\alpha_\bullet$ satisfy. □

4.7.3 We are given monoids A,B and wish to produce the coproduct $A \amalg B$ in ***Mon***? We do this in three steps.

For the first step we forget the structure, we drop down to ***Set*** and produce the coproduct of the two carrying sets in ***Set***. This, of course, is just the disjoint union of the two sets. Let

$$\begin{tikzcd}
A \arrow[d, "i"'] \\
A \dot\cup B \\
B \arrow[u, "j"]
\end{tikzcd}$$

be the coproduct in ***Set***. Here $A \dot\cup B$ is merely a set and i,j are merely functions, but with a certain property.

For the second step we remember that each set X freely generates a monoid FX. This is saying the forgetful functor from ***Mon*** to ***Set*** has a left adjoint. We don't need an explicit description of FX. Let F be the free monoid generated by $A \dot\cup B$ via a function η. Thus we now have a commuting diagram in ***Set***

$$\begin{tikzcd}
A \arrow[d, "i"'] \arrow[dr, "\lambda"] & \\
A \dot\cup B \arrow[r, "\eta"] & F \\
B \arrow[u, "j"] \arrow[ur, "\rho"'] &
\end{tikzcd}$$

where A,B,F are monoids but the arrows are merely functions.

For the third step we take a certain monoid quotient

$$F \xrightarrow{\ k\ } A \amalg B$$

and so obtain a commuting diagram in $\boldsymbol{Set}$.

$$\begin{tikzcd}
A \arrow[d, "i"'] \arrow[dr, "\lambda"] \arrow[drr, "l"] & & \\
A \dot\cup B \arrow[r, "\eta"] & F \arrow[r, "k"] & A \amalg B \\
B \arrow[u, "j"] \arrow[ur, "\rho"] \arrow[urr, "r"'] & &
\end{tikzcd}$$

The idea is that we take a quotient k so that the induced composite functions l, r are monoid morphism. Furthermore, we take the smallest quotient (that is, the quotient that causes the least amount of collapse) for which the produced functions l, r are monoid morphisms.

Let's see what this works. Consider a wedge in $\boldsymbol{Mon}$, as on the left.

$$\begin{tikzcd}
A \arrow[dr, "f"] & \\
& C \\
B \arrow[ur, "g"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[d, "i"'] \arrow[dr, "f"] & \\
A \dot\cup B \arrow[r, "h"] & C \\
B \arrow[u, "j"] \arrow[ur, "g"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[d, "i"'] \arrow[dr, "\lambda"] \arrow[drr, "f"] & & \\
A \dot\cup B \arrow[r, "\eta"] & F \arrow[r, "h^\sharp"] & C \\
B \arrow[u, "j"] \arrow[ur, "\rho"] \arrow[urr, "g"'] & &
\end{tikzcd}$$

Here C is a monoid and f, g are monoid morphisms. We must show there is a unique morphism m to obtain a commuting diagram in $\boldsymbol{Mon}$.

$$\begin{tikzcd}
A \arrow[d, "l"'] \arrow[dr, "f"] & \\
A \amalg B \arrow[r, "m"] & C \\
B \arrow[u, "r"] \arrow[ur, "g"'] &
\end{tikzcd}$$

By dropping down to $\boldsymbol{Set}$ we see there is a unique function h such that the central diagram commutes in $\boldsymbol{Set}$. This is because $A \dot\cup B$ is the coproduct in $\boldsymbol{Set}$.

We now have a function h from the set $X = A \dot\cup B$ to a monoid C. This must factor uniquely through the monoid F freely generated by X. Thus we obtain a commuting diagram as on the right where $h^\sharp$ is a *monoid* morphism.

Notice that

$$h^\sharp \circ \lambda = f \qquad h^\sharp \circ \rho = g$$

so that these two composites functions are monoid morphisms. But k is the smallest quotient for which $k \circ \lambda$ and $k \circ \rho$ are monoid morphism. Thus $h^\sharp$ factors uniquely through k to produce a commuting diagram

$$\begin{tikzcd}
A \arrow[d, "i"'] \arrow[dr, "\lambda"] \arrow[drr, "l"] \arrow[drrr, "f"] & & & \\
A \dot\cup B \arrow[r, "\eta"] & F \arrow[r, "k"] & A \amalg B \arrow[r, "m"] & C \\
B \arrow[u, "j"] \arrow[ur, "\rho"] \arrow[urr, "r"] \arrow[urrr, "g"'] & & &
\end{tikzcd}$$

where m is a monoid morphism.

This doesn't quite finish the proof for we must show that m is the only morphism that does this job. To this end suppose

$$n \circ l = f \quad n \circ r = g \qquad m \circ l = f \quad m \circ r = g$$

for some morphism n. We require $n = m$. By tracking through the various commuting diagrams we have

$$n \circ k \circ \eta \circ i = f \qquad n \circ k \circ \eta \circ j = g$$

and the same equalities hold with n replaced by m. Now i, j determine the coproduct $A \dot{\cup} B$ of A, B in $\boldsymbol{Set}$. Thus

$$n \circ k \circ \eta = m \circ k \circ \eta$$

by the uniqueness of the mediator. Next we remember that η freely generates F from $A \dot{\cup} B$. Thus

$$n \circ k = m \circ k$$

by the uniqueness of the mediator for that construction. Finally, since k is surjective we have $n = m$, as required. $\square$

4.7.4 We describe the construction of a multi-coequalizer.

Let $\mathcal{F}$ be a family of monotone maps f between two posets.

$$T \overset{\longrightarrow}{\underset{\longrightarrow}{\quad\mathcal{F}\quad}} S$$

The coequalizer case is when $\mathcal{F}$ has just two members. We require a certain monotone map

$$S \xrightarrow{\ \eta\ } C$$

which makes equal the family $\mathcal{F}$, that is for each $f \in \mathcal{F}$ the composite $g \circ f$ is independent of f. We require a universal example of such a map.

Consider the *pre-orders* $\sqsubseteq$ on S with the following two properties.

(i) For each $a, b \in S$ we have $a \leq b \Longrightarrow a \sqsubseteq b$.

(ii) For each $c \in T$ and each $f_1, f_2 \in \mathcal{F}$ both the comparisons $f_1(c) \sqsubseteq f_2(c)$ and $f_2(c) \sqsubseteq f_1(c)$ hold.

There is at least one such special pre-order. For instance the indiscrete pre-order for which $a \sqsubseteq b$ for all $a, b \in S$. This is the largest special pre-order. We want the smallest special pre-order.

Let

$$\{\sqsubseteq_i \mid i \in I\}$$

be the family of all special pre-orders. Let $\preceq$ be the intersection of this family, that is

$$a \preceq b \Longleftrightarrow (\forall i \in I)[a \sqsubseteq_i b]$$

for $a, b \in S$. It is easy to check that $\preceq$ is a special pre-order. For instance, consider $c \in T$ and $f_1, f_2 \in \mathcal{F}$. Then $f_1(c) \sqsubseteq_i f_2(c)$ for each $i \in I$, and hence $f_1(c) \preceq f_2(c)$. This show that $\preceq$ satisfies (ii).

Let $\simeq$ be the equivalence relation on S given by $\preceq$, that is

$$a \simeq b \iff a \preceq b \text{ and } b \preceq a$$

for $a, b \in S$. Let $S/\simeq$ be the set of blocks $[a]$ of $\simeq$, and let

$$S \xrightarrow{\ \eta\ } S/\simeq$$

be the quotient function, that is

$$\eta(a) = [a]$$

for each $a \in S$. We convert $S/\simeq$ into a poset with η monotone.

Let

$$[a] \leq [b] \iff a \preceq b$$

for $a, b \in S$. Of course, we must check that this is well-defined.

Suppose

$$[a_1] = [a_2] \qquad [b_1] = [b_2]$$

with $a_1 \preceq b_1$. Then

$$a_2 \simeq a_1 \preceq b_1 \simeq b_2$$

so that

$$a_2 \preceq a_1 \preceq b_1 \preceq b_2$$

to give $a_2 \preceq b_2$ to verify the well-defined property.

By (i) we have

$$a \leq b \Longrightarrow a \preceq b \Longrightarrow [a] \leq [b]$$

to show that η is monotone. By (ii) we have

$$f_1(c) = f_2(c)$$

for each $c \in T$ and $f_1, f_2 \in \mathcal{F}$. Thus for each $f \in \mathcal{F}$ the value

$$(\eta \circ f)(c)$$

is independent of f. Thus η makes equal the family $\mathcal{F}$. It remains to show that η is a universal example of such a monotone map.

Consider any monotone map g which makes equal the family $\mathcal{F}$.

$$\begin{tikzcd} T \arrow[r, shift left] \arrow[r, shift right, "\mathcal{F}"] & S \arrow[r, "g"] \arrow[d, "\eta"'] & R \\ & S/\preceq & \end{tikzcd}$$

We show there is a unique monotone map

$$S/\preceq \xrightarrow{\ g^\sharp\ } R$$

with $g^\sharp \circ \eta = g$. Since η is surjective there is at most one such map $g^\sharp$. Thus it suffices to exhibit such a map.

Consider the relation $\sqsubseteq$ on S given by

$$a \sqsubseteq b \Longleftrightarrow g(a) \leq g(b)$$

for $a, b \in S$. Almost trivially this is a pre-order. We check that it is special, that is it satisfies (i, ii). For $a, b \in S$ we have

$$a \leq b \Longrightarrow g(a) \leq g(b) \Longrightarrow a \preceq b$$

to verify (i). For $f_1, f_2 \in \mathcal{F}$ we have

$$g \circ f_1 = g \circ f_2$$

which leads to (ii).

Since this relation $\sqsubseteq$ is special and $\preceq$ is the smallest special pre-order, we have

$$a \preceq b \Longrightarrow a \sqsubseteq b \Longrightarrow g(a) \leq g(b)$$

for each $a, b \in S$. From this we have

$$[a] = [b] \Longrightarrow a \preceq b \text{ and } b \preceq a \Longrightarrow g(a) = g(b)$$

for $a, b \in S$. Thus we may set

$$g^\sharp([a]) = g(a)$$

to obtain a well-defined function of the required type. By definition we have $g^\sharp \circ \eta = g$. Also, for $a, b \in S$ we have

$$[a] \leq [b] \Longrightarrow a \preceq b \Longrightarrow g(a) \leq g(b) \Longrightarrow g^\sharp([a]) \leq g^\sharp([b])$$

to show that $g^\sharp$ is monotone, This completes the proof. $\square$

5
Adjunctions

5.1 Adjunctions defined

5.1.1 Let S, T be a pair of posets viewed as categories.

A functor $S \longrightarrow T$ is a function (the object function)

$$f : S \longrightarrow T$$

such that for each comparable pair $a_1 \leq a_2$ of elements of S (arrow of S) the corresponding pair of T are comparable $f(a_1) \leq f(a_2)$ (an arrow of T). Thus

$$a_1 \leq a_2 \Longrightarrow f(a_1) \leq f(a_2)$$

for $a_1, a_2 \in S$. This is just a monotone map.

Consider a pair of monotone maps

$$\begin{tikzcd} S \arrow[r, shift left, "f"] & T \arrow[l, shift left, "g"] \end{tikzcd}$$

going in the opposite direction. These form a categorical adjunction if for each $a \in S$ and $b \in T$ there is an appropriate correspondence between the two arrow sets

$$S[a, g(b)] \qquad T[f(a), b]$$

given by the two elements. Each of these is no more than a singleton. Thus the correspondence says that one is non-empty precisely when the other is non-empty. This rephrases as

$$f(a) \leq b \Longleftrightarrow a \leq g(a)$$

for $a \in S, b \in T$, which is the defining property of a poset adjunction. $\square$

5.1.2 Consider the forgetful functor.

$$\textbf{\textit{Set}} \xleftarrow{\;U\;} \textbf{\textit{Pre}}$$

We produce a left adjoint and a right adjoint to this functor.

$$D \dashv U \qquad \begin{tikzcd} \textbf{\textit{Set}} \arrow[r, shift left=2, "D"] \arrow[r, shift right=2, "I"'] & \textbf{\textit{Pre}} \arrow[l, "U" description] \end{tikzcd} \qquad U \dashv I$$

Each set X can be converted into a preset in two extreme ways.

$$DX = (X, =) \qquad IX = (X, \|)$$

On the left we use equality as the comparison. This gives a poset. On the right any two elements are comparable. This is not a poset if X has least two elements. We call these the

discrete indiscrete

presets, respectively. This gives the object assignments of two functors.

Consider any function.

$$X \xrightarrow{\;f\;} Y$$

We observe that f is monotone relative to both the discrete and the indiscrete comparisons, that is

$$x = y \Longrightarrow f(x) = f(y) \qquad f(x) \parallel f(y)$$

for all $x, y \in X$. This gives us the arrow assignments of two functors. We use the same function but view it as a monotone map in two ways.

We now check the two adjunctions separately.

For the adjunction $D \dashv U$ we require an inverse pair of assignments

$$\begin{array}{ccc} f & \longmapsto & f^\sharp \\ \textbf{\textit{Set}}[X, US] & & \textbf{\textit{Pre}}[DX, S] \\ \phi_\flat & \longleftarrow\!\shortmid & \phi \end{array}$$

for each set X and each preset S. In fact both $(\cdot)^\sharp$ and $(\cdot)_\flat$ return the same function, so do form an inverse pair. The only thing we have to check is that

$$DX \xrightarrow{\;f^\sharp\;} S$$

is monotone for each function f, as above. This is trivial. Finally, for $D \dashv U$, we must check that the two assignments are natural. However, nothing much is happening as we pass across a functor or an assignment, so the naturality is immediate.

For the adjunction $U \dashv I$ we require an inverse pair of assignments

$$\begin{array}{ccc} \phi & \longmapsto & \phi^\sharp \\ \textbf{\textit{Pre}}[S, IX] & & \textbf{\textit{Set}}[US, X] \\ f_\flat & \longleftarrow\!\shortmid & f \end{array}$$

for each preset S and each set X. In fact both $(\cdot)^\sharp$ and $(\cdot)_\flat$ return the same function, so do form an inverse pair. The only thing we have to check is that

$$S \xrightarrow{f_\flat} IX$$

is monotone. But since IX is indiscrete, this is trivial. Finally, for $U \dashv I$, we must check that the two assignments are natural. However, nothing much is happening as we pass across a functor or an assignment, so the naturality is immediate. $\square$

5.1.3 This is more or less the same as Solution 5.1.2 except we now use topologies rather than pre-orders.

Let

$$\boldsymbol{Set} \xleftarrow{\;U\;} \boldsymbol{Top}$$

be the forgetfully functor. We produce a left adjoint and a right adjoint to this functor.

$$\begin{tikzcd} \boldsymbol{Set} \arrow[r, shift left=3ex, "D"] \arrow[r, shift right=3ex, "I"'] & \boldsymbol{Top} \arrow[l, "U" description] \end{tikzcd}$$

Each set X can be converted into a topological space in two extreme ways.

$$DX = (X, \mathcal{P}X) \qquad IX = (X, \{\emptyset, X\})$$

On the left we use discrete topology in which each subset is open. On the right we use the indiscrete topology in which only the two extreme subsets are open. Naturally, we call these the

discrete indiscrete

space, respectively. This gives the object assignments of two functors.

Consider any function.

$$X \xrightarrow{f} Y$$

We observe that f is continuous relative to both the discrete and the indiscrete topologies. This gives the arrow assignments of two functors. We use the same function but view it as a continuous map in two ways.

We now check the two adjunctions separately.

$$D \dashv U \qquad U \dashv I$$

For the adjunction $D \dashv U$ we require an inverse pair of assignments

$$\begin{array}{ccc} f & \longmapsto & f^\sharp \\ \boldsymbol{Set}[X, US] & & \boldsymbol{Top}[DX, S] \\ \phi_\flat & \longleftarrow\!\!\shortmid & \phi \end{array}$$

for each set X and each space S. In fact both $(\cdot)^\sharp$ and $(\cdot)_\flat$ return the same function, so do form an inverse pair. The only thing we have to check is that

$$DX \xrightarrow{f^\sharp} S$$

is continuous for each function f, as above. This is trivial. Finally, for $D \dashv U$, we must check that the two assignments are natural. However, nothing much is happening as we pass across a functor or an assignment, so the naturality is immediate.

For the adjunction $U \dashv I$ we require an inverse pair of assignments

$$\begin{array}{ccc} \phi & \longmapsto & \phi^\sharp \\ \boldsymbol{Top}[S, IX] & & \boldsymbol{Set}[US, X] \\ f_\flat & \longleftarrow\!\!\dashv & f \end{array}$$

for each space S and each set X. In fact both $(\cdot)^\sharp$ and $(\cdot)_\flat$ return the same function, so do form an inverse pair. The only thing we have to check is that

$$S \xrightarrow{f_\flat} IX$$

is continuous. But since IX is indiscrete, this is trivial. Finally, for $U \dashv I$, we must check that the two assignments are natural. However, nothing much is happening as we pass across a functor or an assignment, so the naturality is immediate. $\square$

5.1.4 This is dealt with in great detail in the first part of Chapter 6. Not all the details are necessary. You may want to decide which of the quicker solutions you prefer. $\square$

5.1.5 As in Chapter 4 we let $i, j, \ldots$ range over the objects of the template category ∇ and refer to these as nodes. We use

$$i \xrightarrow{e} j$$

as a typical arrow of ∇ and refer to this as an edge.

An object of $\boldsymbol{C}^\nabla$, a ∇-diagram in $\boldsymbol{C}$, is a functor

$$\nabla \xrightarrow{F} \boldsymbol{C}$$

and so consists of a family of objects of $\boldsymbol{C}$

$$F(i)$$

indexed by the nodes and a collection of arrows of $\boldsymbol{C}$

$$F(i) \xrightarrow{F(e)} F(j)$$

indexed by the edges. Various triangles in $\boldsymbol{C}$ are required to commute.

An arrow of $\boldsymbol{C}^\nabla$

$$F \xrightarrow{\eta_\bullet} G$$

is just a natural transformation between the two functors. These are composed in the obvious way.

For each object A of $\boldsymbol{C}$ we set

$$(\Delta A)(i) = A \qquad (\Delta A)(e) = id_A$$

for each node i and each edge e. This gives a constant diagram. Each arrow of $\boldsymbol{C}$

$$A \xrightarrow{\eta} B$$

gives a 'constant' natural transformation

$$\Delta A \xrightarrow{\eta_\bullet} \Delta B$$

in the obvious way. This sets up a functor

$$\boldsymbol{C} \xrightarrow{\Delta} \boldsymbol{C}^\nabla$$

and we are interested in the existence or not of a right adjoint to Δ.

Consider first an object A of $\boldsymbol{C}$ and an object F of $\boldsymbol{C}^\nabla$. What does a member of

$$\boldsymbol{C}^\nabla[\Delta A, F]$$

look like? It is a family of arrows

$$A \xrightarrow{\alpha(i)} F(i)$$

indexed by the nodes of ∇ such that

$$\begin{tikzcd} A \arrow[r, "\alpha(i)"] \arrow[d, "id"'] & F(i) \arrow[d, "F(e)"] & & i \arrow[d, dash, "e"] \\ A \arrow[r, "\alpha(j)"'] & F(j) & & j \end{tikzcd}$$

commutes for each edge e, as indicated on the right. In other words, this is nothing more than a left solution to the diagram F.

Suppose now that we fix a particular solution

$$S \xrightarrow{\sigma(i)} F(i)$$

and try to compare S with an arbitrary object A of $\boldsymbol{C}$. How are the two hom-sets

$$\boldsymbol{C}[A, S] \qquad \boldsymbol{C}^\nabla[\Delta A, F]$$

related? There is an obvious assignment

$$\begin{array}{ccc} \mu & \longmapsto & \mu^\sharp \\ \boldsymbol{C}[A, S] & & \boldsymbol{C}^\nabla[\Delta A, F] \end{array}$$

in one direction. Given a $\boldsymbol{C}$-arrow

$$A \xrightarrow{\mu} S$$

for each node i we let $\mu^\sharp(i)$ be the composite

$$A \xrightarrow{\mu} S \xrightarrow{\sigma(i)} F(i)$$

to produce a left solution of F. This gives us a family of assignments

$$\boldsymbol{C}[-,S] \xrightarrow{(\cdot)^\sharp} \boldsymbol{C}^\nabla[\Delta-,F]$$

as we let the object A vary through $\boldsymbol{C}$. Notice that we have a pair of contravariant functors

$$\boldsymbol{C} \longrightarrow \boldsymbol{Set}$$

and it is easy to show $(\cdot)^\sharp$ is a natural transformation between the two.

When is this natural transformation a natural equivalence? Precisely when each solution

$$\Delta A \xrightarrow{\alpha(\bullet)} F$$

arises from a unique arrow

$$A \xrightarrow{\mu} S$$

as $\mu^\sharp$. This is simply saying that (S,σ) is a universal left solution, a limit of F. Every ∇-diagram F in $\boldsymbol{C}$ has a limit precisely when there is an object assignment

$$S \longleftarrow F$$

picking out the object which carries the limit. The required functorial and adjunction properties follow by similar arguments. $\square$

5.1.6 We use various aspects of the gadgetry of the adjunction.

$$\boldsymbol{Src}[-,G-] \underset{(\cdot)_\flat}{\overset{(\cdot)^\sharp}{\rightleftarrows}} \boldsymbol{Trg}[F-,-]$$

Here F is the left adjoint of the given functor G.

Consider some diagram

$$\boldsymbol{T}D \qquad T(i) \qquad T(e)$$

in $\boldsymbol{Trg}$. Here i ranges over the nodes of the template and e ranges over the edges. In this diagram certain triangles are required to commute.

We use the functor G to transport these objects and arrows to $\boldsymbol{Src}$.

$$\boldsymbol{S}D \qquad GT(i) \qquad GT(e)$$

Since G is a functor this is a diagram of the same template in $\boldsymbol{Src}$.

Suppose

$$T \xrightarrow{\tau(i)} T(i)$$

is a limit of the diagram $\boldsymbol{T}D$ in $\boldsymbol{Trg}$. Thus T is a fixed object and there is an arrow $\tau(i)$ for each node i of the template. Various triangles are required to commute, those indexed by the edges e of the template. We use G to transport this to $\boldsymbol{Src}$

$$GT \xrightarrow{G(\tau(i))} GT(i)$$

and since G is a functor we certainly obtain a left solution of the diagram $\boldsymbol{S}D$ in ***Src***.

Consider any left solution

$$X \xrightarrow{\ \xi(i)\ } GT(i)$$

of the diagram $\boldsymbol{S}D$ in ***Src***. We must somehow produce a unique mediator

$$X \xrightarrow{\ \mu\ } GT$$

for which

$$\begin{tikzcd} & GT \arrow[d, "G(\tau(i))"] \\ X \arrow[ur, "\mu"] \arrow[dr, "\xi(i)"'] \arrow[r, phantom, "(\lhd)"] & GT(i) \end{tikzcd}$$

commutes for each node i.

We use the transpositions

$$\boldsymbol{Src}[X, GT(i)] \xrightarrow{\ (\cdot)^\sharp\ } \boldsymbol{Trg}[FX, T(i)]$$

to obtain a family of arrows

$$FX \xrightarrow{\ \xi(i)^\sharp\ } T(i)$$

in ***Trg***. Since $(\cdot)^\sharp$ is natural, this is a left solution of the diagram $\boldsymbol{T}D$ in ***Trg***. Thus, since we have a limit of this diagram, here is a unique arrow ν such that each triangle

$$\begin{tikzcd} & T \arrow[d, "\tau(i)"] \\ FX \arrow[ur, "\nu"] \arrow[dr, "\xi(i)^\sharp"'] & T(i) \end{tikzcd}$$

commutes.

Since the transposition

$$\boldsymbol{Src}[X, GT] \xrightarrow{\ (\cdot)^\sharp\ } \boldsymbol{Trg}[FX, T]$$

is a bijection we have

$$\nu = \mu^\sharp \qquad \text{for some unique arrow} \qquad X \xrightarrow{\ \mu\ } GT$$

of ***Src***. It suffices to show that each triangle $(\lhd)$ of ***Src*** commutes. But this follows by the naturality of $(\cdot)^\sharp$, strictly speaking, by the naturality of the inverse $(\cdot)_\flat$ of $(\cdot)^\sharp$. ☐

5.1.7 Both $\mathfrak{F}$ and $\mathfrak{G}$ are modified versions of the 2-placed hom-functor $\boldsymbol{Trg}[-.-]$. In particular, each is *contravariant* in the left hand argument. Thus we really have a pair of functors

$$\boldsymbol{Src}^{\mathrm{op}} \times \boldsymbol{Trg} \longrightarrow \boldsymbol{Set}$$

using the opposite of ***Src***. The details of the two arrow assignments are given in Table 5.2 in Section 5.3. ☐

5.2 Adjunctions illustrated

5.2.1 An algebraic example

5.2.1 We deal with Σ first. We are given

$$\Sigma X = X + X = \{(x,i) \mid x \in X,\ i = 0,1\}$$

for each set X. The carried involution flips the tag, that is

$$(x,i)^{\bullet} = (x, 1-i)$$

for each $x \in X$ and tag $i \in \{0,1\}$. Since

$$(x,i)^{\bullet\bullet} = (x,1-i)^{\bullet} = (x,i)$$

this does produce an involution algebra. For each function

$$Y \xrightarrow{\ k\ } X$$

the only sensible assignment

$$\begin{array}{rcl} \Sigma Y & \xrightarrow{\ \Sigma(k)\ } & \Sigma X \\ (y,i) & \longmapsto & (k(y),i) \end{array}$$

is to leave the tag alone. We have

$$\begin{aligned} \Sigma(k)\big((y,i)^{\bullet}\big) &= \Sigma(k)\big((y,1-i)\big) \\ &= (k(y),1-i) = (k(y),i)^{\bullet} = \big(\Sigma(k)(y,i)\big)^{\bullet} \end{aligned}$$

so that $\Sigma(k)$ is a morphism. The functorial requirements are immediate.

Next we deal with Π. We are given

$$\Pi X = X \times X = \{(x,y) \mid x,y \in X\}$$

for each set X. The carried involution swaps the components, that is

$$(x,y)^{\bullet} = (x,y)$$

for each $x,y \in X$. Trivially, this does produce an involution algebra. For each function

$$Y \xrightarrow{\ k\ } X$$

the only sensible assignment

$$\begin{array}{rcl} \Pi Y & \xrightarrow{\ \Pi(k)\ } & \Pi X \\ (y,z) & \longmapsto & (k(y),k(z)) \end{array}$$

is to apply the function to both components. We have

$$\begin{aligned} \Pi(k)\big((y,z)^{\bullet}\big) &= \Pi(k)\big((z,y)\big) \\ &= (k(z),k(y)) \\ &= (k(y),k(z))^{\bullet} = \big(\Pi(k)(y,i)\big)^{\bullet} \end{aligned}$$

so $\Pi(k)$ is a morphism. The functorial requirements are immediate. $\square$

5.2.2 We deal with Σ first. We require an inverse pair of assignments

$$\begin{array}{ccc} f & \longmapsto & f^\sharp \\ \boldsymbol{Set}[X, UA] & & \boldsymbol{Inv}[\Sigma X, A] \\ \psi_\flat & \longleftarrow\!\!\mid & \psi \end{array}$$

for each set X and each algebra A. We set

$$f^\sharp\big((x,i)\big) = f(x)^{(i)} \qquad \psi_\flat(x) = \psi(x,0)$$

for each $x \in X$ and tag i. There are some requirements we must check.

We need to show that $f^\sharp$ is a morphism, that is

$$f^\sharp\big((x,i)^\bullet\big) = \big(f^\sharp(x,i)\big)^\bullet$$

for each $x \in X$ and tag i. To do that we remember that

$$a^{(i)\bullet} = a^{(1-i)} = a^{\bullet(i)}$$

for each $a \in A$. With this we have

$$f^\sharp\big((x,i)^\bullet\big) = f^\sharp(x, 1-i) = f(x)^{(1-i)} = f(x)^{(i)\bullet} = \big(f^\sharp(x,i)\big)^\bullet$$

as required.

For each $x \in X$ and tag i we have

$$\begin{aligned} f^\sharp{}_\flat(x) &= f^\sharp(x,0) = f(x)^{(0)} = f(x) \\ \psi_\flat{}^\sharp(x,i) &= \psi_\flat(x)^{(i)} = \psi(x,0)^{(i)} = \psi(x,i) \end{aligned}$$

to show that the two assignments form an inverse pair. The last step in the lower calculations follows since ψ is a morphism.

Next we deal with Π. We require an inverse pair of assignments

$$\begin{array}{ccc} \phi & \longmapsto & \phi^\sharp \\ \boldsymbol{Inv}[A, \Pi X] & & \boldsymbol{Set}[UA, X] \\ g_\flat & \longleftarrow\!\!\mid & g \end{array}$$

for each set X and each algebra A. We set

$$\phi^\sharp(a) = \phi(a)_0 \qquad g_\flat(a) = (g(a), g(a^\bullet))$$

for each $a \in A$. In $\phi^\sharp$ the $(\cdot)_0$ indicates the left hand component is selected. We need to show that $g_\flat$ is a morphism, that is

$$g_\flat(a^\bullet) = g_\flat(a)^\bullet$$

for each $a \in A$. But, remembering how ΠX is structured, we have

$$g_\flat(a^\bullet) = \big(g(a^\bullet), g(a^{\bullet\bullet})\big) = \big(g(a^\bullet), g(a)\big) = \big(g(a), g(a^\bullet)\big)^\bullet = g_\flat(a)^\bullet$$

as required

To show that these two assignments form an inverse pair consider any $a \in A$. Let

$$\phi(a) = (x, y)$$

where $x, y \in X$. Then, since ϕ is a morphism, we have

$$\phi(a^\bullet) = \phi(a)^\bullet = (y, x)$$

so that

$$\phi^\sharp(a) = x \qquad \phi^\sharp(b) = y$$

to give

$$\phi^\sharp{}_\flat(a) = \big(\phi^\sharp(a), \phi^\sharp(a^\bullet)\big) = (x, y) = \phi(a)$$

for one of the required conditions. For the other, since

$$g_\flat(a) = (g(a), -)$$

we have

$$g_\flat{}^\sharp(a) = g(a)$$

as required. □

5.2.2 A set-theoretic example

5.2.3 For this and the next solution let us write L and R for the two endofunctors on $\boldsymbol{Set}$. Thus

$$LX = X \times I \qquad RY = (I \Rightarrow Y)$$

for all sets X and Y. The arrow assignments are given in the subsection.

We require an inverse pair of bijections

$$\begin{array}{ccc} f & \longmapsto & f^\sharp \\ \boldsymbol{Set}[X, RY] & & \boldsymbol{Set}[LX, Y] \\ g_\flat & \longleftarrow\!\dashv & g \end{array}$$

for arbitrary X, Y.

Each member f of $\boldsymbol{Set}[X, RY]$ is a 2-step function which first consumes an element $x \in X$ and then an index $i \in I$ to return an eventual value $f(x)(i)$ in Y. Each member g of $\boldsymbol{Set}[LX, Y]$ is a function which consumes a pair (x, i) where $x \in X$ and $i \in I$ to return a value $g(x, i)$ in Y. The two transpositions merely shuffle brackets around. We have

$$f^\sharp(x, i) = f(x)(i) \qquad g_\flat(x)(i) = g(x, i)$$

for $x \in X$ and $i \in I$. Normally in Mathematics we would hardly distinguish between f and $f^\sharp$, nor between g and $g_\flat$. □

5.2.4 We continue with the notation of Solution 5.2.3.

We require a pair of assignments

$$X \xrightarrow{\eta_X} (R \circ L)X \qquad (L \circ R)Y \xrightarrow{\epsilon_Y} Y$$

where

$$(R \circ L)X = (I \Rightarrow (X \times I)) \qquad (L \circ R)Y = (I \Rightarrow Y) \times I$$

for sets X and Y.

For each $x \in X$ the value $\eta_X(x)$ must be a function which consumes some $i \in I$ and returns a pair in $X \times I$. Thus

$$\eta_X(x)(i) = (x, i)$$

is the only sensible suggestion. The function ϵ_Y must consume a pair (p, i) where $p : I \to Y$ and $i \in I$ to return a member of Y. Thus

$$\epsilon_Y(p, i) = p(i)$$

is the only sensible suggestion. We show that each of these is natural.

Recall that the arrow assignments of L and R are given by

$$\begin{tikzcd}
X_2 \arrow[d, "k"'] & \longmapsto & X_2 \times I \arrow[d, "L(k)"] & (x,i) \arrow[d, mapsto] & & Y_1 \arrow[d, "l"] & \longmapsto & I \Rightarrow Y_1 \arrow[d, "R(l)"] & p \arrow[d, mapsto] \\
X_1 & & X_1 \times I & (k(x), i) & & Y_2 & & I \Rightarrow Y_2 & l \circ p
\end{tikzcd}$$

respectively.

To deal with $\eta_\bullet$ we must show that

$$\begin{tikzcd}
X_2 \arrow[r, "\eta_{X_2}"] \arrow[d, "k"'] & (R \circ L)X_2 \arrow[d, "(R \circ L)(k)"] \\
X_1 \arrow[r, "\eta_{X_1}"'] & (R \circ L)X_1
\end{tikzcd}$$

commutes for an arbitrary function k, as on the left. Thus we require

$$\eta_{X_1} \circ k = (R \circ L)(k) \circ \eta_{X_2}$$

equivalently

$$\eta_{X_1}\big(k(x)\big) = R\big(L(k)\big)\big(\eta_{X_2}(x)\big)$$

for each $x \in X_2$. Each side of this equality is a function

$$I \longrightarrow (X \times I)$$

so we evaluate both at an arbitrary $i \in I$. We have

$$\eta_{X_1}\big(k(x)\big)(i) = (k(x), i)$$

be the definition of $\eta_\bullet$. The behaviour of R gives

$$R\big(L(k)\big)\big(\eta_{X_2}(x)\big) = L(k) \circ \eta_{X_2}(x)$$

so that

$$R\big(L(k)\big)\big(\eta_{X_2}(x)\big)(i) = L(k)\big(\eta_{X_2}(x)(i)\big) = L(k)(x,i) = (k(x),i)$$

by the behaviour of L, to give the required result.

To deal with $\epsilon_\bullet$ we must show that

$$\begin{tikzcd}
(L \circ R)Y_1 \arrow[r, "\epsilon_{Y_1}"] \arrow[d, "(L \circ R)(l)"'] & Y_1 \arrow[d, "l"] \\
(L \circ R)Y_2 \arrow[r, "\epsilon_{Y_2}"'] & Y_2
\end{tikzcd}$$

commutes for an arbitrary function l, as on the right. Thus

$$\epsilon_{Y_2} \circ \big((L \circ R)(l)\big) = l \circ \epsilon_{Y_1}$$

is the required equality. A typical member of $(L \circ R)Y_1$ is a pair (p, i) where $p : I \to Y_1$ and $i \in I$. We have

$$\big(l \circ \epsilon_{Y_1}\big)(p,i) = l\big(\epsilon_{Y_1}(p,i)\big) = l\big(p(i)\big) = (l \circ p)(i)$$

for each such pair. We also have

$$\big(L \circ R\big)(l)(p,i) = L\big(R(l)\big)(p,i) = \big(R(l)(p), i\big) = (l \circ p, i)$$

for each such pair. This gives

$$\Big(\epsilon_{Y_2} \circ \big((L \circ R)(l)\big)\Big)(p,i) = \epsilon_{Y_1}\Big(\big(L \circ R\big)(l)(p,i)\Big) = \epsilon_{Y_1}(l \circ p, i) = (l \circ p)(i)$$

as required. $\square$

5.2.3 A topological example

5.2.5 As in the block, it suffices to show that for a continuous map

$$Y_1 \xrightarrow{\ \psi\ } Y_2$$

between two spaces, the induced assignment

$$\begin{array}{ccc}
(I \Rightarrow Y_1) & \xrightarrow{\ \Psi\ } & (I \Rightarrow Y_2) \\
\theta & \longmapsto & \psi \circ \theta
\end{array}$$

is continuous where each of the two functions spaces carries the compact open topology. To do that we consider an arbitrary subbasic open set $\langle K, V\rangle$ of $(I \Rightarrow Y_2)$ where $K \in \mathcal{K}I$ and $V \in \mathcal{O}Y_2$, and show

$$\Psi^{\leftarrow}\big(\langle K, V\rangle\big) = \langle K, \psi^{\leftarrow}(V)\rangle$$

which is a subbasic open set of $(I \Rightarrow Y_1)$. For each function $\theta : I \longrightarrow Y_1$ we have

$$\begin{aligned}\theta \in \Psi^{\leftarrow}\big(\langle K, V\rangle\big) &\iff \psi \circ \theta \in \langle K, V\rangle \\ &\iff (\forall i \in I)[i \in K \Rightarrow \psi(\theta(i)) \in V] \\ &\iff (\forall i \in I)[i \in K \Rightarrow \theta(i) \in \psi^{\leftarrow}(V)] \\ &\iff \theta \in \langle K, \psi^{\leftarrow}(V)\rangle\end{aligned}$$

for the required result. □

5.2.6 As suggested in the partial proof of Lemma 5.2.5, we consider a typical subbasic open set

$$\langle K, V\rangle$$

of $(I \Rightarrow Y)$, and show that

$$\psi_\flat^{\leftarrow}\big(\langle K, V\rangle\big)$$

is open in X. Here

$$K \in \mathcal{K}I \qquad V \in \mathcal{O}Y$$

are the two components of the subbasic. We consider an arbitrary

$$s \in \psi_\flat^{\leftarrow}\big(\langle K, V\rangle\big)$$

and show that

$$s \in U \subseteq \psi_\flat^{\leftarrow}\big(\langle K, V\rangle\big)$$

for some open $U \in \mathcal{O}X$.

For the considered point s we have

$$\psi_\flat(s) \in \langle K, V\rangle$$

that is

$$\psi(s, i) = \psi_\flat(s)(i) \in V$$

for each $i \in K$. For each $i \in K$ we have

$$(s, i) \in \psi^{\leftarrow}(V)$$

so that, since ψ is continuous, we have

$$(s, i) \in U_i \times W_i \subseteq \psi^{\leftarrow}(V)$$

for some $U_i \in \mathcal{O}X$ and $W_i \in \mathcal{O}I$. As i ranges through K the sets W_i produce an open covering of K. Since K is compact this refines to a finite covering

$$W = W_1 \cup \cdots \cup W_m$$

of K indexed by some $i(1), \ldots, i(m) \in K$. Let

$$U = U_1 \cap \cdots \cap U_m$$

using the same indexes. We have

$$s \in U \qquad K \subseteq W$$

with

$$U \times W \subseteq \psi^{\leftarrow}(V)$$

by construction. Also, for each $x \in X$ we have

$$\begin{aligned} x \in U &\Longrightarrow (\forall i \in K)[\psi(x,i) \in V] \\ &\Longrightarrow \psi_\flat(x)[K] \subseteq V \\ &\Longrightarrow \psi_\flat(x) \in \langle K, V\rangle \Longrightarrow \psi_\flat^{\leftarrow}\big(\langle K, V\rangle\big) \end{aligned}$$

to give

$$U \subseteq \psi_\flat^{\leftarrow}\big(\langle K, V\rangle\big)$$

for the required result. □

5.2.7 We continue with the partial proof of Lemma 5.2.6.

We start from any

$$(s, r) \in \phi^{\sharp\leftarrow}(V)$$

where $V \in \mathcal{O}Y$, and produce

$$U \times W \subseteq \phi^{\sharp\leftarrow}(V)$$

such that both

$$s \in U \in \mathcal{O}X \qquad r \in W \in \mathcal{O}I$$

hold. We already have

$$r \in W \subseteq K \subseteq \phi(s)^{\leftarrow}(V)$$

for some $K \in \mathcal{K}I$ and $W \in \mathcal{O}I$. Observe that for $i \in I$ we have

$$i \in K \Longrightarrow i \in \phi(s)^{\leftarrow}(V) \Longrightarrow \phi(s)(i) \in V$$

for each $i \in I$. This gives

$$\phi(s)[K] \subseteq V$$

so that

$$\phi(s) \in \langle K, V\rangle$$

and hence

$$s \in U \quad \text{where} \quad U = \phi^{\leftarrow}\big(\langle K, V\rangle\big)$$

with U open in X. From the construction of U and W, for each $x \in X$and $i \in I$ we have

$$(x, i) \in U \times W \Longrightarrow (x, i) \in U \times K \Longrightarrow \phi^\sharp(x, i) = \phi(x)(i) \in V$$

and hence

$$U \times W \subseteq \phi^{\sharp\leftarrow}(V)$$

for the final requirement. □

5.2.8 For spaces X, Y we require continuous maps

$$X \xrightarrow{\eta_X} \big(I \Rightarrow (X \times I)\big) \qquad \big((I \Rightarrow Y) \times I\big) \xrightarrow{\epsilon_Y} Y$$

to do a certain job. We use the idea of the set-theoretic example of Block 5.2.2. Thus we set

$$\eta_X(x)(i) = (x, i) \qquad \epsilon_Y(p, i) = p(i)$$

for each $x \in X, i \in I$, and $p : I \to Y$.

It now looks as though we have quite a bit of work to do, but this is an illusion.

Why are

$$\eta_X \qquad \epsilon_Y$$

continuous? Observe that

$$\eta_X = \psi_\flat \quad \text{where} \quad \psi = id_{X\times I} \qquad \epsilon_Y = \phi^\sharp \quad \text{where} \quad \phi = id_{I\Rightarrow Y}$$

and hence Lemmas 5.2.5 and 5.2.6 give the required continuity.

Why are

$$\eta_\bullet \qquad \epsilon_\bullet$$

natural? We require that certain squares commute. But these are squares in $\boldsymbol{Set}$, and we know they commute by the set-theoretic example. □

5.3 Adjunctions uncoupled

5.3.1 We use the notation of this section as in Table 5.1.

Letting only A vary is equivalent to taking $S = T$ with $l = id_S$. For this case ($\sharp$) and ($\flat$) become ($\sharp\uparrow$) and ($\flat\downarrow$), and these are equivalent as in Lemma 5.3.2.

Letting only S vary is equivalent to taking $A = B$ with $k = id_A$. For this case ($\sharp$) and ($\flat$) become ($\sharp\downarrow$) and ($\flat\uparrow$), and these are equivalent as in Lemma 5.3.2. □

5.3.2 We use the notation and results of Solutions 5.2.1 and 5.2.2.

We deal with the Σ-construction first.

$$\begin{array}{ccc} f & \longmapsto & f^\sharp \\ \boldsymbol{Set}[X, UA] & & \boldsymbol{Inv}[\Sigma X, A] \\ \psi_\flat & \longleftarrow\!\shortmid & \psi \end{array}$$

For each pair k (a function) and λ (a morphism), as indicated, we must show that the two squares commute.

$$\begin{tikzcd} X & \boldsymbol{Set}[X, UA] \arrow[r, shift left, "(\cdot)^\sharp"] \arrow[d, "U(\lambda)\circ - \circ k"'] & \boldsymbol{Inv}[\Sigma X, A] \arrow[l, shift left, "(\cdot)_\flat"] \arrow[d, "\lambda\circ - \circ \Sigma(k)"] & A \arrow[d, "\lambda"] \\ Y \arrow[u, "k"] & \boldsymbol{Set}[Y, UB] \arrow[r, shift left, "(\cdot)^\sharp"] & \boldsymbol{Inv}[\Sigma Y, B] \arrow[l, shift left, "(\cdot)_\flat"] & B \end{tikzcd}$$

By tracking round the various squares we require

$$\big(U(\lambda)\circ f\circ k)\big)^\sharp = \lambda\circ f^\sharp\circ\Sigma(k) \qquad \lambda\circ\psi_\flat\circ k = \big(\lambda\circ\psi\circ\Sigma(k)\big)_\flat$$

for each arrow

$$X \xrightarrow{\ f\ } UA \qquad\qquad \Sigma \xrightarrow{\ \psi\ } A$$

from the appropriate top corner. To verify these equalities we evaluate at an arbitrary element

$$(y,i)\in\Sigma Y \qquad\qquad y\in Y$$

respectively.

We have

$$\big(U(\lambda)\circ f\circ k)\big)^\sharp(y,i) = \Big(\big(U(\lambda)\circ f\circ k\big)(y)\Big)^{(i)} = \Big(\big(\lambda\circ f\circ k\big)(y)\Big)^{(i)}$$

where at the last step we remember we are dealing with three functions. Finally we have

$$\begin{aligned}
\Big(\lambda\circ f^\sharp\circ\Sigma(k)\Big)(y,i) &= \lambda\Big(f^\sharp\big(\Sigma(k)(y,i)\big)\Big)\\
&= \lambda\Big(f^\sharp\big(k(y),i\big)\Big)\\
&= \lambda\Big(\big(f\circ k\big)(y)^{(i)}\Big)\\
&= \Big(\lambda\big(\big(f\circ k\big)(y)\big)\Big)^{(i)} \ = \Big(\big(\lambda\circ f\circ k\big)(y)\Big)^{(i)}
\end{aligned}$$

to give the required left hand result. We also have

$$\begin{array}{ll}
\quad\big(\lambda\circ\psi_\flat\circ k\big)(y) & \quad\big(\lambda\circ\psi\circ\Sigma(k)\big)_\flat(y)\\
= \big(\lambda\circ\psi_\flat\big)(k(y)) & = \big(\lambda\circ\psi\circ\Sigma(k)\big)(y,0)\\
= \lambda\big(\psi(k(y),0\big) & = \big(\lambda\circ\psi\big)\big(\Sigma(k)(y,0)\big)\\
= \big(\lambda\circ\psi\big)(k(y),0) & = \big(\lambda\circ\psi\big)(k(y),0)
\end{array}$$

to give the required right hand result.

Next we deal with the Π-construction.

$$\begin{array}{ccc}
\phi & \longmapsto & \phi^\sharp\\
\boldsymbol{Inv}[A,\Pi X] & & \boldsymbol{Set}[UA,X]\\
g_\flat & \longleftarrow\!\shortmid & g
\end{array}$$

For each pair κ (a morphism) and l (a function), as indicated, we must show that the two squares commute.

$$\begin{array}{ccccccc}
A & & \boldsymbol{Inv}[A,\Pi X] & \overset{(\cdot)^\sharp}{\underset{(\cdot)_\flat}{\rightleftarrows}} & \boldsymbol{Set}[UA,X] & & X\\
\uparrow\kappa & & \downarrow \Pi(l)\circ - \circ\kappa & & \downarrow l\circ - \circ U(\kappa) & & \downarrow l\\
B & & \boldsymbol{Inv}[B,\Pi Y] & \overset{(\cdot)^\sharp}{\underset{(\cdot)_\flat}{\rightleftarrows}} & \boldsymbol{Set}[UB,Y] & & Y
\end{array}$$

By tracking round the various squares we require

$$\big(\Pi(l)\circ\phi\circ\kappa\big)^{\sharp} = l\circ\phi^{\sharp}\circ U(\kappa) \qquad \Pi(l)\circ g_{\flat}\circ\kappa = \big(l\circ g\circ U(\kappa)\big)_{\flat}$$

for each arrow

$$A \xrightarrow{\ \phi\ } \Pi X \qquad UA \xrightarrow{\ g\ } X$$

from the appropriate top corner. To verify these we evaluate at an arbitrary element

$$b\in UB \qquad b\in B$$

respectively. We have

$$\begin{aligned} &\big(\Pi(l)\circ\phi\circ\kappa\big)^{\sharp}(b) \\ &= \Big(\big(\Pi(l)\circ\phi\circ\kappa\big)(b)\Big)_0 \\ &= \Big(\Pi(l)\big((\phi\circ\kappa)(b)\big)\Big)_0 \\ &= l\Big(\big((\phi\circ\kappa)(b)\big)_0\Big) \end{aligned} \qquad \begin{aligned} &\big(l\circ\phi^{\sharp}\circ U(\kappa)\big)(b) \\ &= l\Big(\phi^{\sharp}\big(\kappa(b)\big)\Big) \\ &= l\Big(\phi\big(\kappa(b)\big)_0\Big) \\ &= l\Big(\big((\phi\circ\kappa)(b)\big)_0\Big) \end{aligned}$$

to give the required left hand result. We also have

$$\begin{aligned} \big(\Pi(l)\circ g_{\flat}\circ\kappa\big)(b) &= \Pi(l)\Big(g_{\flat}\big(\kappa(b)\big)\Big) \\ &= \Pi(l)\Big(g\big(\kappa(b)\big), g\big(\kappa(b)\big)^{\bullet}\Big) \\ &= \Big(l\big(g\big(\kappa(b)\big)\big), l\big(g\big(\kappa(b)\big)^{\bullet}\big)\Big) \\ &= \Big(\big(l\circ g\big)\big(\kappa(b)\big), \big(l\circ g\big)\big(\kappa(b)^{\bullet}\big)\Big) \end{aligned}$$

and

$$\begin{aligned} &\big(l\circ g\circ U(\kappa)\big)_{\flat}(b) \\ &= \Big(\big(l\circ g\circ U(\kappa)\big)(b), \big(l\circ g\circ U(\kappa)\big)(b^{\bullet})\Big) \\ &= \Big(\big(l\circ g\big)\big(\kappa(b)\big), \big(l\circ g\big)\big(\kappa(b^{\bullet})\big)\Big) \end{aligned}$$

and

$$\kappa(b)^{\bullet} = \kappa(b^{\bullet})$$

since κ is a morphism, to give the required right hand result. $\square$

5.3.3 Let us use the notation

$$L = (-\times I) \qquad R = (I \Rightarrow -)$$

of Solutions 5.2.3 and 5.2.4.

To show that $(\cdot)^\sharp$ we must check that the square

$$\begin{tikzcd}
X_1 & {[X_1, RY_1]} \arrow[r, "(\cdot)^\sharp"] \arrow[d, "R(l)\circ - \circ k"'] & {[LX_1, Y_1]} \arrow[d, "l \circ - \circ L(k)"] & Y_1 \arrow[d, "l"] \\
X_2 \arrow[u, "k"] & {[X_2, RY_2]} \arrow[r, "(\cdot)^\sharp"'] & {[LX_2, Y_2]} & Y_2
\end{tikzcd}$$

commutes for each pair of functions k and l. Here we need not indicate $\boldsymbol{Set}$ since it is the only category involved. In terms of equations we must show that

$$\big(R(l) \circ f \circ k\big)^\sharp = \big(l \circ f^\sharp \circ L(k)\big)$$

for each function

$$f : X_1 \longrightarrow (I \longrightarrow Y_1)$$

in the top left hand corner of the diagram. To do that the abbreviation

$$g = f \circ k$$

will be useful.

Each side of this equation is a function that consumes a pair

$$(x, i) \in X_2 \times I = LX_2$$

to return a value in Y_2. We have

$$\begin{aligned}
\big(R(l) \circ f \circ k\big)^\sharp(x, i) &= \big(R(l) \circ g\big)(x)(i) \\
&= R(l)\big(g(x)\big)(i) \\
&= \big(l \circ g(x)\big)(i) \qquad = l\big(g(x)(i)\big)
\end{aligned}$$

which evaluates the left hand side. We also have

$$\begin{aligned}
\big(l \circ f^\sharp \circ L(k)\big)(x, i) &= \big(l \circ f^\sharp\big)\big(L(k)(x, i)\big) \\
&= \big(l \circ f^\sharp\big)\big(k(x), i\big) \\
&= l\big(f^\sharp\big(k(x), i\big)\big) \\
&= l\big(f(k(x))(i)\big) \qquad = l\big(g(x)(i)\big)
\end{aligned}$$

which evaluates the right hand side, and verifies the equality.

The diagram for the naturality of $(\cdot)_\flat$ is similar but with two arrows

$$\xleftarrow{\quad (\cdot)_\flat \quad}$$

pointing the other way. We have to show that

$$\big(R(l) \circ g_\flat \circ k\big) = \big(l \circ g \circ L(k)\big)_\flat$$

for each function

$$g : X_1 \times I \longrightarrow Y_1$$

in the top right hand corner. To prove the equality the abbreviation

$$f = l \circ g$$

will be useful.

Each side of this equation is a 2-step function which first consumes $x \in X_2$ and then $i \in I$ to return a value in Y_2. We have

$$\begin{aligned}
\big(R(l) \circ g_\flat \circ k\big)(x)(i) &= R(l)\big(g_\flat(k(x))\big)(i) \\
&= \big(l \circ g_\flat(k(x))\big)(i) \\
&= l\big(g_\flat(k(x)(i)\big) \\
&= l\big(g(k(x), i)\big) \qquad = f\big(k(x), i\big)
\end{aligned}$$

which evaluates the left hand side. We also have

$$\begin{aligned}
\big(l \circ g \circ L(k)\big)_\flat(x)(i) &= \big(f \circ L(k)\big)_\flat(x)(i) \\
&= \big(f \circ L(k)\big)(x, i) \\
&= f\big(L(k)(x, i)\big) \qquad = f\big(k(x), i\big)
\end{aligned}$$

which evaluates the right hand side, and verifies the equality. □

5.3.4 At the function level the two assignment $(\cdot)^\sharp$ and $(\cdot)_\flat$ are just the same as those used in the $\boldsymbol{Set}$-theoretic example of Block 5.2.2. The naturality of these topological versions requires that certain squares in $\boldsymbol{Set}$ must commute. These are just the same as the $\boldsymbol{Set}$-theoretic squares, and are dealt with in Solution 5.3.3. □

5.4 The unit and the co-unit

5.4.1 We must show that the square commutes

$$\begin{tikzcd}
\epsilon_S = \big(id_{GS}\big)^\sharp & (F \circ G)S \arrow[r, "\epsilon_S"] \arrow[d, "(F \circ G)(l)"'] & S \arrow[d, "l"] \\
\epsilon_T = \big(id_{GT}\big)^\sharp & (F \circ G)T \arrow[r, "\epsilon_T"'] & T
\end{tikzcd}$$

for an arbitrary arrow l as on the right. In equational terms we must show that

$$\epsilon_T \circ (F \circ G)(l) = l \circ \epsilon_S$$

holds. To do that we use ($\sharp$) twice. We have

$$\begin{aligned}
\epsilon_T \circ (F \circ G)(l) &= \big(id_{GT}\big)^\sharp \circ F(G(l)) \\
&= id_T \circ \big(id_{GT}\big)^\sharp \circ F(G(l)) \\
&= \big(G(id_T) \circ id_{GT} \circ G(l)\big)^\sharp \;\; = G(l)^\sharp
\end{aligned}$$

where the penultimate step is the first use of ($\sharp$). We also have

$$l \circ \epsilon_S = l \circ \big(id_{GS}\big)^\sharp = l \circ \big(id_{GS}\big)^\sharp \circ F(id_{GS}) = \big(G(l) \circ id_{GS} \circ id_{GS}\big)^\sharp = G(l)^\sharp$$

where the penultimate step is the second use of ($\sharp$). □

5.4.2 For each arrow

$$FA \xrightarrow{\quad g \quad} S$$

we show that

$$g_\flat = G(g)\eta_A$$

and to do that we use ($\flat\downarrow$). Thus

$$G(g) \circ \eta)A = G(g) \circ \big(id_{FA}\big)_\flat = \big(g \circ id_{FA}\big)_\flat = g_\flat$$

where the penultimate step uses ($\flat\downarrow$). □

5.4.3 We must first show that for an arbitrary arrow

$$FA \xrightarrow{\quad g \quad} S$$

the second transpose $g_\flat{}^\sharp$ is just g. To do that we use an instance of the naturality of $\epsilon_\bullet$ as given in Solution 5.4.1. We use the case $l = g$. Thus

$$\begin{aligned} g_\flat{}^\sharp &= \big(G(g) \circ \eta_A\big)^\sharp \\ &= \epsilon_S \circ F\big(G(g) \circ \eta_A\big) \\ &= \epsilon_S \circ \big(F \circ G\big)(g) \circ F\big(\eta_A\big) = g \circ \epsilon_{FA} \circ F\big(\eta_A\big) = g \end{aligned}$$

where the penultimate step uses the naturality of ϵ and the ultimate step uses one of the given conditions on η and ϵ.

It remains to verify ($\flat$). Using the notation of Table 5.1, a use of the definition of $(\cdot)_\flat$ gives

$$\begin{aligned} \big(l \circ g \circ F(k)\big)_\flat &= G\big(l \circ g \circ F(k)\big) \circ \eta_B \\ &= G\big(l \circ g\big) \circ \big(G \circ F\big)(k)\big) \circ \eta_B = G\big(l \circ g\big) \circ \eta_A \circ k \end{aligned}$$

where this last step use the naturality of η. Continuing we have

$$\big(l \circ g \circ F(k)\big)_\flat = G\big(l \circ g\big) \circ \eta_A \circ k = G(l) \circ G(g) \circ \eta_A \circ k = G(l) \circ g_\flat \circ k$$

using the definition of $g_\flat$. □

5.4.4 We continue with the notation of Solutions 5.2.1, 5.2.2, and 5.3.2.

We deal with the Σ-case first. For each element of a set $x \in X$, each element of an algebra $a \in A$, and each tag i we let

$$\eta_X(x) = (x, 0) \qquad \delta_A(a, i) = a^{(i)}$$

to obtain two functions of the required type. We need to check that δ_A is a morphism, that is

$$\delta_A\big((a, i)^\bullet\big) = \delta_A(a, i)^\bullet$$

for each $a \in A$ and tag i. But

$$\delta_A\big((a,i)^\bullet\big) = \delta_A(a, 1-i) = a^{(1-i)} = a^{(i)\bullet} = \delta_A(a,i)^\bullet$$

to give the required equality.

To show that η, δ are natural we must check that a pair of squares commute. These are induced by a function f in $\boldsymbol{Set}$ and a morphism ϕ in $\boldsymbol{Inv}$, as indicated.

$$\begin{tikzcd}
x \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & (x,0) \arrow[ddd, mapsto] \\
& X \arrow[r, "\eta_X"] \arrow[d, "f"'] \arrow[dr, phantom, "\eta"] & (U\circ\Sigma)X \arrow[d, "(U\circ\Sigma)(f)"] & \\
& Y \arrow[r, "\eta_Y"] & (U\circ\Sigma)Y & \\
f(x) \arrow[rrr, mapsto] & & & (f(x),0)
\end{tikzcd}$$

$$\begin{tikzcd}
a,i) \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & a^{(i)} \arrow[ddd, mapsto] \\
& (\Sigma\circ U)A \arrow[r, "\delta_A"] \arrow[d, "(\Sigma\circ U)(\phi)"'] \arrow[dr, phantom, "\delta"] & A \arrow[d, "\phi"] & \\
& (\Sigma\circ U)B \arrow[r, "\delta_B"] & B & \\
(\phi(a),i) \arrow[rrr, mapsto] & & & \phi(a)^{(i)} = \phi(a^{(i)})
\end{tikzcd}$$

In both cases we take an arbitrary element of the top left hand corner and track it both ways to the bottom right hand corner. The two resulting elements must be the same. That condition for the η-square is trivial. For the δ-square we need to recall that ϕ is a morphism.

Next we deal with the Π-case. For each elements of a set $x, y \in X$ and each element of an algebra $a \in A$ we let

$$\epsilon_X(x,y) = x \qquad \zeta_A(a) = (a, a^\bullet)$$

to obtain two functions of the required type. We need to check that ζ_A is a morphism. But

$$\zeta_A(a^\bullet) = (a^\bullet, a^{\bullet\bullet}) = (a^\bullet, a) = (a, a^\bullet)^\bullet = \zeta_A(a)^\bullet$$

as required.

To show that ϵ, ζ are natural we must check that a pair of squares commute. These are induced by a function f in $\boldsymbol{Set}$ and a morphism ϕ in $\boldsymbol{Inv}$, as indicated.

$$
\begin{tikzcd}
(x,y) \arrow[rrr, mapsto] \arrow[dddd, mapsto] & & & x \arrow[dddd, mapsto] \\
& (U\circ\Pi)X \arrow[r, "\epsilon_X"] \arrow[dd, "(U\circ\Pi)(f)"'] \arrow[ddr, phantom, "\epsilon"] & X \arrow[dd, "f"] & \\
& & & \\
& (U\circ\Sigma)Y \arrow[r, "\epsilon_Y"] & Y & \\
(f(x),f(y)) \arrow[rrr, mapsto] & & & f(x)
\end{tikzcd}
$$

$$
\begin{tikzcd}
a \arrow[rrr, mapsto] \arrow[dddd, mapsto] & & & (a,a^\bullet) \arrow[ddd, mapsto] \\
& A \arrow[r, "\zeta_A"] \arrow[dd, "\phi"'] \arrow[ddr, phantom, "\zeta"] & (\Pi\circ U)A \arrow[dd, "(\Pi\circ U)(\phi)"] & \\
& & & \\
& B \arrow[r, "\zeta_B"] & (\Pi\circ U)B & (\phi(a),\phi(a^\bullet)) \\
\phi(a) \arrow[rr, mapsto] & & (\phi(a),\phi(a)^\bullet) &
\end{tikzcd}
$$

In both cases we take an arbitrary element of the top left hand corner and track it both ways to the bottom right hand corner. The two resulting elements must be the same. That condition for the ϵ-square is trivial. For the ζ-square we need to recall that ϕ is a morphism. □

5.4.5 We deal first with the identities of Lemma 5.4.3.

We start with a set X, an algebra A, two functions f, g and two morphisms ψ, ψ, as indicated,

$$
X \xrightarrow{f} UA \qquad \Sigma X \xrightarrow{\psi} A
$$

$$
A \xrightarrow{\phi} \Pi A \qquad UA \xrightarrow{g} X
$$

and must show that

$$
\begin{aligned}
f^\sharp &= \delta_A \circ \Sigma(f) & \qquad \psi_\flat &= U(\psi) \circ \eta_X \\
\phi^\sharp &= \epsilon_X \circ U(\phi) & \qquad g_\flat &= \Pi(g) \circ \zeta_A
\end{aligned}
$$

hold. In other words, we must show that the following triangles commute.

$$
\begin{tikzcd}
\Sigma X \arrow[r, "\Sigma(f)"] \arrow[dr, "f^\sharp"'] & (\Sigma \circ U)A \arrow[d, "\delta_A"] \\
& A
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "\psi_\flat"'] & (U \circ \Sigma)X \arrow[d, "U(\psi)"] \\
& UA
\end{tikzcd}
$$

$$
\begin{tikzcd}
UA \arrow[r, "U(\phi)"] \arrow[dr, "\phi^\sharp"'] & (U \circ \pi)X \arrow[d, "\epsilon_X"] \\
& X
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\zeta_A"] \arrow[dr, "g_\flat"'] & (\pi \circ U)A \arrow[d, "\Pi(g)"] \\
& \Pi X
\end{tikzcd}
$$

To do that we take an arbitrary element

$$
\begin{array}{cc}
(x,i) \in \Sigma X & x \in X \\
a \in UA & a \in A
\end{array}
$$

for the top left hand corner, track it both ways to produce

$$
\begin{array}{cc}
f(x)^{(i)} \in A & \psi(x,i) \in UA \\
\phi(a)_0 \in X & (g(a), g(a^\bullet)) \in \Pi X
\end{array}
$$

to give the required result.

Next we deal with the identities of Corollary 5.4.4. For each set X and algebra A we must show that each of the composites

$$
\begin{array}{cc}
\delta_{\sigma X} \circ \Sigma(\eta_X) & U(\delta_A) \circ \eta_{UA} \\
\epsilon_{UA} \circ U(\zeta_A) & \Pi(\epsilon_X) \circ \zeta_{\Pi X}
\end{array}
$$

is the identity arrow on the relevant carrier. To do that we calculate.

$$
\begin{array}{ccccc}
\Sigma X & \xrightarrow{\Sigma(\eta_X)} & (\Sigma \circ U \circ \Sigma)X & \xrightarrow{\delta_{\Sigma X}} & \Sigma X \\
(x,i) & \longmapsto & (\eta_X(x), i) & \longmapsto & \eta_X(x)^{(i)}
\end{array}
$$

$$
\begin{array}{ccccc}
UA & \xrightarrow{\eta_{UA}} & (U \circ \Sigma \circ U)A & \xrightarrow{U(\delta)} & UA \\
a & \longmapsto & (a,0) & \longmapsto & a^{(0)}
\end{array}
$$

$$
\begin{array}{ccccc}
UA & \xrightarrow{U(\zeta_A)} & (U \circ \Pi \circ U) & \xrightarrow{\epsilon_{UA}} & UA \\
a & \longmapsto & (a, a^\bullet) & \longmapsto & a
\end{array}
$$

$$
\begin{array}{ccccc}
\Pi X & \xrightarrow{\zeta_{\Pi X}} & (\Pi \circ U \circ \Pi)X & \xrightarrow{\Pi(\epsilon_X)} & \Pi X \\
(x,y) & \longmapsto & \big((x,y),(y,x)\big) & \longmapsto & (?,?)
\end{array}
$$

At the top we have $\eta_X(x) = (x,0)$ so that

$$
\eta_X(x)^{(i)} = (x,0)^{(i)} = (x,i)
$$

by considering the two cases for the tag. Next we have $a^{(0)} = a$ by definition. The third composite is trivial. Finally we have

$$(?,?) = \big(\epsilon_X(x,y), \epsilon_X(y,x)\big) = (x,y)$$

to complete the calculations. □

5.4.6 We use the notation of Solution 5.2.3. Thus we have

$$LX = X \times I \qquad RY = (I \Rightarrow Y)$$

for sets or spaces X, Y.

For the first part we must show that for functions

$$X \xrightarrow{\;f\;} RY \qquad\qquad LX \xrightarrow{\;g\;} Y$$

both

$$f^\sharp = \epsilon_Y \circ L(f) \qquad g_\flat = R(g) \circ \eta_X$$

hold. Thus we must evaluate the composites

$$LX \xrightarrow{L(f)} (L \circ R)Y \xrightarrow{\epsilon_Y} Y \qquad X \xrightarrow{\eta_X} (R \circ L)X \xrightarrow{R(g)} RY$$

and remember that $(\cdot)^\sharp$ and $(\cdot)_\flat$ merely shuffle brackets about. For $(x,i) \in LX$ we have

$$\big(\epsilon_Y \circ L(f)\big)(x,i) = \epsilon_Y\big(L(f)(x,i)\big) = \epsilon_Y\big(f(x),i\big) = f(x)(i)$$

as required. For $x \in X$ we have

$$\big(R(g) \circ \eta_X\big)(x) = R(g)\big(\eta_X(x)\big) = g \circ \eta_X(x)$$

and then for $i \in I$ we have

$$\big(R(g) \circ \eta_X\big)(x)(i) = \big(g \circ \eta_X(x)\big)(i) = g\big(\eta_X(x)(i)\big) = g(x,i)$$

as required.

For the second part we must show that

$$\epsilon_{LX} \circ L(\eta_X) = id_{X \times I} \qquad R(\epsilon_Y) \circ \eta_{RY} = id_{I \Rightarrow Y}$$

for arbitrary sets X and Y. For $(x,i) \in LX$ we have

$$\begin{aligned} \big(\epsilon_{LX} \circ L(\eta_X)\big)(x,i) &= \epsilon_{LX}\big(L(\eta_X)(x,i)\big) \\ &= \epsilon_{LX}\big(\eta_X(x),i\big) = \eta_X(x)(i) = (x,i) \end{aligned}$$

as required. For each function $p : I \to Y$ we have

$$\big(R(\epsilon_Y) \circ \eta_{RY}\big)(p) = R(\epsilon_Y)\big(\eta_{RY}(p)\big) = \epsilon_Y \circ \big(\eta_{RY}(p)\big)$$

and this composite is a function $I \to Y$. For each $i \in Y$ we have

$$\big(R(\epsilon_Y) \circ \eta_{RY}\big)(p)(i) = \epsilon_Y\big(\eta_{RY}(p)(i)\big) = \epsilon_Y(p,i) = p(i)$$

to give the required result. □

5.5 Free and cofree constructions

5.5.1 Consider any arrow

$$FA \xrightarrow{g} S$$

of $\boldsymbol{Trg}$. We first check that

$$\begin{tikzcd} FA \arrow[rr, "g"] \arrow[dr, "F(g_\flat)"'] & & S \\ & (F\circ G)S \arrow[ur, "\epsilon_S"'] & \end{tikzcd}$$

does commute (and then consider the required uniqueness). We use the selection of arrows

$$A \xrightarrow{g_\flat} GS \qquad S \xrightarrow{id_S} S$$
$$GS \xrightarrow[id_{GS}]{} GS$$

and then apply ($\sharp$) of (Nat). Thus

$$\epsilon_S \circ F(g_\flat) = id_S \circ (id_{GS})^\sharp \circ F(g_\flat) = (G(id_S) \circ id_{GS} \circ g_\flat)^\sharp = (g_\flat)^\sharp = g$$

as required.

For the uniqueness we consider any arrow

$$A \xrightarrow{f} GS \qquad \text{for which} \qquad g = \epsilon_S \circ F(f)$$

and show that, in fact, $f = g_\flat$. We use the selection of arrows

$$A \xrightarrow{f} GS \qquad S \xrightarrow{\boldsymbol{id}_S} S$$
$$GS \xrightarrow[\boldsymbol{id}_{GS}]{} GS$$

and then apply ($\sharp$) of (Nat). Thus

$$\begin{aligned} g &= \epsilon_S \circ F(f) \\ &= (\boldsymbol{id}_{GS})^\sharp \circ F(f) \\ &= \boldsymbol{id}_S \circ (\boldsymbol{id}_{GS})^\sharp \circ F(f) \\ &= (G(\boldsymbol{id}_S) \circ \boldsymbol{id}_{GS}\sharp \circ f)^\sharp = f^\sharp \end{aligned}$$

and hence

$$f = (f^\sharp)_\flat = g_\flat$$

by a use of (Bij). $\square$

5.5.2 Let us first state the result we must obtain,

Let

$$\boldsymbol{Src} \xrightarrow{F} \boldsymbol{Trg}$$

be a functor, and suppose

$$G \quad \epsilon \quad (\cdot)_\flat$$

is the data that provides a F-cofree solution. Then the object assignment G fills out to a functor for which

$$F \dashv G$$

with $(\cdot)_\flat$ *as the transposition assignment and* ϵ *as the counit.*

On several occasions we invoke the *unique* factorization provided by the G-cofree property. Our first job is to produce an arrow assignment to create the functor G. Consider any arrow

$$S \xrightarrow{l} T$$

of $\boldsymbol{Trg}$. Let g be the composite

$$(F \circ G)S \xrightarrow{\epsilon_S} S \xrightarrow{l} T$$

and consider the commuting square

$$\begin{tikzcd}
(F \circ G)S \arrow[r, "F(g_\flat)"] \arrow[d, "\epsilon_S"'] & (F \circ G)T \arrow[d, "\epsilon_T"] \\
S \arrow[r, "l"'] & T
\end{tikzcd}$$

provided by the F-cofree solution. We set

$$G(l) = g_\flat = (l \circ \epsilon_S)_\flat$$

for each $\boldsymbol{Trg}$-arrow l, as above. In other words, for each such arrow l we take $G(l)$ to be the unique $\boldsymbol{Src}$-arrow such that

$$\begin{tikzcd}
(F \circ G)S \arrow[r, "(F \circ G)(l)"] \arrow[d, "\epsilon_S"'] & (F \circ G)T \arrow[d, "\epsilon_T"] \\
S \arrow[r, "l"'] & T
\end{tikzcd}$$

commutes. This uniqueness ensures that we have produced a functor G.

Consider arrows

$$R \xrightarrow{k} S \xrightarrow{l} T$$

of ***Trg***. We have arranged that the following diagrams commute.

$$\begin{tikzcd}
(F\circ G)R \arrow[r, "(F\circ G)(k)"] \arrow[d, "\epsilon_R"'] & (F\circ G)S \arrow[r, "(F\circ G)(l)"] \arrow[d, "\epsilon_S"] & (F\circ G)T \arrow[d, "\epsilon_T"] \\
R \arrow[r, "k"'] & S \arrow[r, "l"'] & T
\end{tikzcd}$$

$$\begin{tikzcd}
(F\circ G)R \arrow[r, "(F\circ G)(l\circ k)"] \arrow[d, "\epsilon_R"'] & (F\circ G)T \arrow[d, "\epsilon_T"] \\
R \arrow[r, "l\circ k"'] & T
\end{tikzcd}$$

We are given that F is a functor, so that top composite is

$$F(G(l)\circ G(k))$$

and hence

$$G(l)\circ G(k) = G(l\circ k)$$

by the given uniqueness. A similar argument shows that G preserves identity arrows, and hence we do have a functor. Furthermore, the commuting square we have produced ensures that ϵ is natural.

We now begin to show that $F \dashv G$ using the given assignment $(\cdot)_\flat$.

For the time being fix $A \in$ ***Src*** and $S \in$ ***Trg***, and consider the given assignment

$$\begin{array}{ccc} \textbf{\textit{Src}}[A, GS] & \longrightarrow & \textbf{\textit{Trg}}[FA, S] \\ g_\flat & \longleftarrow\!\shortmid & g \end{array}$$

between the two arrow sets.We show that this is a bijection.

By definition of F-cofree, for each arrow

$$FA \xrightarrow{\ g\ } S \qquad \text{the associated arrow} \qquad A \xrightarrow{\ g_\flat\ } GS$$

is the unique arrow such that

$$\begin{tikzcd}
FA \arrow[rr, "g"] \arrow[dr, "F(g_\flat)"'] & (\bigtriangledown) & S \\
& (F\circ G)S \arrow[ur, "\epsilon_S"'] &
\end{tikzcd}$$

commutes. Suppose

$$g^1_\flat = g^2_\flat$$

for two arrows g^1, g^2 from ***Trg***$[FA, S]$. Then

$$g^1 = \epsilon_S \circ F(g^1_\flat) = \epsilon_S \circ F(g^2_\flat) = g^2$$

to show that $(\cdot)_\flat$ is injective. Consider any arrow f from $\boldsymbol{Src}[A, GS]$. With

$$g = \epsilon_S \circ F(f)$$

we see that

$$\begin{tikzcd}
FA \arrow[rr, "g"] \arrow[dr, "F(f)"'] & & S \\
& (F \circ G)S \arrow[ur, "\epsilon_S"'] &
\end{tikzcd}$$

commutes, and hence $g_\flat = f$ by the uniqueness in $(\bigtriangledown)$. This shows that $(\cdot)_\flat$ is surjective, and hence we do have a bijection.

To show that $(\cdot)_\flat$ is natural consider any square

$$\begin{tikzcd}
A & \boldsymbol{Src}[A, GS] \arrow[d, "G(l) \circ - \circ k"] & \boldsymbol{Trg}[FA, S] \arrow[l, "(\cdot)_\flat"'] \arrow[d, "l \circ - \circ F(k)"] & S \arrow[d, "l"] \\
B \arrow[u, "k"] & \boldsymbol{Src}[B, GT] & \boldsymbol{Trg}[FB, T] \arrow[l, "(\cdot)_\flat"] & T
\end{tikzcd}$$

induced by a pair of arrows k and l, as indicated. We must show that this square commutes, that is

$$G(l) \circ g_\flat \circ k = \big(l \circ g \circ F(k)\big)_\flat$$

where g is an arbitrary arrow from the top right hand corner. Let

$$f = G(l) \circ g_\flat \circ k \qquad h = l \circ g \circ F(k)$$

so that

$$f = h_\flat$$

is required. To verify this we show that f satisfies the unique property of $h_\flat$, namely that

$$\begin{tikzcd}
FB \arrow[rr, "h"] \arrow[dr, "F(f)"'] & & T \\
& (F \circ G)T \arrow[ur, "\epsilon_T"'] &
\end{tikzcd}$$

commutes. To verify this we use the commuting properties of two earlier diagrams. Thus

$$\begin{aligned}
\epsilon_T \circ F(f) &= \epsilon_T \circ (F \circ G)(l) \circ F(g_\flat) \circ F(k) \\
&= l \circ \epsilon_T \circ F(g_\flat) \circ F(k) = l \circ g \circ F(k) = h
\end{aligned}$$

as required.

This shows that we do have an adjunction $F \dashv G$ with $(\cdot)_\flat$ as one of the transpositions. It remains to show that the given ϵ is the counit of this adjunction. We require

$$\epsilon_S = \big(\boldsymbol{id}_{GS}\big)^\sharp$$

or, equivalently,

$$(\epsilon_S)_\flat = \boldsymbol{id}_{GS}$$

since, by definition, $(\cdot)^\sharp$ is the inverse of $(\cdot)_\flat$. Since

$$\begin{tikzcd}
(F \circ G)S \arrow[rr, "\epsilon_S"] \arrow[dr, "F(\boldsymbol{id}_{GS})"'] & & S \\
& (F \circ G)S \arrow[ur, "\epsilon_S"'] &
\end{tikzcd}$$

commutes, the required equality follows by the given property of $(\cdot)_\flat$. $\square$

5.5.3 Recall that we have

$$\Sigma X = \{(x, i) \mid x \in X, i = 0, 1\} \qquad \eta_X(x) = (x, 0)$$

for each set X and $x \in X$. Consider any function

$$X \xrightarrow{f} A$$

from X to an algebra A. We require a morphism

$$\Sigma X \xrightarrow{f^\sharp} A \qquad \text{such that} \qquad f^\sharp \circ \eta_X = f$$

and we must show there is only one such morphism.

Consider any $x \in X$ with the two corresponding members

$$(x, 0) = \eta_X(x) \qquad (x, 1) = \eta_X(x)^\bullet$$

of ΣX. If there is such a morphism $f^\sharp$ then

$$f^\sharp(x, 0) = (f^\sharp \circ \eta_X)(x) = f(x)$$

and

$$f^\sharp(x, 1) = f^\sharp\big((x, 0)^\bullet\big) = \big(f^\sharp(x, 0)\big)^\bullet = f(x)^\bullet$$

where the third step uses the morphism property. This shows that there is at most one such morphism $f^\sharp$. Exercise 5.2.2 shows that this $f^\sharp$ is a morphism, and we have checked that the triangle does commute. $\square$

5.5.4 Recall that we have

$$\Pi X = \{(x, y) \mid x, y \in X\} \qquad \epsilon_X(x, y) = x$$

for each set X and $x, y \in X$. Consider any function

$$A \xrightarrow{g} X$$

from an algebra A to X. We require a morphism

$$A \xrightarrow{g_\flat} \Pi X \qquad \text{such that} \qquad \epsilon_X \circ g_\flat = g$$

and we must show there is only one such morphism.

Consider any $a \in A$. We have

$$g_\flat(a) = (x, y)$$

for some $x, y \in X$, and then

$$x = \epsilon_X(x, y) = (\epsilon \circ g_\flat)(a) = g(a)$$

to determine x. But now

$$(y, x) = (x, y)^\bullet = g_\flat(a)^\bullet = g_\flat(a^\bullet) = \big(g(a^\bullet), z\big)$$

for some $z \in X$. The third step uses the morphism property of $g_\flat$, and the last step uses the previous observation. This gives

$$x = g(a) \qquad y = g(a^\bullet)$$

that is

$$g_\flat(a) = \big(g(a), g(a^\bullet)\big)$$

for each $a \in A$. This shows there is at most one such morphism $g_\flat$. Exercise 5.2.2 shows that this $g_\flat$ is a morphism, and we have checked that the triangle does commute. □

5.5.5 We first deal with the $\boldsymbol{Set}$ example of Block 5.2.2. To do that we gather together all the bits of gadgetry that we need.

We have a pair of functors

$$\begin{tikzcd} \boldsymbol{Set} \arrow[r, shift left, "F = (- \times I)"] & \boldsymbol{Set} \arrow[l, shift left, "G = (I \Rightarrow -)"] \end{tikzcd}$$

although for the free case we need only the object assignment of F, and for the cofree case we need only the object assignment of G.

For the free case the arrow assignment of G is given by composition

$$\begin{array}{ccc} Z \xrightarrow{\ l\ } Y & \longmapsto & GZ \longrightarrow GY \\ & & p \longmapsto l \circ p \end{array}$$

for functions as indicated. We also have functions

$$X \xrightarrow{\eta_X} (G \circ F)X = (I \Rightarrow (X \times I))$$

where

$$\eta_X(x)(i) = (x, i)$$

for each $x \in X, i \in I$. We do not need the naturality of this. For an arbitrary function

$$X \xrightarrow{\ f\ } (I \Rightarrow Y)$$

we must show there is a unique function

$$X \times I \xrightarrow{\ f^\sharp\ } Y$$

such that

$$G(f^\sharp) \circ \eta_X = f$$

holds. If there is such a function $f^\sharp$ then for each $x \in X, i \in I$ we have

$$\begin{aligned} f^\sharp(x,i) &= f^\sharp\big(\eta_X(x)(i)\big) \\ &= \big(f^\sharp \circ \eta_X(x)\big)(i) \\ &= \Big(G(f^\sharp)\big(\eta_X(x)\big)\Big)(i) \\ &= \Big(\big(G(f^\sharp) \circ \eta_X\big)(x)\Big)(i) = f(x)(i) \end{aligned}$$

to show there is at most one such function $f^\sharp$. Almost the same calculation shows that this function does make the triangle commute.

For the cofree case the arrow assignment of F is given by

$$\begin{array}{ccc} Z \xrightarrow{\ k\ } Y & \longmapsto & FZ \longrightarrow FY \\ & & (z,i) \longrightarrow (k(z),i) \end{array}$$

for functions k, as indicated, and $z \in Z, i \in I$. We also have functions

$$(F \circ G)Y = (I \Rightarrow Y) \times I \xrightarrow{\ \epsilon_Y\ } Y$$

given by evaluation, that is

$$\epsilon_Y(p,i) = p(i)$$

for $p \in (I \Rightarrow Y), i \in I$. We do not need the naturality of this. For an arbitrary function

$$X \times I \xrightarrow{\ g\ } Y$$

we must show there is a unique function

$$X \xrightarrow{\ g_\flat\ } (I \Rightarrow Y)$$

$$\epsilon_Y \circ F(g_\flat) = g$$

holds. If there is such a function $g_\flat$ then for each $x \in X, i \in I$ we have

$$\begin{aligned} g_\flat(x)(i) &= \epsilon_Y\big(g_\flat(x), i\big) \\ &= \epsilon_Y\big(F\big(g_\flat\big)(x,i)\big) \\ &= \big(\epsilon_Y \circ F(g_\flat)\big)(x,i) = g(x,i) \end{aligned}$$

to show there is at most one such function g_flat. Almost the same calculation shows that this function does make the triangle commute.

For the ***Top*** case of Block 5.2.3 we use the same functions, but must show that certain of them are continuous. This is straight forward. □

5.6 Contravariant adjunctions

5.6.1 (a) This functor occurred in Block 3.3.2 and the exercises there.

(b) For finite subsets a, b of A we have

$$\langle \mathsf{a} \rangle \cap \langle \mathsf{b} \rangle = \langle \mathsf{a} \cup \mathsf{b} \rangle$$

and hence these subsets do form a base for a topology.

(c) For an arbitrary monotone function

$$A \xrightarrow{\ f\ } B$$

between poset, the inverse image map

$$\Upsilon B \xrightarrow{\ \phi = f^{\leftarrow}\ } \Upsilon A$$

does send upper sections to upper section, and hence is a function of the indicated type. To show that ϕ is continuous we show that the inverse image function $\phi^{\leftarrow}$ sends basic open sets of ΥA to basic open sets of ΥB. Consider any finite subset

$$\mathsf{a} = \{a_1, \dots, a_m\}$$

of A. Then, for each $q \in \Upsilon B$ we have

$$\begin{aligned} q \in \phi^{\leftarrow}(\langle \mathsf{a} \rangle) &\iff \phi(q) \in \langle \mathsf{a} \rangle \\ &\iff a_1, \dots, a_m \in f^{\leftarrow}(q) \\ &\iff f(a_1), \dots, f(a_m) \in q \\ &\iff f[\mathsf{a}] \subseteq q \qquad\qquad \iff q \in \langle f[\mathsf{a}] \rangle \end{aligned}$$

and hence

$$\phi^{\leftarrow}(\langle \mathsf{a} \rangle) = \langle f[\mathsf{a}] \rangle$$

which is enough to show that ϕ is continuous.

(d) For an arbitrary poset A and space S we set up an inverse pair of bijections

$$\begin{array}{ccc} \boldsymbol{Pos}[A, \mathcal{O}S] & & \boldsymbol{Top}[S, \Upsilon A] \\ f & \longmapsto & f^{\sigma} \\ \phi^{\alpha} & \longleftarrow\!\dashv & \phi \end{array}$$

between the indicated arrow sets.

Consider any monotone function f, as indicated. Let

$$a \in f^{\sigma}(s) \iff s \in f(a)$$

for each $s \in S$ and $a \in A$. We first check that f^{σ} always returns an upper section of A. Consider elements $a \le b$ of A, and a point s of S. Then, since f is monotone, we have

$$a \in f^{\sigma}(s) \Longrightarrow s \in f(a) \subseteq f(b) \Longrightarrow b \in f^{\sigma}(s)$$

to show that $f^\sigma(s) \in \Upsilon A$. Now consider a finite subset

$$\mathsf{a} = \{a_1, \dots, a_m\}$$

of A. For each $s \in S$ we have

$$\begin{aligned} s \in f^{\sigma\leftarrow}(\langle \mathsf{a} \rangle) &\iff f^\sigma(s) \in \langle \mathsf{a} \rangle \\ &\iff \mathsf{a} \subseteq f^\sigma(s) \\ &\iff a_1, \dots a_m \in f^\sigma(s) \iff s \in f(a_1) \cap \cdots f(a_m) \end{aligned}$$

and hence

$$f^{\sigma\leftarrow}(\langle \mathsf{a} \rangle) = f(a_1) \cap \cdots f(a_m)$$

is open (since each $f(a_i)$ is open). Thus f^σ is continuous.

This gives us an assignment in one direction. To obtain an assignment in the other direction consider any continuous map ϕ, as indicated. Let

$$s \in \phi^\alpha(a) \iff a \in \phi(s)$$

for each $a \in A$ and $s \in S$. We first check that ϕ^α always returns an open set of S. Consider element $a \in A$. Then for each $s \in S$ we have

$$\begin{aligned} s \in \phi^\alpha(a) &\iff a \in \phi(s) \\ &\iff \{a\} \subseteq \phi(s) \\ &\iff \phi(s) \in \langle \{a\} \rangle \iff s \in \phi^{\leftarrow}(\langle \{a\} \rangle) \end{aligned}$$

to show that $\phi^\alpha(a)$ is open (since ϕ is continuous). Now consider elements $a \leq b$ of A. For each $s \in S$ we have

$$s \in \phi^\alpha(a) \Longrightarrow a \in \phi(s) \Longrightarrow b \in \phi(s) \Longrightarrow s \in \phi^\alpha(b)$$

to show that ϕ^α is monotone.

This gives us the two assignments. We show they form an inverse pair. Consider any monotone function f, as above. For $a \in A, s \in S$ we have

$$s \in f^{\sigma\alpha}(a) \iff a \in f^\sigma(s) \iff s \in f(a)$$

to show that $f^{\sigma\alpha} = f$ for one of the inverse properties. A similar calculation shows the other inverse properties.

(e) For each element $a \in A$ let

$$h(a) = \langle \{a\} \rangle$$

that is

$$p \in h(a) \iff a \in p$$

for each $p \in \Upsilon A$. This gives a function

$$h : A \longrightarrow \mathcal{O}(\Upsilon A)$$

and almost trivially it is monotone.

For each element $s \in S$ let $\eta(s)$ be the set of open sets neighbourhoods of s, that is

$$U \in \eta(s) \Longleftrightarrow s \in U$$

for each $U \in \mathcal{O}S$. This gives a function

$$\eta : S \longrightarrow \Upsilon(\mathcal{O}S)$$

for almost trivially $\eta(s)$ is an upper section of $\mathcal{O}S$. We need to check that η is continuous.

We show that for each basic open set of $\Upsilon(\mathcal{O}S)$ the inverse image across η is open in S. Each such basic open set has the form

$$\langle\{U_1, \dots, U_m\}\rangle$$

for $U_1, \dots, U_m \in \mathcal{O}S$. Thus

$$P \in \langle\{U_1, \dots, U_m\}\rangle \Longleftrightarrow U_1, \dots, U_m \in P$$

for each upper section P of $\mathcal{O}S$. For each $s \in S$ we have

$$\begin{aligned} s \in \eta^{\leftarrow}\big(\langle\{U_1, \dots, U_m\}\rangle\big) &\Longleftrightarrow \eta(s) \in \langle\{U_1, \dots, U_m\}\rangle \\ &\Longleftrightarrow U_1, \dots, U_m \in \eta(s) \\ &\Longleftrightarrow s \in U_1 \cap \cdots \cap U_m \end{aligned}$$

so that

$$\eta^{\leftarrow}\big(\langle\{U_1, \dots, U_m\}\rangle\big) = U_1 \cap \cdots \cap U_m$$

which is open in S.

(f) So far we have hardly mentioned the required functorality and naturality conditions. That is because they are ensured by a more general construction. Consider first the material of Subsection 3.3.2 and Exercise 3.5.2. Let $\mathsf{2} = \{0, 1\}$ with the sierpinski topology (that is $\{1\}$ is open but $\{0\}$ is not). Let

$$\Xi S = \boldsymbol{Top}[S, \mathsf{2}]$$

the set of continuous characters of S. We partially order ΞS with the pointwise comparison. There is an obvious bijection between

$$\mathcal{O}S \qquad \Xi S$$

and Exercise 3.5.2 shows that this is natural. Thus the two functors

$$\boldsymbol{Top} \xrightarrow{\ \mathcal{O}, \Xi\ } \boldsymbol{Pos}$$

are naturally equivalent. In particular, we can replace $\mathcal{O}$ by the enriched hom-functor $\Xi = \boldsymbol{Top}[-, \mathsf{2}]$.

Now view $\mathsf{2}$ as a poset with $0 < 1$. Let

$$\Pi A = \boldsymbol{Pos}[A, \mathsf{2}]$$

the set of 'monotone characters' of A. There is an obvious bijection between

$$\Upsilon A \qquad \Pi A$$

for we simply match each upper section of A with is characteristic function. We now use the sierpinski topology on $\mathsf{2}$ to furnish ΠA as a space, the subspace of the product space. We check that the bijection above is a homeomorphism. Thus we have two functors

$$\boldsymbol{Pos} \xrightarrow{\Upsilon, \Pi} \boldsymbol{Top}$$

and with a little bit of work, we see these are naturally equivalent. In particular, we can replace $\mathcal{O}$ by the enriched hom-functor $\Pi = \boldsymbol{Pos}[-, \mathsf{2}]$.

The object $\mathsf{2}$ lives in both categories. It is both a poset and a topological space. It is a schizophrenic object. Furthermore, it induces both of the functors. With this observation we can check all the functorality and naturality required for the contravariant adjunction. In fact, all the calculation can be done down in $\boldsymbol{Set}$. The details are given in the next exercise. □

5.6.2 (a) Consider arrows

$$A \xrightarrow{f} \mathfrak{A}S \qquad\qquad S \xrightarrow{\phi} \mathfrak{S}A$$

from the two arrow sets. These are functions in curried form.

$$f : A \longrightarrow S \longrightarrow \bigstar \qquad\qquad \phi : S \longrightarrow A \longrightarrow \bigstar$$

By uncurrying these are essentially the same as the 2-placed functions

$$f : A \times S \longrightarrow \bigstar \qquad\qquad \phi : S \times A \longrightarrow \bigstar$$

each of which consumes its inputs as a pair rather than one after the other. We chip the order of these two inputs. Thus we say f and ϕ correspond precisely when

$$f(a)(s) = \phi(s)(a)$$

for each $a \in A$ and $s \in S$. I bet you didn't know that curry and chips are part of the bread and butter of certain parts of mathematics.

(b) Consider a diagram induced by a $\boldsymbol{Alg}$-arrow l and a $\boldsymbol{Spc}$-arrow λ.

$$\begin{tikzcd}
 & f \arrow[rr, leftrightarrow] & & \phi & \\
A & \boldsymbol{Alg}[A, \mathfrak{A}S] \arrow[d, "\mathfrak{A}(\lambda) \circ - \circ l"] & & \boldsymbol{Spc}[S, \mathfrak{S}A] \arrow[d, "\mathfrak{S}(l) \circ - \circ \lambda"] & S \\
B \arrow[u, "l"] & \boldsymbol{Alg}[B, \mathfrak{A}T] & & \boldsymbol{Spc}[T, \mathfrak{S}B] & T \arrow[u, "\lambda"'] \\
 & g \arrow[rr, leftrightarrow, "?"] & & \psi &
\end{tikzcd}$$

Across the top we have a corresponding pair f, ϕ of arrows, that is

$$f(a)(s) = \phi(s)(a)$$

for each $a \in A$ and $s \in S$. The functors give us a pair g, ψ of arrows at the bottom. We must show that these correspond, that is

$$(?) \quad g(b)(t) = \psi(t)(b) \quad (?)$$

for each $b \in B$ and $t \in T$.

We have

$$\mathfrak{A}(\lambda) = - \circ \lambda \qquad \mathfrak{S}(l) = - \circ l$$

since both $\mathfrak{A}$, $\mathfrak{S}$ are enriched hom-functors. The two functions

$$g = \mathfrak{A}(\lambda) \circ f \circ l \qquad \psi = \mathfrak{S}(l) \circ \phi \circ \lambda$$

must be matched. For each $b \in B$ and $t \in T$ we have

$$g(b) = \mathfrak{A}(\lambda)\Big(f\big(l(b)\big)\Big) = f\big(l(b)\big) \circ \lambda \quad \psi(t) = \mathfrak{S}(l)\Big(\phi\big(\lambda(t)\big)\Big) = \phi\big(\lambda(t)\big) \circ l$$

so that

$$g(b)(t) = f\big(l(b)\big)\big(\lambda(t)\big) \qquad \psi(t)(b) = \phi\big(\lambda(t)\big)\big(l(b)\big)$$

and hence the given correspondence between f, ϕ ensures the required correspondence between g, ψ.

(c) For $A \in \boldsymbol{Alg}$ unit is induced by the identity arrow on $\mathfrak{S}A$.

$$A \xrightarrow{h_A} (\mathfrak{A} \circ \mathfrak{S})A \qquad \mathfrak{S}A \xrightarrow{id_{\mathfrak{S}A}} \mathfrak{S}$$

Thus for each

$$p \in \mathfrak{S}A \quad \text{that is} \quad p : A \longrightarrow \bigstar$$

we have

$$h_A(a)(p) = id_{\mathfrak{S}A}(p)(a) = p(a)$$

so that $h_A(a)$ is ‘evaluation at a’. In the same way

$$\eta_S(s)(\pi) = \pi(s)$$

for each $s \in S$ and $\pi \in \mathfrak{A}S$. $\square$

6
Posets and monoid sets

6.1 Posets and complete posets

6.1.1 We show that f has a right adjoint precisely when f preserves suprema, that is

$$f(\bigvee X) = \bigvee f[X]$$

for each subset X of S.

Suppose first that $f \dashv g$, that is

$$f(a) \leq b \iff a \leq g(b)$$

for all $a \in S$ and $b \in T$. Consider any subset $X \subseteq S$. For each $b \in T$ we have

$$\begin{aligned} f(\bigvee X) \leq b &\iff \bigvee X \leq g(b) \\ &\iff (\forall x \in X)[x \leq g(b)] \\ &\iff (\forall x \in X)[f(x) \leq b] \iff \bigvee f[X] \leq b \end{aligned}$$

to show that

$$f(\bigvee X) = \bigvee f[X]$$

as required.

Conversely, suppose f does preserve suprema. For each $b \in T$ let

$$g(b) = \bigvee X \quad \text{where} \quad x \in X \iff f(x) \leq b$$

to produce a function $g : T \longrightarrow S$. We easily check that g is monotone, and we show that $f \dashv g$ as follows. Consider any $b \in T$ with the associated set X. For each $a \in S$ we have

$$f(a) \leq b \Longrightarrow a \in X \Longrightarrow a \leq g(b)$$

to give one of the required implications. For the other we use the preservation property of f. Thus if

$$a \leq g(b) = \bigvee X$$

then

$$f(a) \leq f(\bigvee X) = \bigvee f[X] \leq b$$

as required.

The map f has a left adjoint precisely when it preserves infima. $\square$

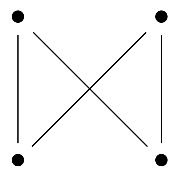
6.2 Two categories of complete posets

6.2.1 When they exists we have

$$\bigvee \emptyset = \bot = \bigwedge S \qquad \bigwedge \emptyset = \top = \bigvee S$$

respectively. □

6.2.2 Consider the following 4-element poset.



The set of two lower nodes has two upper bounds but no supremum. □

6.2.3 We require two implications, but by symmetry it suffices to verify just one of them. Suppose the poset S has all suprema. We show that S has all infima. Let X be an arbitrary subset of X. Let $\ell(X)$ be the set of all lower bounds of X. We must show that $\ell(X)$ has a largest member.

Since S has all suprema we may take

$$a = \bigvee \ell(X)$$

and show that a is the infimum of X. Consider any $x \in X$ and $y \in \ell(X)$. We have

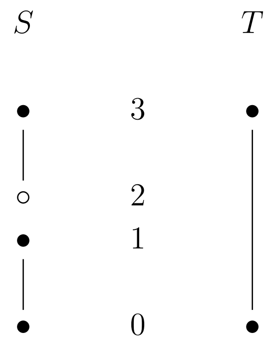
$$y \leq x$$

by construction of $\ell(X)$. Letting y range over $\ell(X)$ we see that x is an upper bound of $\ell(X)$, and hence

$$a \leq x$$

since a is the least upper bound of $\ell(X)$. Letting x range over X this shows that $a \in \ell(X)$, as required. □

6.2.4 We use two subsets of the reals as posets. We draw these sets pointing upwards, with larger number in higher positions. Consider the following two posets.



On the right we have all the real numbers from 0 to 3 including these two end points. This poset T is complete (by the Dedekind completeness of the reals). On the left we omit the central third, we include 1 but exclude 2. This poset S is also complete. The only problem is to find the infimum of its top half. That infimum is 1, whereas the infimum of the corresponding set in T is 2. This observation shows that the inclusion of S into T is a $\bigvee$-morphisms but not a $\bigwedge$-morphism. □

6.3 Sections of a poset

6.3.1 Let $\mathcal{X}$ be any family of lower sections of the poset S. We show that each of $\bigcup\mathcal{X}$ and $\bigcap\mathcal{X}$ is a lower section.

Consider any $a \le x \in \bigcup\mathcal{X}$. We have $a \le x \in X$ for some $X \in \mathcal{X}$. But now $a \in X \subseteq \bigcup\mathcal{X}$ to give $a \in \bigcup\mathcal{X}$.

Consider any $a \le x \in \bigcap\mathcal{X}$, and consider any $X \in \mathcal{X}$. We have $a \le x \in X$, so that $a \in X$, and hence $a \in \bigcap\mathcal{X}$.

Consider any lower section X and any $x \in X$. We have $\downarrow x \subseteq X$, so that

$$\bigcup\{\downarrow x \mid x \in X\} \subseteq X$$

and the converse inclusion is immediate. □

6.3.2 For a, b in the parent poset we have

$$a \le b \Longrightarrow a \in \downarrow b \Longrightarrow \downarrow a \subseteq \downarrow b \qquad a \le b \Longrightarrow b \in \uparrow a \Longrightarrow \uparrow b \subseteq \uparrow a$$

to show that $\eta^{\exists}$ is monotone, but $a \longmapsto \uparrow a$ is antitone. Taking complements is antitone, so $\eta^{\forall}$ is monotone. □

6.4 The two completions

6.4.1 For the second part we have

$$\begin{aligned} y \in (\uparrow X)' &\iff y \notin \uparrow X \\ &\iff \neg(\exists x)[x \in X \;\&\; x \le y] \\ &\iff (\forall x)[x \in X \Rightarrow x \not\le y] \end{aligned}$$

and

$$\begin{aligned} y \in \bigcap \eta^{\forall}[X] &\iff (\forall x)[x \in X \Rightarrow y \in \eta^{\forall}(x)] \\ &\iff (\forall x)[x \in X \Rightarrow y \in \uparrow(x)'] \\ &\iff (\forall x)[x \in X \Rightarrow x \not\le y] \end{aligned}$$

for the required result. □

6.4.2 For the second part consider any $Y \in \mathcal{L}S$ and let $X = Y'$ so that

$$\uparrow X = Y'$$

to give

$$Y = (\uparrow X)' = \bigcap \eta^{\forall}[X]$$

by Exercise 6.4.1. Thus, assuming that g, h are $\bigwedge$-morphisms, we have

$$\begin{aligned} g(Y) &= g\big(\bigcap \eta^{\forall}[X]\big) = \bigwedge(g \circ \eta^{\forall})[X] \\ h(Y) &= h\big(\bigcap \eta^{\forall}[X]\big) = \bigwedge(h \circ \eta^{\forall})[X] \end{aligned}$$

which leads to the required result. □

6.4.3 We deal with the $\forall$-version, in other words we show

$$f^\sharp\big(\bigcap\mathcal{X}\big) = \bigwedge f^\sharp[\mathcal{X}]$$

for each $\mathcal{X} \subseteq \mathcal{L}S$.

Remembering this definition of this $f^\sharp$, for each $t \in T$ we have

$$\begin{aligned}
t \le f^\sharp\big(\bigcap\mathcal{X}\big) &\Longleftrightarrow t \le \bigwedge f\big[\big(\bigcap\mathcal{X}\big)'\big]\\
&\Longleftrightarrow (\forall s \in S)\big[s \in \big(\bigcap\mathcal{X}\big)' \Rightarrow t \le f(s)\big]\\
&\Longleftrightarrow (\forall s \in S)\big[t \not\le f(s) \Rightarrow s \in \big(\bigcap\mathcal{X}\big)\big]\\
&\Longleftrightarrow (\forall s \in S)(\forall X \in \mathcal{X})[t \not\le f(s) \Rightarrow s \in X]
\end{aligned}$$

and

$$\begin{aligned}
t \le \bigwedge f^\sharp[\mathcal{X}] &\Longleftrightarrow (\forall X \in \mathcal{X})[t \le f^\sharp(X)]\\
&\Longleftrightarrow (\forall X \in \mathcal{X})(\forall s \in S)[s \notin X \Rightarrow t \le f(s)]\\
&\Longleftrightarrow (\forall X \in \mathcal{X})(\forall s \in S)[t \not\le f(s) \Rightarrow s \in X]
\end{aligned}$$

which gives the required result. □

6.5 Three endofunctors on $\boldsymbol{Pos}$

6.5.1 In each case the $\exists$-version is straight forward but the $\forall$-version need a little more care.

To produce the explicit description of $\forall(f)(X)$ we remember how to take a negation through quantifiers and connectives. We have

$$\begin{aligned}
b \in \forall(f)(X) &\Longleftrightarrow b \notin {\uparrow}f[X']\\
&\Longleftrightarrow \neg[b \in {\uparrow}f[X']]\\
&\Longleftrightarrow \neg(\exists x \in S)[x \in X' \;\&\; f(b) \le x]\\
&\Longleftrightarrow \neg(\exists x \in S)[x \notin X \;\&\; f(b) \le x]\\
&\Longleftrightarrow (\forall x \in S)[x \in X \Rightarrow f(b) \le x]
\end{aligned}$$

as required.

To show that $\forall(f)$ is monotone consider lower sections $X_1 \subseteq X_2$ of S. For each $b \in \forall(f)(X_1)$ and $x \in S$ we have

$$f(x) \le b \Longrightarrow x \in X_1 \Longrightarrow x \in X_2$$

to verify that $b \in \forall(f)(X_2)$. The required implication also follows directly from the definition of $\forall(f)$.

To show that $\forall$ is a functor consider any pair

$$R \xrightarrow{\quad g\quad} S \xrightarrow{\quad f\quad} T$$

of monotone maps between posets. We require

$$\forall(f \circ g) = \forall(f) \circ \forall(g)$$

that is

$$\forall(f \circ g)(X) = \forall(f) \circ \forall(g)(X)$$

for all $X \in \mathcal{L}R$. Unravelling the definition we see that

$$\uparrow(f \circ g)[X'] = \uparrow f[\uparrow g[X']]$$

is required. Consider any $c \in \uparrow f[\uparrow g[X']]$. We have $f(b) \le c$ for some $b \in \uparrow g[X']$ which gives $g(a) \le b$ for some $a \in X'$. Then

$$f(g(a)) \le f(b) \le c$$

to show that

$$c \in \forall(f \circ g)(X)$$

and so obtain one of the two required inclusions.

The other inclusion follows by a similar argument. $\square$

6.5.2 A proof of the left hand equivalence is straight forward.

For the right hand equivalence consider any $X \in \mathcal{L}S$ and $Y \in \mathcal{L}T$. Then

$$\begin{aligned}
Y \subseteq \forall(f)(X) &\iff Y \subseteq (\uparrow f[X'])' \\
&\iff \uparrow f[X'] \subseteq Y' \\
&\iff f[X'] \subseteq Y' \\
&\iff (\forall x \in S)[x \in X' \Rightarrow f(x) \in Y'] \\
&\iff (\forall x \in S)[f(x) \in Y \Rightarrow x \in X] \\
&\iff (\forall x \in S)[x \in f^{\leftarrow}(Y) \Rightarrow x \in X] \iff f^{\leftarrow}(Y) \subseteq X
\end{aligned}$$

as required. $\square$

6.6 Long strings of adjunctions

6.6.1 There are many possible examples all with small posets. For instance we have

$$\exists(f)(\emptyset) = \emptyset \qquad \forall(f)(\emptyset) = \big(\uparrow f[S]\big)'$$

so if $\uparrow f[S]$ is not the whole of T then the two induced maps are different. Thus

$$\begin{tikzcd}
S \arrow[r, "f"] & T \\
\bullet \arrow[r, mapsto] & \bullet \\
& \bullet
\end{tikzcd}$$

is a very small example. $\square$

6.6.2 For $Y \in \mathcal{L}T$ and $a \in S$ we have

$$\begin{aligned}
a \in \exists(g)(Y) &\iff (\exists y \in T)[a \le g(y) \;\&\; y \in Y] \\
&\iff (\exists y \in T)[f(a) \le y \;\&\; y \in Y] \\
&\iff f(a) \in Y \qquad\qquad \iff a \in \mathsf{I}(f)(Y)
\end{aligned}$$

to show $\exists(g) = \mathsf{I}(f)$.

For $X \in \mathcal{L}S$ and $b \in T$ we have

$$\begin{aligned} b \in \forall(f)(X) &\iff (\forall x \in S)[f(x) \leq b \Rightarrow x \in X] \\ &\iff (\forall x \in S)[x \leq g(b) \Rightarrow x \in X] \\ &\iff g(b) \in X \qquad\qquad \iff b \in \mathsf{I}(g)(X) \end{aligned}$$

to show $\forall(f) = \mathsf{I}(g)$. □

6.6.3 You will probably find that the components are listed the other way up with a left adjoint below its right adjoint. Show

$$\delta_{i+1}^{n+1} \dashv \sigma_i^{n+1} \dashv \delta_i^{n+1}$$

and

$$\begin{aligned} \exists(\delta_i^n) &= \delta_{i+1}^{n+1} & \qquad \forall(\delta_i^n) &= \delta_i^{n+1} \\ \exists(\delta_i^{n+1}) &= \delta_{i+1}^{n+2} & \qquad \forall(\delta_i^{n+1}) &= \delta_i^{n+2} \end{aligned}$$

for all $0 \leq i \leq n$. This is a bit fiddly, but not difficult. □

6.7 Two adjunctions for R-sets

6.7.1 (a) Let $R = \{\pm 1\}$ under multiplication. For an R-set A let

$$a^\bullet = a(-1)$$

for each $a \in A$.

(b) Let $R = \{1, 0\}$ under multiplication. For an R-set A let

$$a^\bullet = a0$$

for each $a \in A$.

(c) Let $R = \{\pm 1, \pm i\}$ under multiplication. For an R-set A let

$$a^\bullet = ai \qquad {}^\bullet a = a(-1)$$

for each $a \in A$.

(d) With $\omega^3 = 1$ let $R = \{1, \omega, \omega^2\}$ under multiplication. For an R-set A let

$$a^\bullet = a\omega \qquad {}^\bullet a = a\omega^2$$

for each $a \in A$.

Other roots of unity give many other examples of this kind. These algebras usually need more than two carried 1-placed operations. As a bit of entertainment look for a description using $\vartheta^{42} = 1$. □

6.8 The upper left adjoint

6.8.1 We require
$$\big((x,r)s\big)t = (x,r)(st)$$
for all $\in X$ and $r,s,t \in R$. However, since R is associative, we see that both sides evaluate to (x,rst) for the required result. □

6.8.2 For an arbitrary function

$$Y \xrightarrow{\ g\ } X$$

we require

$$\Sigma Y \xrightarrow{\ \Sigma(g)\ } \Sigma X$$

to be a morphism, that is
$$\Sigma(g)\big((y,r)s\big) = \big(\Sigma(g)(y,r)\big)s$$
for each $y \in Y$ and $r,s \in R$. But, remembering how ΣY and ΣX are structured, we have

$$\Sigma(g)\big((y,r)s\big) = \Sigma(g)(y,rs) = \big(g(y),rs\big) = \big((g(y),r)\big)s = \big(\Sigma(g)(y,r)\big)s$$

as required.

We also require that the arrow assignment Σ passes across composition, that is

$$\Sigma(h \circ g) = \Sigma(h) \circ \Sigma(g)$$

for each pair

$$Z \xrightarrow{\ g\ } Y \xrightarrow{\ g\ } X$$

of composible sets. By evaluating at an arbitrary pair $(z,r) \in \Sigma Z$, we see that this is almost immediate. □

6.8.3 Consider any morphism

$$\Sigma X \xrightarrow{\ f\ } A$$

to an arbitrary R-set. We have
$$(x,r) = \eta_X(x)r$$
for each $(x,r) \in \Sigma X$, and hence

$$f\big((x,r)\big) = f\big(\eta_X(x)r\big) = f\big(\eta_X(x)\big)r = \big(f \circ \eta_X\big)(x)r$$

since f is a morphism.

Applying this observation to a parallel pair of morphisms gives the required result. □

6.8.4 We have seen that there is only one possible function $g^\sharp$, that given by

$$g^\sharp(x, r) = g(x)r$$

for $x \in X$ and $r \in R$. Since

$$(g^\sharp \circ \eta_X)(x) = g^\sharp\big(\eta_X(x)\big) = g^\sharp(x, 1) = g(x)1 = g(x)$$

we see that this function does make the triangle commute (in $\boldsymbol{Set}$). Thus it suffices to show that $g^\sharp$ is a morphism.

We require

$$g^\sharp\big((x, r)s\big) = g^\sharp(x, r)s$$

for each $x \in X$ and $r, s \in R$. Remembering the way ΣX is structured we have

$$g^\sharp\big((x, r)s\big) = g^\sharp(x, rs) = g(x)rs = \big(g(x)r\big)s = g^\sharp(x, r)s$$

as required. □

6.8.5 For an arbitrary function

$$Y \xrightarrow{\ g\ } X$$

we apply Theorem 6.8.5 to the composite

$$Y \xrightarrow{\ g\ } X \xrightarrow{\ \eta_Y\ } \Sigma X$$

to obtain $\Sigma(g)$. Thus $\Sigma(g)$ is the unique morphism for which the square

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "\eta_Y"'] & X \arrow[d, "\eta_X"] \\ \Sigma Y \arrow[r] & \Sigma X \end{tikzcd} \qquad \Sigma(g) = \big(\eta_X \circ g\big)^\sharp$$

commutes. This is defined in equational form on the right. Evaluating at an arbitrary $(y, r) \in \Sigma Y$ we have

$$\begin{aligned} \Sigma(g)(y, r) &= \big(\eta_X \circ g\big)^\sharp(y, r) \\ &= \big(\eta_X \circ g\big)(y)r \\ &= \eta_X\big(g(y)\big)r \\ &= (g(y), 1)r = (g(y), r) \end{aligned}$$

which agrees with Definition 6.8.2. At the last step of this calculation we remember how ΣX is structured. □

6.8.6 As in the subsection, we must show that

$$\eta_X \circ g = \Sigma(g) \circ \eta_Y$$

where

$$Y \xrightarrow{\quad g \quad} X$$

is an arbitrary function. To do that we evaluate both sides at an arbitrary $y \in Y$. We have

$$\big(\eta_X \circ g\big)(y) = \eta_X\big(g(y)\big) = \big(g(y), 1\big)$$

and

$$\big(\Sigma(g) \circ \eta_Y\big)(y) = \Sigma(g)\big(\eta_Y(y)\big) = \Sigma(g)\big(y, 1\big) = \big(g(y), 1\big)$$

to give the required result. □

6.8.7 We require

$$\epsilon_A\big((a, r)s\big) = \big(\epsilon_A(a, r)\big)s$$

for each $a \in A$ and $r, s \in R$. But

$$\epsilon_A\big((a, r)s\big) = \epsilon_A(a, rs) = a(rs) \qquad \big(\epsilon_A(a, r)\big)s = (ar)s$$

so it suffices to remember that A is an R-set. □

6.8.8 As in the subsection, we must show that

$$\epsilon_B \circ \Sigma(f) = f \circ \epsilon_A$$

where

$$A \xrightarrow{\quad f \quad} b$$

is an arbitrary morphism. To do that we evaluate both sides at an arbitrary $(a, r) \in \Sigma A$. We have

$$\big(\epsilon_B \circ \Sigma(f)\big)(a, r) = \epsilon_B\big(\Sigma(f)(a, r)\big) = \epsilon_B\big(f(a), r\big) = f(a)r$$

and

$$f\big(\circ\, \epsilon_A\big)(a, r) = f\big(\epsilon_A(a, r)\big) = f(ar)$$

so remembering that f is a morphism gives the required result. □

6.8.9 For a given morphism

$$\Sigma X \xrightarrow{\quad f \quad} A$$

suppose there is a function

$$X \xrightarrow{\quad g \quad} A \qquad\qquad \text{such that} \quad \epsilon_A \circ \Sigma(g) = f$$

holds. For each $(x, r) \in \Sigma X$ we have

$$\big(\epsilon_A \circ \Sigma(g)\big)(x, r) = \epsilon_A\big(\Sigma(g)(x, r)\big) = \epsilon_A\big(g(x), r\big) = g(x)r$$

to give

$$g(x)r = f(x, r)$$

for each such x and r. In particular, we have

$$g(x) = f(x, 1)$$

to show there is at most one such function g.

It suffices to show that this particular function does make the triangle commute.

For this function g we have

$$\big(\epsilon_A \circ \Sigma(g)\big)(x, r) = g(x)r = f(x, 1)r = f(x, r)$$

where at the last step we remember that f is a morphism and the way that ΣX is structured. □

6.9 The upper adjunction

6.9.1 Consider first the composite

$$\Sigma X \xrightarrow{\ \Sigma(\eta_X)\ } (\Sigma \circ U \circ \Sigma)X \xrightarrow{\ \epsilon_{\Sigma X}\ } \Sigma X$$

on ΣX. We must show that this is the identity function on ΣX.

For each

$$(x, r) \in \Sigma X$$

we have

$$\Sigma(\eta_X)(x, r) = (\eta_X(x), r)$$

where

$$\eta_X(x) = (x, 1)$$

by the definitions of Σ and η. Thus

$$\big(\epsilon_{\Sigma X} \circ \Sigma(\eta_X)\big)(x, r) = \epsilon_{\Sigma X}(\eta_X(x), r) = \eta_X(x)r$$

by the definition of η. We now remember the action on ΣX to get

$$\big(\epsilon_{\Sigma X} \circ \Sigma(\eta_X)\big)(x, r) = \eta_X(x)r = (x, 1)r = (x, r)$$

for the required result.

Next consider the composite

$$UA \xrightarrow{\eta_{UA}} (U \circ \Sigma \circ U)A \xrightarrow{U(\epsilon_A)} UA$$

on UA. In terms of functions he two components are

$$\begin{array}{ccc} A & \xrightarrow{\eta} & \Sigma A \\ a & \longmapsto & (a,1) \end{array} \qquad\qquad \begin{array}{ccc} \Sigma A & \xrightarrow{\epsilon} & A \\ (a,r) & \longmapsto & ar \end{array}$$

so the composite is

$$\begin{array}{ccccc} A & \xrightarrow{\eta} & \Sigma A & \xrightarrow{\epsilon} & A \\ a & \longmapsto & (a,1) & \longmapsto & a1 = a \end{array}$$

the identity on A, as required. □

6.9.2 For an arbitrary function

$$X \xrightarrow{g} A$$

and $x \in X$ we have

$$\big(\epsilon_A \circ \Sigma(g)\big)(x) = \epsilon_A\big(\Sigma(g)(x)\big) = \epsilon_A\big(g(x), r\big) = g(x)r$$

which agrees with the suggested $g^\sharp$.

For an arbitrary function

$$X \times R \xrightarrow{f} A$$

and $(x, r) \in X \times R$ we have

$$\big(U(f) \circ \eta_X\big)(x,r) = f\big(\eta_X(x,r)\big) = f(x,1)$$

which agrees with the suggested $f_\flat$. □

6.9.3 Remembering how ΣX is structured, we require

$$\big(g^\sharp(x,r)\big)s = g^\sharp\big((x,r)s\big)$$

for each $x \in X$ and $r, s \in R$. But

$$\big(g^\sharp(x,r)\big)s = \big(g(x)r\big)s \qquad g^\sharp\big((x,r)s\big)g^\sharp(x,rs) = g(x)(rs)$$

so that fact that the target of g is an R-set gives the required result. □

6.9.4 For an arbitrary function

$$X \xrightarrow{\ g\ } A$$

the equality

$$g^{\sharp}{}_{\flat} = g$$

is almost trivial.

For an arbitrary morphism

$$\Sigma X \xrightarrow{\ f\ } A$$

and $(x, r) \in \Sigma X$ we have

$$f_{\flat}{}^{\sharp}(x, r) = f_{\flat}(x)r = f(x, 1)r = f\big((x, 1)r\big) = f(x, r)$$

as required. At the last two steps remember that f is a morphism and how ΣX is structured. □

6.9.5 For $(\sharp)$ we must show that

$$\big(U(l) \circ g \circ k\big)^{\sharp} = \big(l \circ g^{\sharp} \circ \Sigma(k)\big)$$

for each pair k, g of functions and morphism l, as indicated in the section. To do that we evaluate both sides at an arbitrary $(y, r) \in \Sigma Y$. Using the definition of $(\cdot)^{\sharp}$ we have

$$\big(U(l) \circ g \circ k\big)^{\sharp}(y, r) = \big(l \circ g \circ k\big)(y)r = l\big((g \circ k)(y)\big)r = l\big((g \circ k)(y)r\big)$$

since l is a morphism. We also have

$$\begin{aligned} \big(l \circ g^{\sharp} \circ \Sigma(k)\big)(y, r) &= \Big(l \circ g^{\sharp}\Big)\big(\Sigma(k)(y, r)\big) \\ &= \Big(l \circ g^{\sharp}\Big)\big(k(y), r\big) \\ &= l\Big(g^{\sharp}\big(k(y), r\big)\Big) = l\Big(g\big(k(y)\big)r\Big) \end{aligned}$$

to give the required result.

For $(\flat)$ we must show

$$\big(l \circ f_{\flat} \circ k\big) = \big(l \circ f \circ \Sigma(k)\big)_{\flat}$$

for each function k and pair f, l of morphisms, as indicated in the section. To do that we evaluate both sides at an arbitrary $y \in Y$. Using the definition of $(\cdot)_{\flat}$ and Σ we have

$$\big(l \circ f_{\flat} \circ k\big)(y) = l\Big(f_{\flat}\big(k(y)\big)\Big) = l\Big(f\big(k(y), 1\big)\Big) = \big(l \circ f\big)\big(k(y), 1\big)$$

and

$$\begin{aligned} \big(l \circ f \circ \Sigma(k)\big)_{\flat}(y) &= \big(l \circ f \circ \Sigma(k)\big)(y, 1) \\ &= \big(l \circ f\big)\big(\Sigma(k)(y, 1)\big) = \big(l \circ f\big)\big(k(y), 1\big) \end{aligned}$$

to give the required result. □

6.10 The lower right adjoint

6.10.1 We require

$$(h^r)^s = h^{rs}$$

for each function $h : R \to X$ and $r, s \in R$. But for each $r, s, t \in R$ we have

$$(h^r)^s(t) = h^r(st) = h(r(st)) = h((rs)t) = h^{rs}(t).$$

for the required result. □

6.10.2 For each pair of functions

$$R \xrightarrow{\ h\ } X \xrightarrow{\ g\ } Y$$

we require

$$\Pi(g)(h^r) = \big(\Pi(g)(h)\big)^r$$

that is

$$g \circ h^r = (g \circ h)^r$$

for each $r \in R$. But for each $r, s \in R$ we have

$$(g \circ h^r)(s) = g\big(h^r(s)\big) = g\big(h(rs)\big) = \big(g \circ h\big)(rs) = (g \circ h)^r(s)$$

for the required result. □

6.10.3 Remembering how ΠA is structured we require

$$\eta_A(as) = \eta_A(a)^s$$

for each $a \in A$ and $s \in R$. To verify this we evaluate both sides at an arbitrary $r \in R$. Thus

$$\eta_A(as)(r) = (as)r = a(sr) = \eta_A(a)(sr) = \eta_A(a)^s(r)$$

as required. □

6.10.4 Referring to the diagram we require

$$\eta_B \circ f = \Pi(f) \circ \eta_A$$

for an arbitrary morphisms f, as in the section. Thus we require

$$\eta_B\big(f(a)\big) = \Pi(f)\big(\eta_A(a)\big)$$

for each $a \in A$. Remembering the definition of Π this is

$$\eta_B\big(f(a)\big) = f \circ \eta_A(a)$$

so we evaluate both sides at an arbitrary $r \in R$. We have

$$\eta_B\big(f(a)\big)(r) = f(a)r \qquad \big(f \circ \eta_A(a)\big)(r) = f(\eta_A(r)) = f(ar)$$

so that remembering that f is a morphism gives the required result. □

6.10.5 It suffices to show that the composite

$$\Pi(f^\sharp) \circ \eta_A$$

is the given morphism f. To do that we first evaluate at an arbitrary $a \in A$. This gives

$$\big(\Pi(f^\sharp) \circ \eta_A\big)(a) = \Pi(f^\sharp)\big(\eta_A(a)\big) = f^\sharp \circ \eta_A(a)$$

by the construction of Π. This is a function $R \longrightarrow A$, so we evaluate at an arbitrary $r \in R$ to get

$$\big(\Pi(f^\sharp) \circ \eta_A\big)(a)(r) = f^\sharp\big(\eta_A(a)(r)\big) = f^\sharp(ar) = f(ar)(1)$$

by the construction of $f^\sharp$.

We now remember that f is a morphism, and the way ΠX is structured. Thus

$$\big(\Pi(f^\sharp) \circ \eta_A\big)(a)(r) = f(ar)(1) = f(a)^r(1) = f(a)(r1) = f(a)(r)$$

to give the required result. □

6.10.6 Consider any morphism

$$A \xrightarrow{\ k\ } \Pi X$$

from an R-sets to a cofree R-set. Remembering how ΠX is structured we have

$$k(a)(r) = k(a)(r1) = k(a)^r(1) = k(ar)(1) = \big(\epsilon_X \circ k\big)(ar)$$

where the third equality holds since k is a morphism. □

6.10.7 We require

$$\epsilon_Y \circ \Pi(g) = g \circ \epsilon_X$$

for an arbitrary function

$$X \xrightarrow{\ g\ } Y$$

between sets. To verify this we evaluate at an arbitrary function $h : R \to X$. Thus

$$\begin{aligned}
\big(\epsilon_Y \circ \Pi(g)\big)(h) &= \epsilon_Y\big(\Pi(g)(h)\big) \\
&= \epsilon_Y(g \circ h) \\
&= (g \circ h)(1) \\
&= g\big(h(1)\big) \\
&= g\big(\epsilon_Y(h)\big) = \big(g \circ \epsilon_X\big)(h)
\end{aligned}$$

for the required result. □

6.10.8 Consider the 2-step function $g_\flat$ given by

$$g_\flat(a)(r) = g(ar)$$

for each $a \in A$ and $r \in R$. We have

$$\big(\epsilon_X \circ g_\flat\big)(a) = \epsilon_X\big(g_\flat(a)\big) = g_\flat(a)(1) = g(a)$$

so the triangle does commute. It remains to show that $g_\flat$ is a morphism.

Remembering the way ΠX is structured, we require

$$g_\flat(ar) = g_\flat(a)^r$$

for each $a \in A$ and $r \in R$. To check this we evaluate both sides at an arbitrary $s \in R$. Thus

$$g_\flat(ar)(s) = g\big((ar)s\big) \qquad g_\flat(a)^r(s) = g_\flat(a)(rs) = g\big(a(rs)\big)$$

which, since A is an R-set, gives the required result. □

6.11 The lower adjunction

6.11.1 For each $a \in A$ we have

$$\big(\epsilon_{\Pi A} \circ U(\eta_A)\big)(a) = \epsilon_{\Pi A}\big(\eta_A(a)\big) = \eta_A(a)(1) = a1 = a$$

to verify the left hand equality.

For each function $h : R \to X$ we have

$$\big(\Pi(\epsilon_X) \circ \eta_{\Pi X}\big)(h) = \Pi(\epsilon_X)\big(\eta_{\Pi X}(h)\big) = \epsilon_X \circ \eta_{\Pi X}(h)$$

and we must show that this is just the function h. To do that we evaluate at an arbitrary $r \in R$. Thus

$$\big(\Pi(\epsilon_X) \circ \eta_{\Pi X}\big)(h)(r) = \epsilon_X\big(\eta_{\Pi X}(h)(r)\big) = \epsilon_X\big(h^r\big) = h^r(1) = h(r)$$

to give the required result. □

6.11.2 For the left hand equality we evaluate the compound

$$A \xrightarrow{\ f\ } \Pi X \xrightarrow{\ \epsilon_X\ } X$$

at an arbitrary $a \in A$. Thus

$$\big(\epsilon_X \circ f\big)(a) = \epsilon_X\big(f(a)\big) = f(a)(1)$$

as required.

For the right hand equality we first observe that the compound

$$A \xrightarrow{\ \eta_A\ } (\Pi \circ U)A \xrightarrow{\ \Pi(g)\ } \Pi X$$

is a 2-step function

$$A \longrightarrow R \longrightarrow X$$

so we evaluate first at $a \in A$ and then at $r \in R$. Thus we obtain

$$\big(\Pi(g) \circ \eta_A\big)(a) = \Pi(g)\big(\eta_A(a)\big) = g \circ \eta_A)(a)$$

followed by

$$\big(\Pi(g) \circ \eta_A\big)(a)(r) = g\big(\eta_A(a)(r)\big) = g(ar)$$

for the required result. □

6.11.3 For each morphism

$$A \xrightarrow{\ f\ } \Pi X$$

we require

$$f^{\sharp}{}_{\flat} = f$$

that is

$$f^{\sharp}{}_{\flat}(a)(r) = f(a)(r)$$

for each $a \in A$ and $r \in R$. But we have

$$f^{\sharp}{}_{\flat}(a)(r) = f^{\sharp}(ar) = f(ar)(1) = f(a)^{r}(1) = f(a)(r1) = f(a)(r)$$

as required. At the final couple of steps we remember that f is a morphism and how ΠX is structured.

For each function

$$A \xrightarrow{\ g\ } X$$

we require

$$g_{\flat}{}^{\sharp} = g$$

that is

$$g_{\flat}{}^{\sharp}(a) = g(a)$$

for each $a \in A$. But we have

$$g_{\flat}{}^{\sharp}(a) = g_{\flat}(a)(1) = g(a1) = g(a)$$

as required. □

6.11.4 For the given function g we require

$$g_{\flat}(ar) = g_{\flat}(a)^{r}$$

for each $a \in A$ and $r \in R$. To verify this we evaluate both sides at an arbitrary $s \in R$. Thus

$$g_{\flat}(ar)(s) = g\big((ar)s\big) \qquad g_{\flat}(a)^{r}(s) = g_{\flat}(a)(rs) = g\big(a(rs)\big)$$

to give the required result. □

6.11.5 To prove

$$\big(\Pi(l) \circ f \circ k\big)^{\sharp} = \big(l \circ f^{\sharp} \circ U(k)\big)$$

it is convenient to set

$$m = (f \circ k)$$

so that m is a morphism. For each $b \in B$ we have

$$\begin{aligned}
\big(\Pi(l) \circ f \circ k\big)^{\sharp}(b) &= \big(\Pi(l) \circ m\big)^{\sharp}(b) \\
&= \big(\Pi(l) \circ m\big)(b)(1) \\
&= \Pi(l)\big(m(b)\big)(1) \\
&= \big(l \circ m(b)\big)(1)
\end{aligned}
\qquad
\begin{aligned}
\big(l \circ f^{\sharp} \circ U(k)\big)(b) &= l\Big(f^{\sharp}\big(k(b)\big)\Big) \\
&= l\Big(f\big(k(b)\big)(1)\Big) \\
&= l\Big(m(b)(1)\Big) \\
&= \big(l \circ m(b)\big)(1)
\end{aligned}$$

to give the required result.

To prove

$$\Pi(l) \circ g_{\flat} \circ k = \big(l \circ g \circ U(k)\big)_{\flat}$$

it is convenient to set

$$n = (l \circ g)$$

so that n is a mere function. For each $b \in B$ we have

$$\big(\Pi(l) \circ g_{\flat} \circ k\big)(b) = \Pi(l)\Big(g_{\flat}\big(k(b)\big)\Big) = l \circ \Big(g_{\flat}\big(k(b)\big)\Big)$$

and hence for each $r \in R$ we have

$$\begin{aligned}
\big(\Pi(l) \circ g_{\flat} \circ k\big)(b)(r) &= \Big(l \circ \Big(g_{\flat}\big(k(b)\big)\Big)\Big)(r) \\
&= l\Big(g_{\flat}\big(k(b)\big)(r)\Big) \\
&= l\Big(g\big(k(b)r\big)\Big) = n\big(k(b)r\big)
\end{aligned}$$

to evaluate the left hand side. In a similar way, for the right hand side we have

$$\big(l \circ g \circ U(k)\big)_{\flat}(b)(r) = \big(n \circ k\big)(br) = n\big(k(br)\big)$$

which, since k is a morphism, gives the required result. $\square$