ON THE EXPONENTIAL DIOPHANTINE EQUATION RELATED TO POWERS OF TWO CONSECUTIVE TERMS OF LUCAS SEQUENCES

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Abstract. Let $(U_n)_{n\geq 0}$ be the Lucas sequence given by $U_0 = 0,\ U_1 = 1$, and $U_{n+2} = rU_{n+1} + U_n$, for all $n \geq 0$ and $r \geq 1$. In this paper, we show that there is no integer $r \geq 3$ such that the sum of rth powers of two consective terms of a Lucas sequence is a term of a Lucas sequence.

1. Introduction

Let $(U_n)_{n\geq 0}$ be the Lucas sequence given by $U_0 = 0,\ U_1 = 1$, and

$$U_{n+2} = rU_{n+1} + U_n, \tag{1}$$

for all $n \geq 0$ and $r \geq 1$. When $r = 1$, then U_n coincides with the nth term of the Fibonacci sequence, F_n, when $r = 2$, U_n coincides with the nth term of the Pell sequence, P_n, and so on. It is well-known that

$$U_n^2 + U_{n+1}^2 = U_{2n+1}, \quad \text{for all} \quad n \geq 0. \tag{2}$$

In particular, the identity (2) tells us that the sum of the squares of two consecutive terms of a Lucas sequence is also a term of a Lucas sequence.

We consider the Diophantine equation

$$U_n^x + U_{n+1}^x = U_m, \tag{3}$$

in nonnegative integers (n, m, x). For $r = 1$, Luca and Oyono [11] studied Eq. (3) and they proved that Eq. (3) has no integer solutions (n, m, x) with $n \geq 2$ and $x \geq 3$. For $r = 2$, Rihane, et al [14] also studied Eq. (3) and they proved that all the integer solutions to Eq. (3) in nonnegative integers (n, m, x) are $(n, m, x) \in \{(1, 0, x), (2n + 1, n, 2), (2, n, 0)\}$. That is, they proved that for $r = 2$, Eq. (3) has no integer solutions (n, m, x) with $n \geq 2$ and $x \geq 3$. In the same spirit, Gómez Ruiz and Luca [6] studied Eq. (3) with $U_n = F_n^{(k)}$, where $(F_n^{(k)})_{n\geq k-2}$ is the $k-$generalized Fibonacci sequence given by the recurrence

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)} \quad \text{for all} \quad n \geq 2 \quad \text{and} \quad k \geq 2,$$

with the initial conditions $F_{-(k-2)}^{(k)} = F_{-(k-2)}^{(k)} = \cdots = F_0^{(k)} = 0$ and $F_1^{(k)} = 1$. They proved that Eq. (3) has no positive integer solutions (k, n, m, x) with $k \geq 3$, $n \geq 2$, and $x \geq 2$.

In this paper, we study Eq. (3) in nonnegative integers (n, m, x) with $r \geq 3$, since the cases for $r = 1$ and $r = 2$ have been already studied by Luca and Oyono [11], and Rihane, et al. [14], respectively. The main purpose of this paper is to prove the following result.

Theorem 1. *All solutions of the Diophantine equation* (3) *in nonnegative integers* (n, m, x)*, with* $r \geq 3$ *are*

$$(n, m, x) \in \{(0, 1, x), (n, 2n + 1, 2)\}. \tag{4}$$

Namely, we have

$$U_0^x + U_1^x = U_1, \quad U_n^2 + U_{n+1}^2 = U_{2n+1}.$$

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2. Preliminary Results

2.1. **Lucas sequence.** Let

$$(\alpha, \beta) = \left(\frac{r+\sqrt{r^2+4}}{2}, \frac{r-\sqrt{r^2+4}}{2}\right),$$

be the roots of the characteristic equation $x^2 - rx - 1 = 0$ of the Lucas sequence $(U_n)_{n\geq 0}$. The Binet formula for its general terms is given by

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{for all} \quad n \geq 0. \tag{5}$$

We can prove by induction that the above formula implies that the forllowing inequality

$$\alpha^{n-2} \leq U_n \leq \alpha^{n-1}, \tag{6}$$

holds for all positive integers n. It is also easy to show that

$$\frac{U_n}{U_{n+1}} \leq \frac{r+1}{r(r+1)+1} < \frac{1}{r}, \tag{7}$$

holds for all $n \geq 2$.

The following lemma is useful. For further details we refer the reader to the book of Koshy [9].

Lemma 1. *Let $\{U_n(x)\}_{n\geq 0}$ be the polynomial Lucas sequence defined by $U_0(x) = 0$, $U_1(x) = 1$, and*

$$U_{n+2}(x) = xU_{n+1}(x) + U_n(x), \quad \text{for all} \quad n \geq 0.$$

Then,

$$U_n(x) = \sum_{\substack{0\leq k\leq n \\ k\not\equiv n (mod\ 2)}} \binom{\frac{n+k-1}{2}}{k} x^k, \tag{8}$$

that is, when n and k have opposite parity.

2.2. **Logarithmic height.** Let η be an algebraic number of degree d with minimal primitive polynomial over the integers

$$a_0x^d + a_1x^{d-1} + \cdots + a_d = a_0\prod_{i=1}^{d}(x - \eta^{(i)}),$$

where the leading coefficient a_0 is positive and the $\eta^{(i)}$'s are the conjugates of η. Then the *logarithmic height* of η is given by

$$h(\eta) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d}\log\left(\max\{|\eta^{(i)}|, 1\}\right)\right).$$

In particular, if $\eta = p/q$ is a rational number with $\gcd(p, q) = 1$ and $q > 0$, then $h(\eta) = \log\max\{|p|, q\}$. The following are some of the properties of the logarithmic height function $h(\cdot)$, which will be used in the next sections of this paper without reference:

$$\begin{aligned} h(\eta \pm \gamma) &\leq h(\eta) + h(\gamma) + \log 2,\\ h(\eta\gamma^{\pm 1}) &\leq h(\eta) + h(\gamma),\\ h(\eta^s) &= |s|h(\eta) \quad (s \in \mathbb{Z}). \end{aligned} \tag{9}$$

2.3. **Linear forms in logarithms and continued fractions.** The following result is due to Mignotte [13].

Theorem 2. *Consider three nonzero algebraic numbers γ_1, γ_2, and γ_3, which are either all real and greater than 1 or all complex of modulus 1 and all $\neq 1$. Moreover, assume that either the three numbers γ_1, γ_2, and γ_3 are multiplicatively independent, or two of these numbers are multiplicatively independent and the third is a root of unity. Put*

$$\mathcal{D} := [\mathbb{Q}(\gamma_1, \gamma_2, \gamma_3) : \mathbb{Q}]/[\mathbb{R}(\gamma_1, \gamma_2, \gamma_3) : \mathbb{R}].$$

Also, consider three coprime positive rational integers b_1, b_2, b_3, and the linear form

$$\Gamma := b_2 \log \gamma_2 - b_1 \log \gamma_1 - b_3 \log \gamma_3,$$

where the logarithms of the γ_i are arbitrary determinants of the logarithm, but which are all real or purely imaginary. Also, assume that

$$b_2 |\log \gamma_2| = b_1 |\log \gamma_1| + b_3 |\log \gamma_3| \pm |\Gamma|.$$

Put

$$d_1 = \gcd(b_1, b_2) = \frac{b_1}{b_1^{'}} = \frac{b_2}{b_2^{'}}, \quad d_3 = \gcd(b_3, b_2) = \frac{b_2}{b_2^{''}} = \frac{b_3}{b_3^{''}}$$

Let A_1, A_2, and A_3 be real numbers such that

$$A_i \geq \max\{4,\ 5.296|\log \gamma_i| - \log|\gamma_i| + 2\mathcal{D}h(\gamma_i)\}, \quad i = 1, 2, 3, \quad \textit{and} \quad \Omega := A_1 A_2 A_3 \geq 100.$$

Put

$$b^{'} := \left(\frac{b_1^{'}}{A_2} + \frac{b_2^{'}}{A_1}\right)\left(\frac{b_3^{''}}{A_2} + \frac{b_2^{''}}{A_3}\right) \quad \textit{and} \quad \log \mathcal{B} := \max\left\{0.882 + \log b^{'}, \frac{10}{\mathcal{D}}\right\}.$$

Then, either

$$\log|\Gamma| > -790.95 \Omega \mathcal{D}^2 (\log \mathcal{B})^2 > -307187 \mathcal{D}^5 (\log \mathcal{B})^2 \prod_{i=1}^{3} \max\left\{0.55, h(\gamma_i), \frac{|\log \gamma_i|}{\mathcal{D}}\right\},$$

or the following conditions hold:

(i) *there exist two nonzero rational integers r_0 and s_0 such that*

$$r_0 b_2 = s_0 b_1$$

with

$$|r_0| \leq 5.61 A_2 (\mathcal{D} \log \mathcal{D})^{\frac{1}{3}} \quad \textit{and} \quad |s_0| \leq 5.61 A_1 (\mathcal{D} \log \mathcal{D})^{\frac{1}{3}},$$

(ii) *there exist rational integers r_1, s_1, t_1, and t_2, with $r_1 s_1 \neq 0$, such that*

$$(t_1 b_1 + r_1 b_3) s_1 = r_1 b_2 t_2, \quad \gcd(r_1, t_1) = \gcd(s_1, t_2) = 1,$$

which also satisify

$$|r_1 s_1| \leq 5.61 \delta A_3 (\mathcal{D} \log \mathcal{D})^{\frac{1}{3}}, \quad |s_1 t_1| \leq 5.61 \delta A_1 (\mathcal{D} \log \mathcal{D})^{\frac{1}{3}}, \quad |r_1 t_2| \leq 5.61 \delta A_2 (\mathcal{D} \log \mathcal{D})^{\frac{1}{3}},$$

where

$$\delta := \gcd(r_1, s_1).$$

Moreover, when $t_1 = 0$ we can take $r_1 = 1$, and when $t_2 = 0$ we can take $s_1 = 1$.

Let γ_1 and γ_2 be positive and multiplicatively independent, we use a result of Laurent, Mignotte, and Nesterenko [10]. Namely, in this case let B_1 and B_2 be real numbers larger than 1 such that

$$\log B_i \geq \max\left\{h(\gamma_i), \frac{|\log \gamma_i|}{D}, \frac{1}{D}\right\}, \qquad \text{for} \quad i = 1, 2,$$

and put

$$b' = \frac{|b_1|}{D \log B_2} + \frac{|b_2|}{D \log B_1}.$$

Put

$$\Gamma = b_1 \log \gamma_1 + b_2 \log \gamma_2. \tag{10}$$

We note that $\Gamma \neq 0$ because γ_1 and γ_2 are multiplicatively independent. The following result is Corollary 2 in [10].

Theorem 3. *With the above notations, assuming that γ_1 and γ_2 are positive and multiplicatively independent, then*

$$\log |\Gamma| > -24.34 D^4 \left(\max \left\{ \log b' + 0.14, \frac{21}{D}, \frac{1}{2} \right\} \right)^2 \log B_1 \log B_2. \tag{11}$$

During the calculations, we get upper bounds on our variables which are too large, thus we need to reduce them. To do so, we use some results from the theory of continued fractions.

For the treatment of linear forms homogeneous in two integer variables, we use the well-known classical result in the theory of Diophantine approximation. The following lemma is the criterion of Legendre.

Lemma 2. *Let τ be an irrational number, $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots$ be all the convergents of the continued fraction expansion of τ and M be a positive integer. Let N be a nonnegative integer such that $q_N > M$. Then putting $a(M) := \max\{a_i : i = 0, 1, 2, \ldots, N\}$, the inequality*

$$\left| \tau - \frac{r}{s} \right| > \frac{1}{(a(M)+2)s^2},$$

holds for all pairs (r, s) of positive integers with $0 < s < M$.

For a nonhomogeneous linear form in two integer variables, we use a slight variation of a result due to Dujella and Pethő (see [7], Lemma 5a). For a real number X, we write $\|X\| := \min\{|X - n| : n \in \mathbb{Z}\}$ for the distance from X to the nearest integer.

Lemma 3. *Let M be a positive integer, $\frac{p}{q}$ be a convergent of the continued fraction expansion of the irrational number τ such that $q > 6M$, and A, B, μ be some real numbers with $A > 0$ and $B > 1$. Furthermore, let $\varepsilon := \|\mu q\| - M\|\tau q\|$. If $\varepsilon > 0$, then there is no solution to the inequality*

$$0 < |u\tau - v + \mu| < AB^{-w},$$

in positive integers u, v, and w with

$$u \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

Finally, the following lemma is also useful. It is Lemma 7 in [8].

Lemma 4 (Gúzman, Luca)**.** *If $m \geq 1$, $T > (4m^2)^m$, and $T > x/(\log x)^m$, then*

$$x < 2^m T (\log T)^m.$$

3. Proof of Theorem 1

3.1. **Calculations in the ranges $1 \leq n \leq 100$ and $1 \leq x \leq 100$.** We assume that $n \geq 1$, as the solution with $n = 0$ is trivial. Since $U_{n+1} < U_{n+1} + U_n < U_{n+2}$, then the Diophantine equation (3) has no solution with $x = 1$. Furthermore, when $n = 1$ we get that

$$U_m = 1 + r^x. \tag{12}$$

We rewrite (12) as

$$r^x = U_m - 1 = U_{\frac{m-\delta}{2}} V_{\frac{m+\delta}{2}}, \quad \text{where} \quad \delta \in \{\pm 1\}, \delta \equiv n (\text{mod } 4).$$

So $r^x | U_{m+1}U_{m-1}$, and by the Primitive Divisor Theorem for Lucas sequences, we have that $m-1 \leq 12$ or $m \leq 13$. After a simple computer search, we found no other solutions to (12) apart from the solution $(n, m, x) = (1, 3, 2)$ which is part of the solutions already lisited in Eq. 4.

We now assume that $x \geq 3$ and $r \geq 3$. Using the equation (3) and the inequality (6), we get

$$\alpha^{(n-1)x} \leq U_{n+1}^x < U_n^x + U_{n+1}^x = U_m \leq \alpha^{m-1},$$

and

$$\alpha^{m-2} \leq U_m = U_n^x + U_{n+1}^x < (U_n + U_{n+1})^x < U_{n+2}^x \leq \alpha^{(n+1)x}.$$

Thus, we have

$$(n-1)x + 1 < m < (n+1)x + 2. \tag{13}$$

Then, we consider Eq. (8) given in Lemma 1. We write Eq. 3 as

$$\left(\sum_{\substack{0\leq k\leq n\\ k\not\equiv n \pmod 2}} \binom{\frac{n+k-1}{2}}{k} r^k\right)^x + \left(\sum_{\substack{0\leq k\leq n+1\\ k\not\equiv n+1 \pmod 2}} \binom{\frac{n+k}{2}}{k} r^k\right)^x = \sum_{\substack{0\leq k\leq m\\ k\not\equiv m \pmod 2}} \binom{\frac{m+k-1}{2}}{k} r^k. \tag{14}$$

We assume that n is even. Thus, Eq. (14) becomes

$$\left(\frac{n}{2}r + \binom{\frac{n+2}{2}}{3} r^3 + \cdots\right)^x + \left(1 + \binom{\frac{n+2}{2}}{2} r^2 + \binom{\frac{n+4}{2}}{4} r^4 + \cdots\right)^x = 1 + \binom{\frac{m+1}{2}}{2} r^2 + \binom{\frac{m+3}{2}}{3} r^4 + \cdots,$$

which is equivalent to

$$\binom{\frac{n+2}{2}}{2} r^2 x + \binom{\frac{n+4}{2}}{4} r^4 x + \cdots \equiv \binom{\frac{m+1}{2}}{2} r^2 + \binom{\frac{m+3}{2}}{3} r^4 + \cdots \pmod{r^x}. \tag{15}$$

Thus, when n is even, we need to check that

$$r^{\min\{x,4\}} \mid \binom{\frac{m+1}{2}}{2} - x\binom{\frac{n+2}{2}}{2}. \tag{16}$$

Similarly, when n is odd, we need to check that

$$r^{\min\{x,4\}} \mid \binom{\frac{m+1}{2}}{2} - x\binom{\frac{n+1}{2}}{2}. \tag{17}$$

After a computer search in Mathematica, on Eq. (3) with the conditions (16) and (17), and the ranges $3 \leq x \leq 100$ and $3 \leq r \leq 100$, we found no other solutions.

From now on, we assume that $n \geq 100$ and $x \geq 100$.

3.2. **An inequality for x in terms of n, m, and r.** Now, we rewrite the equation (3) as

$$\frac{\alpha^m}{\alpha - \beta} - U_{n+1}^x = U_n^x + \frac{\beta^m}{\alpha - \beta}. \tag{18}$$

Dividing both sides of the above equation by U_{n+1}^x and using the inequality (7), we obtain

$$\left|\alpha^m(\alpha-\beta)^{-1}U_{n+1}^{-x} - 1\right| = \left(\frac{U_n}{U_{n+1}}\right)^x + \frac{\beta^m}{(\alpha-\beta)U_{n+1}^x} < 2\left(\frac{U_n}{U_{n+1}}\right)^x < \frac{2}{r^x}. \tag{19}$$

Put

$$\Lambda := \alpha^m(\alpha-\beta)^{-1}U_{n+1}^{-x} - 1 \quad \text{and} \quad \Gamma := m\log\alpha - \log(\alpha-\beta) - x\log U_{n+1}. \tag{20}$$

We observe that $\Lambda = e^\Gamma - 1$, where Λ and Γ is given by (20). Since $|\Lambda| < 1/2$, we have that $e^\Gamma < 2$ and using the inequality (19) we obtain

$$|\Gamma| = \left|m\log\alpha - \log(\sqrt{r^2+4}) - x\log U_{n+1}\right| \leq e^{\Gamma}|e^{\Gamma} - 1| < 2|\Lambda| < \frac{4}{r^x}, \tag{21}$$

We apply Theorem 2 with the following data:

$$\gamma_1 = \alpha - \beta = \sqrt{r^2+4}, \quad \gamma_2 = \alpha, \quad \gamma_3 = U_{n+1}, \quad b_1 = 1, \quad b_2 = m, \quad b_3 = x.$$

We need to check that $\Gamma \neq 0$, that is, we need to check that γ_1, γ_2, and γ_3 are multiplicatively independent. ....

Since $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{Q}(\alpha)$, with degree 2, then $\mathcal{D} = 2$. Since

$$h(\gamma_1) = h(\alpha-\beta) = \frac{1}{2}\log(r^2+4), \quad h(\gamma_2) = \frac{1}{2}\log\alpha, \quad \text{and} \quad h(\gamma_3) = h(U_{n+1}) \leq h(\alpha^n) = \frac{1}{2}n\log\alpha,$$

we take

$$A_1 := 4.148\log(r^2+4), \quad A_2 := 6.296\log\alpha, \quad \text{and} \quad A_3 := 6.296n\log\alpha.$$

We also take $\Omega = A_1A_2A_3 = 164.43n(\log\alpha)^2\log(r^2+4) \geq 100$. Then,

$$\begin{aligned} b^{'} &= \left(\frac{1}{6.296\log\alpha} + \frac{m}{4.148\log(r^2+4)}\right)\left(\frac{x}{6.296\log\alpha} + \frac{m}{6.296n\log\alpha}\right) \\ &< \frac{(m+1)}{\log r}\left(\frac{1}{8.296} + \frac{1}{6.296}\right)\frac{(x+2)}{\log r}\left(\frac{1}{6.296} + \frac{1}{6.296}\right) \\ &< \frac{(m+1)(x+2)}{7(\log r)^2}, \end{aligned}$$

and

$$\log\mathcal{B} = \max\{0.882 + \log b^{'}, 5\}.$$

If $0.882 + \log b^{'} < 5$, then we have

$$\begin{aligned} \log|\Gamma| &> -790.95 \times 164.43n(\log\alpha)^2\log(r^2+4) \times 2^2 \times 5^2 \\ &> -1.3 \times 10^7 n(\log\alpha)^2\log(r^2+4). \end{aligned}$$

By comparing the above inequality with (21), we get that

$$x\log r - 2\log 2 < 1.3 \times 10^7 n(\log\alpha)^2\log(r^2+4),$$

which implies that

$$x < 2.8 \times 10^7 n(\log r)^2. \tag{22}$$

On the other hand, if $0.882 + \log b^{'} > 5$, then

$$\log\mathcal{B} := \log\left(\frac{(m+1)(x+2)}{3(\log r)^2}\right).$$

Thus, we get that

$$\begin{aligned} \log|\Gamma| &> -790.95 \times 164.43n(\log\alpha)^2\log(r^2+4) \times 2^2 \times \left(\log\left(\frac{(m+1)(x+2)}{3(\log r)^2}\right)\right)^2 \\ &> -5.2 \times 10^5 n(\log\alpha)^2\log(r^2+4)\left(\log\left(\frac{(m+1)(x+2)}{3(\log r)^2}\right)\right)^2. \end{aligned}$$

Comparing this inequality with (21), we get that

$$x\log r - 2\log 2 < 5.2 \times 10^5 n(\log\alpha)^2\log(r^2+4)\left(\log\left(\frac{(m+1)(x+2)}{3(\log r)^2}\right)\right)^2,$$

which implies that

$$x < 1.1 \times 10^6 n(\log r)^2\left(\log\left(\frac{(m+1)(x+2)}{3(\log r)^2}\right)\right)^2. \tag{23}$$

Using the inequality (13), we know that $m+1<(n+1)x+3<(n+1)(x+2)$ and substuting this in (23), we get that

$$x<1.1\times 10^6 n(\log r)^2\left(\log\left(\frac{(n+1)(x+2)^2}{3(\log r)^2}\right)\right)^2. \tag{24}$$

3.3. **An absolute upper bound on x.** We consider the element

$$y:=\frac{x}{\alpha^{2n}}.$$

The inequality (24) implies that

$$y<\frac{1}{\alpha^n}, \tag{25}$$

where the last inequality holds for all $n\geq 100$. Now, we write

$$U_n^x=\frac{\alpha^{nx}}{(r^2+4)^{x/2}}\left(1-\frac{(-1)^n}{\alpha^{2n}}\right)^x,$$

and

$$U_{n+1}^x=\frac{\alpha^{(n+1)x}}{(r^2+4)^{x/2}}\left(1-\frac{(-1)^{n+1}}{\alpha^{2(n+1)}}\right)^x.$$

If n is odd, then

$$1<\left(1-\frac{(-1)^n}{\alpha^{2n}}\right)^x=\left(1+\frac{1}{\alpha^{2n}}\right)^x<e^y<1+2y,$$

because y is very small. On the other hand, if n is even, then

$$1>\left(1-\frac{(-1)^n}{\alpha^{2n}}\right)^x=\exp\left(x\log\left(1-\frac{1}{\alpha^{2n}}\right)\right)^x>e^{-2y}>1-2y,$$

because y is very small. Thus, the following inequalities hold in both cases,

$$\left|U_n^x-\frac{\alpha^{nx}}{(r^2+4)^{x/2}}\right|<\frac{2y\alpha^{nx}}{(r^2+4)^{x/2}},$$

and

$$\left|U_{n+1}^x-\frac{\alpha^{(n+1)x}}{(r^2+4)^{x/2}}\right|<\frac{2y\alpha^{(n+1)x}}{(r^2+4)^{x/2}}.$$

Now, we return to (3) and rewrite it as

$$\begin{aligned}\frac{\alpha^m-\beta^m}{(r^2+4)^{1/2}}&=U_m=U_n^x+U_{n+1}^x\\&=\frac{\alpha^{nx}}{(r^2+4)^{x/2}}+\frac{\alpha^{(n+1)x}}{(r^2+4)^{x/2}}+\left(U_n^2-\frac{\alpha^{nx}}{(r^2+4)^{x/2}}\right)+\left(U_{n+1}^x-\frac{\alpha^{(n+1)x}}{(r^2+4)^{x/2}}\right),\end{aligned}$$

or

$$\begin{aligned}\left|\frac{\alpha^m}{(r^2+4)^{1/2}}-\frac{\alpha^{nx}(1+\alpha^x)}{(r^2+4)^{x/2}}\right|&=\left|\frac{\beta^m}{(r^2+4)^{1/2}}+\left(U_n^x-\frac{\alpha^{nx}}{(r^2+4)^{x/2}}\right)+\left(U_{n+1}^x-\frac{\alpha^{(n+1)x}}{(r^2+4)^{x/2}}\right)\right|\\&<\frac{1}{\alpha^m}+\left|U_n^x-\frac{\alpha^{nx}}{(r^2+4)^{x/2}}\right|+\left|U_{n+1}^x-\frac{\alpha^{(n+1)x}}{(r^2+4)^{x/2}}\right|\\&<\frac{1}{\alpha^m}+2y\left(\frac{\alpha^{nx}(1+\alpha^x)}{(r^2+4)^{x/2}}\right).\end{aligned}$$

Multiplying both sides of the above inequality by $\alpha^{-(n+1)x}(r^2+4)^{x/2}$, we obtain that

$$\left|\alpha^{m-(n+1)x}(r^2+4)^{(x-1)/2}-(1+\alpha^{-x})\right|<\frac{(r^2+4)^{x/2}}{\alpha^{m+(n+1)x}}+2y(1+\alpha^{-x})<\frac{1}{2\alpha^n}+3y<\frac{3}{\alpha^n}, \tag{26}$$

where we have used the fact that $(r^2+4)^{x/2}\alpha^{-(n+1)x} < 1/2$ for large n, $m \ge (n-1)x \ge n$, and $\alpha^x \ge \alpha^3 > ...$, as well as the inequality (25). Hence, we conclude that

$$\left|\alpha^{m-(n+1)x}(r^2+4)^{(x-1)/2} - 1\right| < \frac{1}{\alpha^x} + \frac{3}{\alpha^n} \le \frac{4}{\alpha^\ell}, \tag{27}$$

where $\ell := \min\{n, x\}$. Now, we set

$$\Lambda_1 := \alpha^{m-(n+1)x}(r^2+4)^{(x-1)/2} - 1, \tag{28}$$

and observe that $\Lambda_1 \neq 0$. Indeed, if $\Lambda_1 = 0$, then $\alpha^{2(m-(n+1)x)} = (r^2+4)^{x-1} \in \mathbb{Z}$, which is possible only when $m = (n+1)x$. However, if this is so, then we would get that $0 = \Lambda_1 = (r^2+4)^{(x-1)/2} - 1$, which leads to the conclusion that $x = 1$, which is impossible. Hence $\Lambda_1 \neq 0$. Next, we notice that since $x \ge 3$ and $n \ge 100$, we have that

$$|\Lambda_1| < \frac{1}{\alpha^3} + \frac{3}{\alpha^n} < \frac{1}{2}, \tag{29}$$

so that $\alpha^{m-(n+1)x}(r^2+4)^{(x-1)/2} \in [1/2, 3/2]$. In particular,

$$(n+1)x - m < \frac{1}{\log\alpha}\left(\frac{(x-1)}{2}\log(r^2+4) + \log(3/2)\right) < x\left(\frac{\log(r^2+4)}{2\log\alpha}\right) < 1.8x, \tag{30}$$

and

$$(n+1)x - m > \frac{1}{\log\alpha}\left(\frac{(x-1)}{2}\log(r^2+4) - \log 2\right) > 1.5x - 2 > 0. \tag{31}$$

We put

$$\Gamma_1 := (x-1)\log\left(\sqrt{r^2+4}\right) - ((n+1)x - m)\log\alpha.$$

Observe that $\Lambda_1 = e^{\Gamma_1} - 1$, where Λ_1 is given by (28). Since $|\Lambda_1| < 1/2$, we have that $e^{\Gamma_1} < 2$ and using the inequality (27) we obtain

$$|\Gamma_1| = \left|(x-1)\log(\sqrt{r^2+4}) - ((n+1)x-m)\log\alpha\right| \le e^{\Gamma_1}|e^{\Gamma_1} - 1| < 2|\Lambda_1| < \frac{8}{\alpha^\ell}, \tag{32}$$

where $\ell := \min\{n, x\}$. We apply Theorem 3 with the data:

$$t := 2, \quad \gamma_1 := \sqrt{r^2+4}, \quad \gamma_2 := \alpha, \quad b_1 := x-1, \quad b_2 := m - (n+1)x. \tag{33}$$

Since $\gamma_1, \gamma_2 \in \mathbb{Q}(\alpha)$, we take $\mathbb{K} := \mathbb{Q}(\alpha)$ with degree $D := 2$. The fact that γ_1 and γ_2 are multiplicatively independent follows because α is a unit, whereas $\sqrt{r^2+4}$ is not since the norm of $\sqrt{r^2+4}$ is $-r^2-4 \neq \pm 1$.

We take

$$\log B_1 = \max\left\{h(\gamma_1), \frac{|\log\gamma_1|}{2}, \frac{1}{2}\right\} = \frac{1}{2}\log(r^2+4),$$

and

$$\log B_2 = \max\left\{h(\gamma_2), \frac{|\log\gamma_2|}{2}, \frac{1}{2}\right\} = \frac{1}{2}\log\alpha.$$

Thus,

$$b' = \frac{|b_1|}{D\log B_2} + \frac{|b_2|}{D\log B_1} = \frac{x-1}{\log(r^2+4)} + \frac{(n+1)x-m}{\log\alpha} < \frac{1.2x}{\log r}.$$

By Theorem 3, we get that

$$\log|\Gamma_1| > -97.36\left(\max\left\{\log\left(\frac{1.4x}{\log r}\right), 10.5\right\}\right)^2 \log(r^2+4)\log\alpha.$$

If $\log\left(\frac{1.4x}{\log r}\right) < 10.5$, then

$$\log|\Gamma_1| > -10733.94\log(r^2+4)\log\alpha. \tag{34}$$

By comparing the inequalities (32) and (34), we get that

$$\ell \log\alpha - \log 8 < 10733.94 \log(r^2+4)\log\alpha,$$

which implies that

$$\min\{n,x\} = \ell < 3.22 \times 10^4 \log r. \tag{35}$$

On the other hand, if $\log\left(\frac{1.4x}{\log r}\right) > 10.5$, then

$$\log|\Gamma_1| > -97.36\left(\log\left(\frac{1.4x}{\log r}\right)\right)^2 \log(r^2+4)\log\alpha. \tag{36}$$

By comparing the inequalities (32) and (36), we get that

$$\ell\log\alpha - \log 8 < 97.36\left(\log\left(\frac{1.4x}{\log r}\right)\right)^2 \log(r^2+4)\log\alpha,$$

which implies that

$$\min\{n,x\} = \ell < 300\left(\log\left(\frac{x}{\log r}\right)\right)^2 \log r,$$

where we have used that $\log\left(\frac{1.2x}{\log r}\right)+0.14 < \log\left(\frac{1.4x}{\log r}\right) < 1.01\log\left(\frac{x}{\log r}\right)$ for all $x \geq 3$ and $\log(r^2+4) < 3\log r$ for all $r \geq 3$. If $\ell = x$, then we have that

$$x < 300\left(\log\left(\frac{x}{\log r}\right)\right)^2 \log r. \tag{37}$$

Then we apply Lemma 4 to (37) with the data: $m := 2$, $x := x/\log r$, and $T := 300$. Then

$$x < 3.90 \times 10^4 \log r.$$

If $\ell = n$, then

$$n < 300\left(\log\left(\frac{x}{\log r}\right)\right)^2 \log r. \tag{38}$$

We recall that, by (24), we have

$$x < 1.1 \times 10^6 n(\log r)^2 \left(\log\left(\frac{(n+1)(x+2)^2}{3(\log r)^2}\right)\right)^2.$$

Then, using (38), we get that

$$x < 3.3 \times 10^8 \left(\log\left(\frac{x}{\log r}\right)\right)^2 (\log r)^3 \left(\log\left(100\left(\log\left(\frac{x}{\log r}\right)\right)^2 \frac{(x+2)^2}{\log r}\right)\right)^2. \tag{39}$$

Using the fact that $x+2 < 1.0001x$ for $x \geq 10^6$, we rewrite (39) as

$$\frac{x}{\log r} < 3.3 \times 10^8 \left(\log\left(\frac{x}{\log r}\right)\right)^2 (\log r)^2 \left(\log\left(100.02\left(\log\left(\frac{x}{\log r}\right)\right)^2 \left(\frac{x}{\log r}\right)^2 \log r\right)\right)^2. \tag{40}$$

Then, we take the substitution $z := \frac{x}{\log r}$ in (40) to get

$$z < 3.3 \times 10^8 (\log z)^2 (\log r)^2 \left(\log(100.02\log r) + 2\log z + 2\log\log z\right)^2,$$

which can also be written as

$$\frac{z}{(\log z)^2\left(\frac{1}{2}\log(100.02\log r) + \log z + \log\log z\right)^2} < 1.32 \times 10^9 (\log r)^2. \tag{41}$$

We now consider the function

$$f(z) = \frac{z}{(\log z)^2\left(\frac{1}{2}\log(100.02\log r) + \log z + \log\log z\right)^2},$$

whose first derivative with respect to z is given by

$$f'(z) = \frac{1}{(\log z)^4 \left(\frac{1}{2}\log(100.02\log r) + \log z + \log\log z\right)^2} \times \left[1 - \frac{2}{\log z} - 2\left(1 + \frac{1}{\log z}\right)\frac{1}{\frac{1}{2}\log(100.02\log r) + \log z + \log\log z}\right].$$

We notice that if $\log z \leq 4$, then the function $f(z)$ is decreasing. Thus, in this case, we have that

$$\log\left(\frac{x}{\log r}\right) \leq 4,$$

which implies that

$$x \leq e^4 \log r. \tag{42}$$

On the other hand, if $\log z > 4$, then the function $f(z)$ is increasing. Thus, we return to (32) and rewrite it as

$$\left|\frac{(x-1)}{2}\log(r^2+4) - ((n+1)x - m)\log\left(\frac{1}{2}(r + \sqrt{r^2+4})\right)\right| < \frac{8}{\alpha^\ell}. \tag{43}$$

Then, using the fact that $r \geq 3$, we take

$$\begin{aligned}
\log(r^2+4) &= \log\left(r^2\left(1 + \left(\frac{2}{r}\right)^2\right)\right) = 2\log r + \log\left(1 + \left(\frac{2}{r}\right)^2\right) \\
&= 2\left(\log r + \frac{2}{r^2} - \frac{4}{r^4} + \frac{32}{3r^6} - \frac{32}{r^8} + \cdots\right) \\
&= 2\left(\log r + \frac{2}{r^2} + \zeta(r)\right),
\end{aligned}$$

where

$$|\zeta(r)| \leq \frac{4}{r^4}\sum_{k=0}^{\infty}\left|\frac{2}{r}\right|^{2k} \leq \frac{4}{r^4}\sum_{k=0}^{\infty}\left(\frac{4}{9}\right)^k = \frac{36}{5r^4} < \frac{8}{r^4}.$$

Similary,

$$\begin{aligned}
\log\left(\frac{r + \sqrt{r^2+4}}{2}\right) &= \log\left(r\left(\frac{1 + \sqrt{1 + \left(\frac{2}{r}\right)^2}}{2}\right)\right) = \log r + \log\left(1 + \frac{1}{2}\left(\left(1 + \left(\frac{2}{r}\right)^2\right)^{\frac{1}{2}} - 1\right)\right) \\
&= \log r + \frac{1}{2}\left(\left(1 + \left(\frac{2}{r}\right)^2\right)^{\frac{1}{2}} - 1\right) - \frac{1}{4}\left(\left(1 + \left(\frac{2}{r}\right)^2\right)^{\frac{1}{2}} - 1\right)^2 + \cdots.
\end{aligned}$$

But,

$$\begin{aligned}
\left(1 + \left(\frac{2}{r}\right)^2\right)^{\frac{1}{2}} &= 1 + \frac{1}{2}\left(\frac{2}{r}\right)^2 - \frac{1}{8}\left(\frac{2}{r}\right)^4 + \frac{3}{48}\left(\frac{2}{r}\right)^6 + \cdots \\
&= 1 + \frac{2}{r^2} - \frac{2}{r^4} + \cdots.
\end{aligned}$$

Then,

$$\begin{aligned}
\log\left(\frac{r + \sqrt{r^2+4}}{2}\right) &= \log r + \frac{1}{2}\left(\frac{2}{r^2} - \frac{2}{r^4} + \cdots\right) - \frac{1}{4}\left(\frac{2}{r^2} - \frac{2}{r^4} + \cdots\right)^2 + \cdots \\
&= \log r + \frac{1}{r^2} + \zeta_1(r),
\end{aligned}$$

where

$$
\begin{aligned}
|\zeta_1(r)| &\le \left|\frac{1}{2}\left(\frac{-2}{r^4}+\frac{4}{r^6}-\cdots\right)-\frac{1}{4}\left(\left(1+\left(\frac{2}{r}\right)^2\right)^{\frac{1}{2}}-1\right)^2\sum_{k=0}^{\infty}\left|\left(1+\left(\frac{2}{r}\right)^2\right)^{\frac{1}{2}}-1\right|^{2k}\right|\\
&\le \left|\frac{1}{2}\left(\frac{-2}{r^4}+\frac{4}{r^6}-\cdots\right)-\frac{1}{4}\left(\frac{2}{r^2}-\frac{2}{r^4}+\cdots\right)^2\sum_{k=0}^{\infty}\left(\frac{1}{64}\right)^k\right|\\
&\le \left|\frac{1}{2}\left(\frac{-2}{r^4}+\frac{4}{r^6}-\cdots\right)-\frac{1}{2}\left(\frac{2}{r^2}-\frac{2}{r^4}+\cdots\right)^2\right|\\
&\le \left|\frac{-2}{r^4}+\frac{4}{r^6}-\cdots\right| \le \frac{2}{r^4}\sum_{k=0}^{\infty}\left(\frac{4}{9}\right)^k < \frac{4}{r^4}.
\end{aligned}
$$

Thus, we have

$$
\left|(x-1)\left(\log r+\frac{2}{r^2}+\zeta(r)\right)-((n+1)x-m)\left(\log r+\frac{1}{r^2}+\zeta_1(r)\right)\right| < \frac{8}{\alpha^\ell},
$$

which implies that

$$
\begin{aligned}
\left|(m-nx-1)\log r+\frac{(m-nx-1)+(x-1)}{r^2}\right| &< (x-1)|\zeta(r)|+((n+1)x-m)|\zeta_1(r)|+\frac{8}{\alpha^\ell}\\
&< \frac{8(x-1)}{r^4}+\frac{4((x-1)-(m-nx-1))}{r^4}+\frac{8}{\alpha^\ell}.
\end{aligned}
$$

We let $-\kappa := m-nx-1$.

Lemma 5. *The following holds:*

(i) *$\kappa \neq 1$;*

(ii) *If $\kappa = 2$ and $n \ge 3$ then $x \ge r^{\max\{2,n-3\}}$;*

(iii) *If $\kappa \ge 3$ and $r > 257$, then $\kappa > n/4$.*

Proof. (i). If $\kappa = 1$, then $m = nx$. So the equation (3) becomes

$$
U_n^x + U_{n+1}^x = U_{nx}.
$$

If p is prime dividing U_n (which exists since $n > 1$), then $p \mid U_n^x$ and $p \mid U_n \mid U_{nx}$, so from the above equation we get $p \mid U_{n+1}$, a contradiction since $\gcd(U_n, U_{n+1}) = U_{\gcd(n,n+1)} = 1$.

(ii) In this case $m = nx-1$ so the equation (3) becomes

$$
U_n^x + U_{n+1}^x = U_{nx-1}.
$$

In particular, $U_{nx-1} - U_{n+1}^x \equiv 0 \pmod{U_n^2}$. We study this congruence. In what follows for three algebraic integers a, b, c, we write $a \equiv b \pmod{c}$ if $(a-b)/c$ is an algebraic integer. Write

$$
U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad \text{as} \qquad \alpha^n = \beta^n + \sqrt{\Delta}U_n,
$$

where $\sqrt{\Delta} = \alpha - \beta = \sqrt{r^2+4}$. Then

$$
\alpha^{nx} = (\beta^n + \sqrt{\Delta}U_n)^x \equiv \beta^{nx} + x\beta^{n(x-1)}\sqrt{\Delta}U_n \pmod{\Delta U_n^2}.
$$

Thus,

$$\begin{aligned} U_{nx-1} &= \frac{\alpha^{nx}\alpha^{-1}-\beta^{nx-1}}{\sqrt{\Delta}} \\ &\equiv \frac{(\beta^{nx}+x\beta^{n(x-1)}\sqrt{\Delta}U_n)\alpha^{-1}-\beta^{nx-1}}{\alpha-\beta} \pmod{\sqrt{\Delta}U_n^2} \\ &\equiv \frac{\beta^{nx}\alpha^{-1}-\beta^{nx}\beta^{-1}}{\alpha-\beta}+x\beta^{n(x-1)}\alpha^{-1}U_n \pmod{U_n^2} \\ &\equiv \beta^{nx}+x\beta^{n(x-1)}\alpha^{-1}U_n \pmod{U_n^2}. \end{aligned}$$

On the other hand

$$\begin{aligned} U_{n+1}^x &= \left(\frac{\alpha^{n+1}-\beta^{n+1}}{\sqrt{\Delta}}\right)^x \\ &\equiv \left(\frac{(\beta^n+\sqrt{\Delta}U_n)\alpha-\beta^{n+1}}{\sqrt{\Delta}}\right)^x \pmod{U_n^2} \\ &\equiv (\beta^n+U_n\alpha)^x \pmod{U_n^2} \\ &\equiv \beta^{nx}+x\beta^{n(x-1)}\alpha U_n \pmod{U_n^2}. \end{aligned}$$

Thus,

$$\begin{aligned} U_{nx-1}-U_{n+1}^x &\equiv (\beta^{nx}+x\beta^{n(x-1)}\alpha^{-1}U_n)-(\beta^{nx}-x\beta^{n(x-1)}\alpha U_n) \pmod{U_n^2} \\ &\equiv x\beta^{n(x-1)}(\alpha^{-1}-\alpha)U_n \pmod{U_n^2} \\ &\equiv -x\beta^{n(x-1}rU_n \pmod{U_n^2}. \end{aligned}$$

In the last step above we used the fact that $\alpha^{-1}-\alpha=-\beta-\alpha=-r$. Since the expression $U_{nx-1}-U_{n+1}^x$ is divisible by U_n^2, we get that $U_n^2 \mid \beta^{n(x-1)}xrU_n$. Since β is a unit, we get that $U_n \mid xr$. For $n=2$, this gives us nothing since $U_2=r$. For $n=3$, $U_3=r^2+1$ is coprime to r, so $U_3 \mid x$, which gives $x \ge r^2+1>r^2$. For $n=4$, we have $U_4=r(r^2+2)$ divides rx, so $r^2+2 \mid x$ giving $x \ge r^2+2>r^2$. Finally, for $n \ge 5$, we have that $U_n>\alpha^{n-2}>r^{n-2}$ and so $x \ge U_n/r \ge r^{n-3}$. This proves (ii).

(iii) We put $V_n=\alpha^n+\beta^n$. It is well-known that

$$U_{n+1}^2-U_nU_{n+2}=(-1)^n \qquad \text{and} \qquad V_n^2-\Delta U_n^2=4(-1)^n.$$

In particular, $U_{n+1}^4 \equiv 1 \pmod{U_n}$ and $V_n^4 \equiv 16 \pmod{U_n}$. We also use that $U_{-m}=(-1)^{m-1}U_m$, $V_{-m}=(-1)^mV_m$ and

$$2U_{m+n}=U_mV_n+U_nV_m.$$

Armed with these facts, writing $m=nx-(\kappa-1)$ and

$$U_n^x+U_{n+1}^x=U_{nx-(\kappa-1)}$$

we multiply both sides of the above equation by 2 and write

$$2U_n^2+2U_{n+1}^x=2U_{nx-(\kappa-1)}=U_{nx}V_{-(\kappa-1)}+V_{nx}U_{-(\kappa-1)}.$$

We raise both sides of the above equation to the forth power and reduce it modulo U_n taking into account that $U_n \mid U_{nx}$ and $V_{nx}^4 \equiv 16 \pmod{U_{nx}} \equiv 16 \pmod{U_n}$, and get

$$\begin{aligned} 16 &\equiv 16(U_{n+1}^4)^x \pmod{U_n} \equiv (U_{nx}V_{-(\kappa-1)} \pm V_{nx}U_{\kappa-1})^4 \pmod{U_n} \\ &\equiv V_{nx}^4U_{\kappa-1}^4 \pmod{U_n} \equiv 16U_{\kappa-1}^4 \pmod{U_n}. \end{aligned}$$

Thus, $U_n \mid 16(U_{\kappa-1}^4-1)$. The right–hand side is nonzero since $\kappa>2$. Thus,

$$\alpha^{n-2} \le U_n < 16U_{\kappa-1}^4 < 16(2\alpha^{\kappa-1})^4 = 256\alpha^{4(\kappa-1)} < \alpha^{4(\kappa-1)+1}$$

for $r>257$. We thus get that $4(\kappa-1) \ge n-2$, so $\kappa \ge 1+(n-2)/4=(n+2)/4>n/4$, which is what we wanted.

$\square$

Then, we have that

$$\begin{aligned}\left|-\kappa\log r+\frac{x-1-\kappa}{r^2}\right| &< \frac{8(x-1)}{r^4}+\frac{4((x-1)+\kappa)}{r^4}+\frac{8}{\alpha^\ell}\\ &< \frac{8x}{r^3}+\frac{8\max\{x-1,|\kappa|\}}{r^4}\\ &< \frac{16\max\{x-1,|\kappa|\}}{r^3},\end{aligned}$$

where we have used that $\ell\ge 3$. Thus,

$$\left|-\kappa\log r+\frac{x-1-\kappa}{r^2}\right| < \frac{16\max\{x-1,|\kappa|\}}{r^3}. \tag{44}$$

Now the argument is splitted into three cases.

Case 1. If $\kappa\le 0$, then

$$\left|-\kappa\log r+\frac{x-1-\kappa}{r^2}\right| = |\kappa|\log r+\frac{|\kappa|}{r^2}+\frac{x-1}{r^2}.$$

So,

$$\frac{16\max\{x-1,|\kappa|\}}{r^3}\ge\frac{\max\{x-1,|\kappa|\}}{r^2},$$

which implies that $r\le 16$.

Case 2. If $\kappa>0$ and $x-1-\kappa\le 0$, then

$$\begin{aligned}\left|-\kappa\log r+\frac{x-1-\kappa}{r^2}\right| &= \kappa\log r+\frac{\kappa+1-x}{r^2}\ge\max\left\{\kappa\log r,\frac{\kappa+1-x}{r^2}\right\}\\ &\ge\max\left\{(x-1)\log r,\frac{\kappa+1-x}{r^2}\right\}.\end{aligned}$$

Thus, if $\kappa\le 2(x-1)$, then

$$\frac{32(x-1)}{r^3}\ge\frac{16\max\{x-1,\kappa\}}{r^3}\ge(x-1)\log r,$$

which gives $r^3\log r\le 32$, which implies that $r\le 3$. On the other hand, if $\kappa\ge 2x-1$, then $x\le(\kappa+1)/2$, which implies that

$$\kappa+1-x\ge\kappa+1-\frac{\kappa+1}{2}>\frac{\kappa}{2}.$$

Thus, we have that

$$\frac{16\kappa}{r^3}>\frac{\kappa/2}{r^2},$$

which gives $r<32$.

Case 3. If $\kappa>0$ and $x-1-\kappa>0$, then

$$\left|-\kappa\log r+\frac{x-1-\kappa}{r^2}\right| = \kappa\log r+\frac{x-1-\kappa}{r^2}.$$

Thus, we have that

$$\frac{16(x-1)}{r^3}\ge\kappa\log r+\frac{x-1-\kappa}{r^2}>\kappa\log r+\frac{x-1}{r^2}.$$

This gives that

$$\kappa\log r<\frac{x-1}{r^2}\left(1+\frac{16}{r}\right).$$

Thus,

$$\frac{x}{\log r} > \frac{\kappa r^2}{\left(1+\frac{16}{r}\right)}. \tag{45}$$

Now we replace $z := \frac{x}{\log r}$ in (41) by $z := \frac{\kappa r^2}{\left(1+\frac{16}{r}\right)}$ given in (45) as follows.

$$\frac{\frac{\kappa r^2}{\left(1+\frac{16}{r}\right)}}{\left(\log\left(\frac{\kappa r^2}{\left(1+\frac{16}{r}\right)}\right)\right)^2\left(\frac{1}{2}\log(100.02\log r)+\log\left(\frac{\kappa r^2}{\left(1+\frac{16}{r}\right)}\right)+\log\log\left(\frac{\kappa r^2}{\left(1+\frac{16}{r}\right)}\right)\right)^2} < 1.32\times 10^9(\log r)^2. \tag{46}$$

By Lemma 5, if $\kappa \geq 3$ and $r > 257$, then $\kappa > n/4$. Since $n \geq 100$, we have that $\kappa \geq 25$. A simple computer search in *Mathematica* on (46), reveals that for $\kappa \geq 25$, we have that

$$r < 3.0 \times 10^8. \tag{47}$$

For $\kappa = 2$, we get that $r < 35$, which is much less than 3×10^8. Thus, (47) always holds for $n \geq 100$. Then, substituting (47) into (40), and with the help of a simple computer search in *Mathematica*, we obtained that

$$x < 4.4 \times 10^{19}. \tag{48}$$

To obtain an absolute upper bound on n, we substitute the inequalites (47) and (48) into (38). This gives

$$n < 1.1 \times 10^7. \tag{49}$$

3.4. **Reduction of the bounds on x and r with $n < 100$.** We recall that, from (21) we have that

$$\left|m\log\alpha - \log(\sqrt{r^2+4}) - x\log U_{n+1}\right| < \frac{4}{r^x}.$$

This can be rewritten as

$$\left|(m-(n+1)x)\log\alpha + \frac{x-1}{2}\log(r^2+4) - x\log\left(1-\frac{(-1)^{n+1}}{\alpha^{2(n+1)}}\right)\right| < \frac{4}{r^x}. \tag{50}$$

But

$$\begin{aligned}
|\zeta_2(r)| := \left|\log\left(1-\frac{(-1)^{n+1}}{\alpha^{2(n+1)}}\right)\right| &= \left|-\frac{(-1)^{n+1}}{\alpha^{n+1}} - \frac{1}{\alpha^{4(n+1)}} - \frac{(-1)^{3(n+1)}}{\alpha^{6(n+1)}} - \cdots\right| \\
&\leq \frac{1}{r^{2(n+1)}} + \frac{1}{r^{4(n+1)}} + \frac{1}{r^{6(n+1)}} + \cdots \\
&\leq \frac{1}{r^4}\sum_{k=0}^{\infty}\left|\frac{1}{r}\right|^{2k} = \frac{1}{r^4}\sum_{k=0}^{\infty}\left(\frac{1}{9}\right)^k = \frac{9}{8r^4} < \frac{2}{r^4}.
\end{aligned}$$

We also recall that

$$\log(r^2+4) = 2\left(\log r + \frac{2}{r^2} + \zeta(r)\right) \quad \text{and} \quad \log\alpha = \log r + \frac{1}{r^2} + \zeta_1(r),$$

where

$$|\zeta(r)| < \frac{8}{r^4} \quad \text{and} \quad |\zeta_1(r)| < \frac{4}{r^4}.$$

Thus, (50) becomes

$$\left|(m-(n+1)x)\left(\log r + \frac{1}{r^2} + \zeta_1(r)\right) + (x-1)\left(\log r + \frac{2}{r^2} + \zeta(r)\right) - x\zeta_2(r)\right| < \frac{4}{r^x},$$

which is equivalent to

$$\begin{aligned}
&\left|(m-nx-1)\log r+\frac{(m-nx-1)-(x-1)}{r^2}\right| \\
&< \frac{4}{r^x}+(x-1)|\zeta(r)|+((n+1)x-m)|\zeta_1(r)|+x|\zeta_2(r)| \\
&< \frac{4}{r^x}+\frac{8(x-1)}{r^4}+\frac{4(x-1-(m-nx-1))}{r^4}+\frac{2x}{r^4}.
\end{aligned}$$

We also recall that $-\kappa := m-nx-1$. Thus, we have that

$$\begin{aligned}
\left|-\kappa\log r+\frac{x-1-\kappa)}{r^2}\right| &< \frac{10(x-1)}{r^3}+\frac{8\max\{x-1,|\kappa|\}}{r^4} \\
&< \frac{18\max\{x-1,|\kappa|\}}{r^3}.
\end{aligned}$$

Going through similar arguments as before, we conclude that

$$\frac{x}{\log r} > \frac{\kappa r^2}{\left(1+\frac{18}{r}\right)}. \tag{51}$$

For $n<100$, we return to (24) and write

$$x < 1.1\times 10^6 n(\log r)^2\left(\log\left(\frac{(10(x+2))^2}{3(\log r)^2}\right)\right)^2, \tag{52}$$

which is equivalent to

$$\frac{10(x+2)}{\sqrt{3}(\log r)} < \frac{1.1}{\sqrt{3}}\times 10^7 n(\log r)\left(\log\left(\frac{(10(x+2))^2}{3(\log r)^2}\right)\right)^2. \tag{53}$$

We let $y := \frac{10(x+2)}{\sqrt{3}(\log r)}$, then we can write (53) as

$$y < \frac{4.4}{\sqrt{3}}\times 10^7 n(\log r)(\log y)^2. \tag{54}$$

Since $\frac{x+2}{\log r} > \frac{x}{\log r} > \frac{nr^2}{4\left(1+\frac{18}{r}\right)}$ (by Lemma 5 and (51)), then $y > \frac{10nr^2}{4\sqrt{3}\left(1+\frac{18}{r}\right)}$. Thus, we have that

$$\frac{\frac{r^2}{\left(1+\frac{18}{r}\right)}}{\left(\log\left(\frac{10nr^2}{4\sqrt{3}\left(1+\frac{18}{r}\right)}\right)\right)^2} < y < 1.1\times 10^6(\log r),$$

which also equivalent to

$$r^2 < 1.1\times 10^6(\log r)\left(1+\frac{18}{r}\right)\left(\log\left(\frac{1000r^2}{4\sqrt{3}\left(1+\frac{18}{r}\right)}\right)\right)^2, \tag{55}$$

which gives

$$r < 9.96\times 10^4. \tag{56}$$

Using (56), we return to (52) and rewrite it as

$$x < 1.1\times 10^8(\log(9.96\times 10^4))^2\left(\log\left(\frac{(10(x+2))^2}{3(\log(9.96\times 10^4))^2}\right)\right)^2.$$

This gives that

$$x < 6.08\times 10^{13}. \tag{57}$$

The bound (57) on x is too large, so we need to reduce it, to do so we return to (21) and rewrite it as

$$\left| x\frac{\log U_{n+1}}{\log \alpha} - m + \frac{\log(r^2+4)}{2\log \alpha}\right| < \frac{4}{r^x \log \alpha}, \tag{58}$$

Now, for each $n \in [2, 100]$ and each $r \in [3, 96000]$, we appply Lemma 3 on (58) with the data: $M := 6.08 \times 10^{13}$,

$$\tau(n,r) := \frac{\log U_{n+1}}{\log \alpha}, \quad \mu(r) := \frac{\log(r^2+4)}{2\log \alpha}, \quad A(r) := \frac{4}{\log \alpha}, \quad \text{and} \quad B(r) := r.$$

An intensive computer search in *Mathematica* reveals that $\varepsilon(n,r) := \|\mu(r)q(n,r)\| - M\|\tau(n,r)q(n,r)\| \geq 1.2212 \times 10^{-120} > 0$. Therefore, we calculated each value of $\lfloor \log(A(r)q(n,r)/\varepsilon(n,r))/\log r \rfloor$ and found that all of them are at most 2655. Thus, we have that $x \leq 2655$. Now, using ...(45), we have that $r^2/(1+18/r) < x \leq 2655$. So, we get that $r \leq 58$. We ran the computation again with the updated values; $M := 2655$, and $r \in [3, 58]$. A quick computer search in *Mathematica* revealed that $\varepsilon(n,r) > 2.202 \times 10^{-97}$. So, $x \leq 221$, and $r \leq 20$. Thus, we have

$$x \leq 221 \quad \text{and} \quad r \leq 20. \tag{59}$$

3.5. **Reduction of the bounds on x and r with $n \geq 100$.** The bound obtained on x given in (48) is too large to carry out meaningful computations, so we need to reduce it. To do so, we return to (32) and divide through the inequality by $(x-1)\log \alpha$ as follows.

$$\left| \frac{\log(\sqrt{r^2+4})}{\log \alpha} - \frac{(n+1)x - m}{x-1}\right| < \frac{8}{\alpha^{\ell}(x-1)\log \alpha}, \tag{60}$$

In this case, for each r in the range $[3, 300000000]$ with $M := 4.4 \times 10^{19}$, we apply the classical result from Diophantine approximation given in Lemma 2. We assume that ℓ is so large that the right-hand side of the inequality in (60) is smaller than $1/(2(x-1)^2)$. This certainly holds if

$$\alpha^{\ell} > 16(x-1)/\log \alpha. \tag{61}$$

Since $x - 1 < 4.4 \times 10^{19}$, it follows that the last inequality (61) holds provided that $\ell \geq 44$, which we now assume. In this case $r/s := ((n+1)x - m)/(x-1)$ is a convergent of the continued fraction of $\tau_r := \log(\sqrt{r^2+4})/\log \alpha$ and $x - 1 < M$. We are now set to apply Lemma 2.

We write $\tau_r := [a_0; a_1, a_2, a_3, \ldots]$ for the continued fraction expansion of τ_r and p_k/q_k for the $k-$th convergent. We get that $r/s = p_j/q_j$ for some $j \leq N_r$. Furthermore, putting $a(M) := \max\{a_j : j = 0, 1, \ldots, N_r\}$, we get $a(M) := 1756736372935842814$. By Lemma 2, we get

$$\frac{1}{1756736372935842816(x-1)^2} = \frac{1}{(a(M)+2)(x-1)^2} \leq \left|\tau_r - \frac{r}{s}\right| < \frac{8}{\alpha^{\ell}(x-1)\log \alpha},$$

which gives

$$\alpha^{\ell} < \frac{1756736372935842816 \times 8(x-1)}{\log \alpha} < \frac{1756736372935842816 \times 8 \times 4.4 \times 10^{19}}{\log \alpha}.$$

This implies that $\ell := \min\{n, x\} \leq 81$. Since, we are working under the assumption that $n \geq 100$, then $\min\{n, x\} = x \leq 81$. Then, we return to (45), find an absolute upper bound on r, and compare the bounds obtained in Case 1, Case 2, and Case 3. For $\kappa \geq 2$ and $x \leq 81$, a simple computer search on (45), revealed that $r \leq 9$. On comparing the bounds obtained in Case 1, Case 2, and Case 3, we conclude that $r \leq 32$. Thus, we have

$$x \leq 81 \quad \text{and} \quad r \leq 32. \tag{62}$$

3.6. **Reducing the bound on n.** The bound (49) ontained on x is too large to carry out meaningful computations, thus, we need to reduce it. To do so, we return to the inequality (26) and rewrite it as

$$\left|\alpha^{m-(n+1)x}(r^2+4)^{(x-1)/2}(1+\alpha^{-x})^{-1}-1\right| < \frac{3}{\alpha^n(1+\alpha^{-x})} < \frac{3}{\alpha^n}. \tag{63}$$

We put

$$\Lambda_x := \alpha^{m-(n+1)x}(r^2+4)^{(x-1)/2}(1+\alpha^{-x})^{-1}-1,$$

and

$$\Gamma_x := (m-(n+1)x)\log\alpha - (1-x)\log(\sqrt{r^2+4}) + \log(1/(1+\alpha^{-x})).$$

We use the assumption that $n \geq 100$ and go to (63). Note that $e^{\Gamma_x}-1 = \Lambda_x \neq 0$. Thus, $\Gamma_x \neq 0$. If $\Gamma_x < 0$, then

$$0 < |\Gamma_x| < e^{|\Gamma_x|}-1 = |\Lambda_x| < \frac{3}{\alpha^n}.$$

If $\Gamma_x > 0$, then we have that $|e^{\Gamma_x}-1| < 1/2$. Hence $e^{\Gamma_x} < 2$. Thus, we get that

$$0 < \Gamma_x < e^{\Gamma_x}-1 = e^{\Gamma_x}|\Lambda_x| < \frac{6}{\alpha^n}.$$

Therefore, in both cases, we have that

$$0 < |\Gamma_x| = \left|(m-(n+1)x)\log\alpha - (1-x)\log(\sqrt{r^2+4}) + \log(1/(1+\alpha^{-x}))\right| < \frac{6}{\alpha^n}.$$

Dividing through the above inequality by $\log(\sqrt{r^2+4})$, we get

$$0 < \left|(m-(n+1)x)\frac{\log\alpha}{\log(\sqrt{r^2+4})} - (1-x) + \frac{\log(1/(1+\alpha^{-x}))}{\log(\sqrt{r^2+4})}\right| < \frac{6}{\alpha^n\log(\sqrt{r^2+4})}. \tag{64}$$

We put

$$\tau(r) := \frac{\log\alpha}{\log(\sqrt{r^2+4})} \quad \text{and} \quad \mu(r,x) := \frac{\log(1/(1+\alpha^{-x}))}{\log(\sqrt{r^2+4})},$$

where $1 \leq x \leq 81$ and $3 \leq r \leq 32$. We can rewrite (64) as

$$0 < |(m-(n+1)x)\tau(r) - (1-x) + \mu(r,x)| < \frac{6}{\log(\sqrt{r^2+4})} \cdot \alpha^{-n}. \tag{65}$$

We now apply Lemma 3 on (65). We put $M := 1.1 \times 10^7$. An quick computer search in *Mathematica* reveals that $\varepsilon(r,x) := \|\mu(r,x)q(r)\| - M\|\tau(r)q(r)\| \geq 0.0000100786 > 0$. Therefore, with $A(r) := \lfloor 6/\log(\sqrt{r^2+4})\rfloor$ and $B := \alpha$ we calculated each value of $\log(A(r)q(r)/\varepsilon(r,x))/\log\alpha$ and found that all of them are at most 21. Thus, $n \leq 21$. This contradicts our assumption that $n \geq 100$. Hence, Theorem 1 holds. $\square$

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