The $x-$coordinates of Pell equations and sums of two Fibonacci numbers II

MAHADI DDAMULIRA [1,*] and FLORIAN LUCA[2,3,4]

[1] Institute of Analysis and Number Theory, Graz University of Technology, Kopernikusgasse 24/II, A-8010 Graz, Austria
[2] School of Mathematics, University of the Witwatersrand, Johannesburg, South Africa
[3] Research Group in Algebraic Structures and Applications, King Abdulaziz University, Jeddah, Saudi Arabia
[4] Department of Mathematics, University of Ostrava, 30 Dubna 22, 701 03 Ostrava 1, Czech Republic
[*]Corresponding author.
Email: mddamulira@tugraz.at; mahadi@aims.edu.gh; Florian.Luca@wits.ac.za

Abstract. Let $\{F_n\}_{n\geq 0}$ be the sequence of Fibonacci numbers defined by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. In this paper, for an integer $d \geq 2$ which is square-free, we show that there is at most one value of the positive integer x participating in the Pell equation $x^2 - dy^2 = \pm 4$ which is a sum of two Fibonacci numbers, with a few exceptions that we completely characterize.

Keywords. Fibonacci number; Pell equation; linear form in logarithm; reduction method.

2020 Mathematics Subject Classification. 11B39, 11D45, 11D61, 11J86.

1. Introduction

Let $\{F_n\}_{n\geq 0}$ be the sequence of Fibonacci numbers given by

$$F_0 = 0,\ F_1 = 1 \text{ and } F_{n+2} = F_{n+1} + F_n \text{ for all } n \geq 0.$$

The Fibonacci sequence is sequence $A000045$ on the On-Line Encyclopedia of Integer Sequences (OEIS). The first few terms of this sequence are

$$\{F_n\}_{n\geq 0} = 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, \ldots.$$

In this paper, we let $U := \{F_n + F_m : n \geq m \geq 0\}$ be the sequence of sums of two Fibonacci numbers. The first few members of U are

$$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 18, 21, 22, 23, 24, 26, 29, 34, 35, \ldots\}.$$

Let $d \geq 2$ be a positive integer which is not a square. It is well known that the Pell equation

$$x^2 - dy^2 = \pm 4, \tag{1}$$

has infinitely many positive integer solutions (x, y). By putting (x_1, y_1) for the smallest positive solutions to (1), all solutions are of the forms (x_k, y_k) for some positive integer k, where

$$\frac{x_k + y_k\sqrt{d}}{2} = \left(\frac{x_1 + y_1\sqrt{d}}{2}\right)^k \qquad \text{for all} \quad k \geq 1,$$

Furthermore, the sequence $\{x_k\}_{k\geq 1}$ is binary recurrent. In fact, the following formula

$$x_k = \left(\frac{x_1 + y_1\sqrt{d}}{2}\right)^k + \left(\frac{x_1 - y_1\sqrt{d}}{2}\right)^k,$$

holds for all positive integers k.

Recently, Gómez and Luca [2] studied the Diophantine equation

$$x_k = F_m + F_n, \quad \text{with} \quad n \geq m \geq 0, \tag{2}$$

where x_k are the x−coordinates of the solutions of the Pell equation $x^2 - dy^2 = \pm 1$ for some positive integer k and $\{F_n\}_{n\geq 0}$ is the sequence of Fibonacci numbers. They proved that for each square free integer $d \geq 2$, there is at most one positive integer k such that x_k admits the representation (3) for some nonnegative integers $0 \leq m \leq n$, except for $d \in \{2, 3, 5, 11, 30\}$. Furthermore, they explicitly stated all the solutions for these exceptional cases.

In the same spirit, Bravo et al. [1] studied the Diophantine equation

$$x_k = T_m + T_n, \quad \text{with} \quad n \geq m \geq 0. \tag{3}$$

where x_k are the x−coordinates of the solutions of the Pell equation $x^2 - dy^2 = \pm 1$ for some positive integer k and $\{T_n\}_{n\geq 0}$ is the sequence of Tribonacci numbers given by $T_0 = 0,\ T_1 = 1 = T_2$ and $T_{n+3} = T_{n+2} + T_{n+1} + T_n$ for all $n \geq 0$. They proved that for each square free integer $d \geq 2$, there is at most one positive integer k such that x_k admits the representation (3) for some nonnegative integers $0 \leq m \leq n$, except for $d \in \{2, 3, 5, 15, 26\}$. Furthermore, they explicitly stated all the solutions for these exceptional cases. Several other related problems have been studied where x_k belongs to some interesting positive integer sequences. For example, see [2, 5, 6, 7, 9, 11, 12, 13, 14, 15].

2. Main Result

In this paper, we study a problem related to that of Gómez and Luca [2], but for the Pell equation (1) instead of $x^2 - dy^2 = \pm 1$. Before formulating our main theorem, let us notice that our problem is a bit different from the previous ones in that there are infinitely many d's such that the equation

$$x_k = F_n + F_m \quad \text{with} \quad n \geq m \geq 0$$

has at least two solutions (m, n, k). Indeed, take $d = 5u^2$ with some integer $u \geq 1$. Then positive solutions integer solutions (x, y) to the Diophantine equation

$$x^2 - dy^2 = \pm 4$$

correspond to positive integer solutions $(X, Y) := (x, uy)$ to $X^2 - 5Y^2 = \pm 4$. It is well-known that these are parametrised by $(X, Y) = (L_n, F_n)$, where $\{L_n\}_{n\geq 0}$ is the Lucas companion of the Fibonacci sequence given by $L_0 = 2,\ L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ for $n \geq 0$. Furthermore, in this case $L_n^2 - 5F_n^2 = 4(-1)^n$. Thus, the sign in the right–hand side is given by the parity of n. Now say u is fixed and $F_n = uy$. Then $y = F_n/u$ and F_n must be a multiple of u. It is well-known that $u \mid F_n$ is and only if $z(u) \mid n$, where $z(u)$ is the smallest positive integer ℓ such that $u \mid F_\ell$. This always exists and is called the *index of appearance of* u in the Fibonacci sequence. We conclude that for $d = 5u^2$, we have $(x_k, y_k) = (L_{z(u)k}, F_{z(u)k}/u)$. In particular, $x_k = L_{n_k}$ for some positive integer n_k. Since $L_n = F_{n+1} + F_{n-1}$ holds for all $n \geq 1$, it follows that for all values of k, x_k is a sum of two Fibonacci numbers. This gives an infinite parametric family of exceptions which dod not exist in any of the cases treated by others.

The main aim of this paper is to prove the following result.

Theorem 1. *Let $d \geq 2$ be an integer which is not a square. If $d \neq 5\square$, then is at most one positive integer k such that x_k admits a representation as*

$$x_k = F_n + F_m \tag{4}$$

for some nonnegative integers $0 \leq m \leq n$, except when $d \in \{2, 3, 7, 21, 26\}$.

For the exceptional values of d listed in Theorem 1, all solutions (k, n, m) are listed at the end of the paper. The main tools used in this paper are the lower bounds for linear forms in logarithms of algebraic numbers and the Baker-Davenport reduction procedure, as well as the elementary properties of Fibonacci numbers and solutions to Pell equations.

3. Preliminary results

3.1 The Fibonacci sequence

Here, we recall some important properties of the Fibonacci sequence $\{F_n\}_{n\geq 0}$. The characteristic equation

$$x^2 - x - 1 = 0$$

has roots α and β, where

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}.$$

The Binet formula for its geneneral terms is given by

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \quad \text{for all} \quad n \geq 0. \tag{5}$$

Furthermore, by induction, we can prove that

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \quad \text{holds for all} \quad n \geq 1. \tag{6}$$

Let $\{L_n\}_{n\geq 0}$ be the sequence of Lucas numbers defined by $L_0 = 2$, $L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ for all $n \geq 0$. For all nonnegative integers n, the following hold.

$$L_n = F_{n-1} + F_{n+1} \tag{7}$$

and

$$L_n^2 - 5F_n^2 = 4(-1)^n. \tag{8}$$

The above identities will be useful in the next parts of this paper.

3.2 Linear forms in logarithms

Let η be an algebraic number of degree d with minimal primitive polynomial over the integers

$$a_0 x^d + a_1 x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} (x - \eta^{(i)}),$$

where the leading coefficient a_0 is positive and the $\eta^{(i)}$'s are the conjugates of η. Then the *logarithmic height* of η is given by

$$h(\eta) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\{|\eta^{(i)}|, 1\}\right)\right).$$

In particular, if $\eta = p/q$ is a rational number with $\gcd(p,q) = 1$ and $q > 0$, then $h(\eta) = \log\max\{|p|, q\}$. The following are some of the properties of the logarithmic height function $h(\cdot)$, which will be used in the next sections of this paper without reference:

$$\begin{aligned} h(\eta_1 \pm \eta_2) &\leq h(\eta_1) + h(\eta_2) + \log 2, \\ h(\eta_1\eta_2^{\pm 1}) &\leq h(\eta_1) + h(\eta_2), \\ h(\eta^s) &= |s|h(\eta) \quad (s \in \mathbb{Z}). \end{aligned} \tag{9}$$

We start by recalling the result of Bugeaud, Mignotte and Siksek ([3], Theorem 9.4, pp. 989), which is a modified version of the result of Matveev [16]. This result is one of our main tools in this paper.

Theorem 2. *Let $\eta_1, \ldots, \eta_t$ be positive real numbers in number field $\mathbb{K} \subseteq \mathbb{R}$ of degree $D_{\mathbb{K}}$, $b_1, \ldots, b_t$ be nonzero integers, and assume that*

$$\Lambda := \eta_1^{b_1} \cdots \eta_t^{b_t} - 1, \tag{10}$$

is nonzero. Then

$$\log|\Lambda| > -1.4 \times 30^{t+3} \times t^{4.5} \times D_{\mathbb{K}}^2 (1 + \log D_{\mathbb{K}})(1 + \log B) A_1 \cdots A_t,$$

where

$$B \geq \max\{|b_1|, \ldots, |b_t|\},$$

and

$$A_i \geq \max\{D_{\mathbb{K}} h(\eta_i), |\log \eta_i|, 0.16\}, \qquad \textit{for all} \qquad i = 1, \ldots, t.$$

3.3 Reduction procedure

During the calculations, we get upper bounds on our variables which are too large, thus we need to reduce them. To do so, we use some results from the theory of continued fractions.

For the treatment of linear forms homogeneous in two integer variables, we use the well-known classical result in the theory of Diophantine approximation.

Lemma 3. *Let τ be an irrational number, $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots$ be all the convergents of the continued fraction of τ and M be a positive integer. Let N be a nonnegative integer such that $q_N > M$. Then putting $a(M) := \max\{a_i : i = 0, 1, 2, \ldots, N\}$, the inequality*

$$\left|\tau - \frac{r}{s}\right| > \frac{1}{(a(M)+2)s^2},$$

holds for all pairs (r, s) of positive integers with $0 < s < M$.

For a nonhomogeneous linear form in two integer variables, we use a slight variation of a result due to Dujella and Pethő (see [8], Lemma 5a). For a real number X, we write $||X|| := \min\{|X - n| : n \in \mathbb{Z}\}$ for the distance from X to the nearest integer.

Lemma 4. *Let M be a positive integer, $\frac{p}{q}$ be a convergent of the continued fraction of the irrational number τ such that $q > 6M$, and A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let further $\varepsilon := ||\mu q|| - M||\tau q||$. If $\varepsilon > 0$, then there is no solution to the inequality*

$$0 < |u\tau - v + \mu| < AB^{-w},$$

in positive integers u, v and w with

$$u \le M \quad and \quad w \ge \frac{\log(Aq/\varepsilon)}{\log B}.$$

At various occasions, we need to find a lower bound for linear forms in logarithms with bounded integer coefficients in three and four variables. In this case we use the LLL algorithm that we describe below. Let $\tau_1, \tau_2, \ldots \tau_t \in \mathbb{R}$ and the linear form

$$x_1\tau_1 + x_2\tau_2 + \cdots + x_t\tau_t \quad \text{with} \quad |x_i| \le X_i. \tag{11}$$

We put $X := \max\{X_i\}$, $C > (tX)^t$ and consider the integer lattice Ω generated by

$$\mathbf{b}_j := \mathbf{e}_j + \lfloor C\tau_j \rceil \quad \text{for} \quad 1 \le j \le t-1 \quad \text{and} \quad \mathbf{b}_t := \lfloor C\tau_t \rceil \mathbf{e}_t,$$

where C is a sufficiently large positive constant.

Lemma 5. *Let $X_1, X_2, \ldots, X_t$ be positive integers such that $X := \max\{X_i\}$ and $C > (tX)^t$ is a fixed sufficiently large constant. With the above notation on the lattice Ω, we consider a reduced base $\{\boldsymbol{b}_i\}$ to Ω and its associated Gram-Schmidt orthogonalization base $\{\boldsymbol{b}_i^*\}$. We set*

$$c_1 := \max_{1\le i\le t} \frac{||\boldsymbol{b}_1||}{||\boldsymbol{b}_i^*||}, \quad \theta := \frac{||\boldsymbol{b}_1||}{c_1}, \quad Q := \sum_{i=1}^{t-1} X_i^2 \quad and \quad R := \frac{1}{2}\left(1 + \sum_{i=1}^{t} X_i\right).$$

If the integers x_i are such that $|x_i| \le X_i$, for $1 \le i \le t$ and $\theta^2 \ge Q + R^2$, then we have

$$\left|\sum_{i=1}^{t} x_i\tau_i\right| \ge \frac{\sqrt{\theta^2 - Q} - R}{C}.$$

For the proof and further details, we refer the reader to the book of Cohen. (Proposition 2.3.20 in ([4], pp. 58–63).

Finally, the following lemma is also useful. It is Lemma 7 in [10].

Lemma 6. *If $r \geqslant 1$, $H > (4r^2)^r$ and $H > L/(\log L)^r$, then*

$$L < 2^r H(\log H)^r.$$

4. Proof of Theorem 1

Let (x_1, y_1) be the smallest positive integer solution to the Pell quation (1). We Put

$$\delta := \frac{x_1 + y_1\sqrt{d}}{2} \quad \text{and} \quad \sigma = \frac{x_1 - y_1\sqrt{d}}{2}. \tag{12}$$

From which we get that

$$\delta \cdot \sigma = \frac{x_1^2 - dy_1^2}{4} =: \epsilon, \quad \text{where} \quad \epsilon \in \{\pm 1\}. \tag{13}$$

Then

$$x_k = \delta^k + \sigma^k. \tag{14}$$

Since $\delta \ge \alpha$, it follows that the estimate

$$\frac{\delta^k}{\alpha} < x_k < \alpha\delta^k \quad \text{holds for all} \quad k \ge 1. \tag{15}$$

We assume that (k_1, n_1, m_1) and (k_2, n_2, m_2) are triples of integers such that

$$x_{k_1} = F_{n_1} + F_{m_1} \quad \text{and} \quad x_{k_2} = F_{n_2} + F_{m_2} \tag{16}$$

We assume that $1 \le k_1 < k_2$.

Furthermore, by the well-known properties of solutions to Pell equations, we may assume that $\gcd(k_1, k_2) = 1$. That is, if $\gcd(k_1, k_2) = \ell$, we then write $k_1 = k_1'\ell$, $k_2 = k_2'\ell$. We replace d by $d' := dy_\ell^2$. Then the smallest solution (x_1', y_1') of the Pell equation $x'^2 - d'y'^2 = \pm 4$ is $(x_\ell, 1)$. Furthermore, $x'_{k_1'} = x_{k_1}$ and $x'_{k_2'} = x_{k_2}$. This justifies our claim that we may assume that $\gcd(k_1, k_2) = 1$.

Next, $F_1 = F_2 = 1$, so it follows that we may assume that $m_i \geq 2$ if $m_i \neq 0$. Thus, we either have $(m_i, n_i) = (0, n_i)$ with $n_i \geq 2$ or $2 \leq m_i \leq n_i$. If $m = n$, then $F_m + F_n = 2F_n$. If $n = 2$, then $2F_n = F_3$. Otherwise, $2F_n = F_{n+1} + F_{n-2}$ and $n \geq 3$. Thus, we may always assume that $m_i < n_i$ for $i = 1, 2$. Finally, if $m = n - 1$, then $F_m + F_n = F_{n-1} + F_n = F_{n+1}$. Thus, if $2 \leq m_i < n_i$, we may assume that m_i and n_i are not consecutive. In particular, either $(m_i, n_i) = (0, 2)$. or $n_i \geq 3$. Let us treat the case $(m_i, n_i) = (0, 2)$. In this case, $x_k = F_0^2 + F_2^2 = 1$. Thus, $1^2 - dy^2 = \pm 4$. The only possibility is the sign $-$ in the right–hand side, for which $d = 5$, a case which we have excluded.

Thus, $n_i \geq 3$ for $i = 1, 2$.

Using the inequalities (6) and (15), we get from (16) that

$$\frac{\delta^k}{\alpha} \leq x_k = F_n + F_m \leq F_n + F_{n-2} \leq \alpha^n \quad \text{and} \quad \alpha^{n-2} \leq F_n + F_m = x_k \leq \alpha\delta^k.$$

The above inequalities give

$$(n-3)\log\alpha < k\log\delta < (n+1)\log\alpha.$$

Dividing through by $\log\alpha$ and setting $c_2 := 1/\log\alpha$, we get that

$$-3 < c_2 k\log\delta - n < 1,$$

and since $\alpha^{3/2} > 2$, we get

$$|n - c_2 k\log\delta| < 3. \tag{17}$$

Furthermore, $k \leq n$, for if not, we would then get that

$$\alpha^{n+1} \leq \delta^{n+1} \leq \delta^k < \alpha^{n+1},$$

a contradiction. Besides, given that $k_1 < k_2$, we have by (6) and (16) that

$$\alpha^{n_1-2} \leq F_{n_1} \leq F_{n_1} + F_{m_1} = x_{k_1} < x_{k_2} = F_{n_2} + F_{m_2} \leq F_{n_2} + F_{n_2-2} \leq \alpha^{n_2-1} + \alpha^{n_2-3} < \alpha^{n_2}.$$

Thus, we get that

$$n_1 < n_2 + 2. \tag{18}$$

4.1 An inequality for n and k

Using the equations (5) and (14) and (16), we get

$$\delta^k + \sigma^k = F_n + F_m = \frac{\alpha^n - \beta^n}{\sqrt{5}} + \frac{\alpha^m - \beta^m}{\sqrt{5}}.$$

So,

$$\delta^k - \frac{\alpha^n + \alpha^m}{\sqrt{5}} = -\sigma^k - \frac{\beta^n + \beta^m}{\sqrt{5}},$$

and by (6), we have

$$\begin{aligned}
\left|\delta^k \cdot \sqrt{5} \cdot \alpha^{-n}(1 + \alpha^{m-n})^{-1} - 1\right| &\leq \frac{\sqrt{5}}{\delta^k(\alpha^n + \alpha^m)} + \frac{|\beta|^n + |\beta|^m}{\alpha^n + \alpha^m} \\
&\leq \frac{\sqrt{5}\alpha}{\alpha^n(\alpha^n + \alpha^m)} + \frac{1}{\alpha^{n+m}} \\
&\leq \frac{1}{\alpha^n}\left(\frac{\sqrt{5}}{\alpha^n + \alpha^m} + \frac{1}{\alpha^m}\right) < \frac{2}{\alpha^n}.
\end{aligned}$$

The numerator 1.5 above comes from the fact that $m \geq 0$ and $n \geq 3$. Thus, we have

$$\left|\delta^k(\sqrt{5})\alpha^{-n}(1+\alpha^{m-n})^{-1}-1\right| \quad < \quad \frac{2}{\alpha^n}. \tag{19}$$

Put

$$\Lambda_1 := \delta^k(\sqrt{5})\alpha^{-n}(1+\alpha^{m-n})^{-1}-1$$

and

$$\Gamma_1 := k\log\delta + \log(\sqrt{5}) - n\log\alpha - \log(1+\alpha^{m-n}).$$

Since $|\Lambda_1| = |e^{\Gamma_1}-1| < \frac{1}{2}$ for $n \geq 3$ (because $n \geq 3$ and $\alpha^3 > 4$, so $2/\alpha^n \leq 2/\alpha^3 < 1/2$), and since the inequality $|y| < 2|e^y-1|$ holds for all $y \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, it follows that $e^{|\Gamma_1|} < 2$ and so

$$|\Gamma_1| < e^{|\Gamma_1|}|e^{\Gamma_1}-1| < \frac{4}{\alpha^n}.$$

Thus, we get that

$$\left|k\log\delta + \log(\sqrt{5}) - n\log\alpha - \log(1+\alpha^{m-n})\right| < \frac{4}{\alpha^n}. \tag{20}$$

We apply Theorem 2 on the left-hand side of (19) with the data:

$$t := 4, \quad \eta_1 := \delta, \quad \eta_2 := \sqrt{5}, \quad \eta_3 := \alpha, \quad \eta_4 := 1+\alpha^{m-n},$$
$$b_1 := k, \quad b_2 := 1, \quad b_3 := -n, \quad b_4 := -1.$$

Furthermore, we take the number field $\mathbb{K} := \mathbb{Q}(\sqrt{d}, \alpha)$ which has degree $D_{\mathbb{K}} := 4$. Since $\max\{1, k, n\} \leq n$, we take $B := n$. First we note that the left-hand side of (19) is non-zero, since otherwise,

$$\delta^k = \frac{1}{\sqrt{5}}(\alpha^n + \alpha^m).$$

The left-hand side belongs to the quadratic field $\mathbb{Q}(\sqrt{d})$ and is not rational while the right-hand side belongs to the field $\mathbb{Q}(\sqrt{5})$. This is not possible since $d \neq 5$. Thus, $\Lambda_1 \neq 0$ and we can apply Theorem 2.

We have $h(\eta_1) = h(\delta) = \frac{1}{2}\log\delta$, $h(\eta_2) = h(\sqrt{5}) = \frac{1}{2}\log 5$ and $h(\eta_3) = h(\alpha) = \frac{1}{2}\log\alpha$. On the other hand,

$$\begin{aligned} h(\eta_4) &= h(1+\alpha^{m-n}) \leq h(1) + h(\alpha^{m-n}) + \log 2 \\ &= (n-m)h(\alpha) + \log 2 = \frac{1}{2}(n-m)\log\alpha + \log 2. \end{aligned}$$

Thus, we can take

$$A_1 := 2\log\delta, \quad A_2 := 2\log 5, \quad A_3 := 2\log\alpha, \quad A_4 := 2(n-m)\log\alpha + 4\log 2.$$

Now, Theorem 2 tells us that

$$\begin{aligned} \log|\Lambda_1| &> -1.4\times 30^7\times 4^{4.5}\times 4^2(1+\log 4)(1+\log n)(2\log\delta) \\ &\quad \times(2\log 5)(2\log\alpha)(2(n-m)\log\alpha + 4\log 2) \\ &> -3.4\times 10^{16}(n-m)\log n\log\delta\log\alpha. \end{aligned}$$

Comparing the above inequality with (19), we get

$$n\log\alpha - \log 2 < 3.4\times 10^{16}(n-m)\log n\log\delta\log\alpha.$$

Hence, we get that

$$n < 3.5\times 10^{16}(n-m)\log n\log\delta. \tag{21}$$

We now return to the equation $x_k = F_n + F_m$ and rewrite it as

$$\delta^k - \frac{\alpha^n}{\sqrt{5}} = -\sigma^k - \frac{\beta^n}{\sqrt{5}} + F_m,$$

we obtain

$$\left|\delta^k\cdot\sqrt{5}\cdot\alpha^{-n}-1\right| \leq \frac{1}{\alpha^{n-m}}\left(\frac{1}{\alpha} + \frac{1}{\alpha^{n+m}} + \frac{\sqrt{5}}{\delta^k\alpha^m}\right) < \frac{2}{\alpha^{n-m}}. \tag{22}$$

The numerator 2 in the right–hand side above comes from the fact that $m \geq 0$, $n \geq 3$, $\delta \geq 1 + \sqrt{2}$. Put

$$\Lambda_2 := \delta^k \cdot \sqrt{5} \cdot \alpha^{-n} - 1 \quad \text{and} \quad \Gamma_2 := k \log \delta + \log(\sqrt{5}) - n \log \alpha.$$

If $n - m \geq 3$, then $2/\alpha^{n-m} \leq 2/\alpha^3 < 1/2$, so $|e^{\Lambda_2} - 1| < \frac{1}{2}$. It follows that

$$\left| k \log \delta + \log(\sqrt{5}) - n \log \alpha \right| = |\Gamma_2| < e^{|\Lambda_2|} |e^{\Lambda_2} - 1| < \frac{4}{\alpha^{n-m}}. \tag{23}$$

We show that (23) holds for $n-m = 2$ as well. Well, the case $(m, n) = (0, 2)$ is not allowed (since $d = 5$). The case $(m, n) = (1, 3)$ reduces to $(m, n) = (0, 4)$ by our conventions, for which $x_k = F_4 = 3$, so $3^2 - dy_k^2 = \pm 4$, and since $d \neq 5$, we get $d = 13$, so $\delta^k = (3 + \sqrt{13})/2$. One checks that (23) holds in this particular case as well. In the same way, $(m, n) = (2, 4)$ gives $x_k = F_2 + F_4 = 4$, so $4^2 - dy_k^2 = \pm 4$ and since $d \neq 5$, we get $dy_k^2 = 12$. Thus, $\delta^k = 2 + \sqrt{3}$ and one checks that (23) holds in this case as well. Finally, for $m \geq 3$, we have $n + m = (m + 2) + m \geq 8$, and now the factor 2 in the numerator of the right–hand side of (22) can be replaced by 1. Since $1/\alpha^{n-m} \leq 1/\alpha^2 < 1/2$, it follows that (23) holds also in this case (even with the better numerator of 2 in the right–hand side instead of 4).

Furthermore, $\Lambda_2 \neq 0$ (so $\Gamma_2 \neq 0$), since $\delta^k \notin \mathbb{Q}(\alpha)$ by the previous argument.

We now apply Theorem 2 to the left-hand side of (22) with the data

$$t := 3, \quad \eta_1 := \delta, \quad \eta_2 := \sqrt{5}, \quad \eta_3 := \alpha, \quad b_1 := k, \quad b_2 := 1, \quad b_3 := -n.$$

Thus, we have the same A_1, A_2, A_3, B as before. Then, by Theorem 2, we conclude that

$$\log |\Lambda_2| > -2.4 \times 10^{14} \log n \log \delta \log \alpha.$$

By comparing with (22), we get

$$n - m < 2.5 \times 10^{14} \log n \log \delta. \tag{24}$$

We replace the bound (24) on $n - m$ in (21) and use the fact that $\delta^k < \alpha^{n+1}$, to obtain bounds on n and k in terms of $\log n$ and $\log \delta$. We now record what we have proved so far.

Lemma 7. Let (k, n, m) be a solution to the equation $x_k = F_n + F_m$ with $0 \leq m \leq n$ and $d \neq 5\Box$, then

$$k < 4.2 \times 10^{30} (\log n)^2 \log \delta \quad \text{and} \quad n < 8.8 \times 10^{30} (\log n)^2 (\log \delta)^2. \tag{25}$$

4.2 Absolute bounds

We recall that $(k, n, m) = (k_i, n_i, m_i)$, where $0 \leq m_i \leq n_i$, for $i = 1, 2$ and $1 \leq k_1 < k_2$. Further, $n_i \geq 2$ for $i = 1, 2$. We return to (23) and write

$$\left|\Gamma_2^{(i)}\right| := \left| k_i \log \delta + \log\left(\sqrt{5}\right) - n_i \log \alpha \right| < \frac{4}{\alpha^{n_i - m_i}}, \quad \text{for} \quad i = 1, 2.$$

We do a suitable cross product between $\Gamma_2^{(1)}$, $\Gamma_2^{(2)}$ and k_1, k_2 to eliminate the term involving $\log \delta$ in the above linear forms in logarithms:

$$\begin{aligned} |\Gamma_3| &:= \left| (k_2 - k_1) \log\left(\sqrt{5}\right) + (k_1 n_2 - k_2 n_1) \log \alpha \right| = |k_2 \Gamma_2^{(1)} - k_1 \Gamma_2^{(2)}| \\ &\leq k_2 |\Gamma_2^{(1)}| + k_1 |\Gamma_2^{(2)}| \leq \frac{4k_2}{\alpha^{n_1 - m_1}} + \frac{4k_1}{\alpha^{n_2 - m_2}} \leq \frac{8n_2}{\alpha^\lambda}, \end{aligned} \tag{26}$$

where $\lambda := \min_{1 \leq i \leq 2} \{n_i - m_i\}$.

We need to find an upper bound for λ. If $8n_2/\alpha^\lambda > 1/2$, we then get

$$\lambda < \frac{\log(16n_2)}{\log \alpha} < 3 \log(16n_2). \tag{27}$$

Otherwise, $|\Gamma_3| < \frac{1}{2}$, so

$$\left| e^{\Gamma_3} - 1 \right| = \left| \left(\sqrt{5}\right)^{k_2 - k_1} \alpha^{k_1 n_2 - k_2 n_1} - 1 \right| < 2|\Gamma_3| < \frac{16n_2}{\alpha^\lambda}. \tag{28}$$

We apply Theorem 2 with the data: $t := 2$, $\eta_1 := \sqrt{5}$, $\eta_2 := \alpha$, $b_1 := k_2 - k_1$, $b_2 := k_1 n_2 - k_2 n_1$. We take the number field $\mathbb{K} := \mathbb{Q}(\alpha)$ and $D_{\mathbb{K}} := 2$. We begin by checking that $e^{\Gamma_3} - 1 \neq 0$ (so $\Gamma_3 \neq 0$). This is true because α and $\sqrt{5}$ are multiplicatively independent, since α is a unit in the ring of integers $\mathbb{Q}(\alpha)$ while the norm of $\sqrt{5}$ is $-5 \neq \pm 1$.

We note that $k_2 - k_1 < k_2 < n_2$. Further, from (26), we have

$$|k_2 n_1 - k_1 n_2| < (k_2 - k_1)\frac{\log\left(\sqrt{5}\right)}{\log\alpha} + \frac{8k_2}{\alpha^{\lambda}\log\alpha} < 15k_2 < 15n_2$$

given that $\lambda \geq 1$. So, we can take $B := 15n_2$. By Theorem 2, with $A_1 := \log 5$ and $A_2 := \log\alpha$, we have that

$$\begin{aligned}\log|e^{\Gamma_3} - 1| &> -1.4 \times 30^5 \times 2^{4.5} \times 2 \times (1 + \log 2)(1 + \log(15n_2))(\log 5)(\log\alpha)\\ &> -1.7 \times 10^{10}\log(15n_2)\log\alpha.\end{aligned}$$

By comparing this with (28), we get

$$\lambda\log\alpha - \log(16n_2) < 1.7 \times 10^{10}\log(15n_2)\log\alpha,$$

which implies that

$$\lambda < 1.8 \times 10^{10}\log(15n_2). \tag{29}$$

Note that (29) is better than (27), so (29) always holds. Without loss of generality, we can assume that $\lambda = n_i - m_i$, for $i = 1, 2$ fixed.

We set $\{i, j\} = \{1, 2\}$ and return to (20) to replace $(k, n, m) = (k_i, n_i, m_i)$:

$$|\Gamma_1^{(i)}| = \left|k_i\log\delta + \log(\sqrt{5}) - n_i\log\alpha - \log(1 + \alpha^{m_i - n_i})\right| < \frac{4}{\alpha^{n_i}}, \tag{30}$$

and also return to (23), replacing with $(k, n, m) = (k_j, n_j, m_j)$:

$$|\Gamma_2^{(j)}| = \left|k_j\log\delta + \log(\sqrt{5}) - n_j\log\alpha\right| < \frac{4}{\alpha^{n_j - m_j}}. \tag{31}$$

We perform a cross product on (30) and (31) in order to eliminate the term on $\log\delta$:

$$\begin{aligned}|\Gamma_4| &:= \left|(k_i - k_j)\log(\sqrt{5}) + (k_j n_i - k_i n_j)\log\alpha + k_j\log(1 + \alpha^{m_i - n_i})\right|\\ &= \left|k_i\Gamma_2^{(j)} - k_j\Gamma_1^{(i)}\right| \leq k_i\left|\Gamma_2^{(j)}\right| + k_j\left|\Gamma_1^{(i)}\right|\\ &< \frac{4k_i}{\alpha^{n_j - m_j}} + \frac{4k_j}{\alpha^{n_i}} < \frac{8n_2}{\alpha^{\nu}}\end{aligned} \tag{32}$$

with $\nu := \min\{n_i, n_j - m_j\}$. As before, we need to find an upper bound on ν. If $8n_2/\alpha^{\nu} > 1/2$, then we get

$$\nu < \frac{\log(16n_2)}{\log\alpha} < 3\log(16n_2). \tag{33}$$

Otherwise, $|\Gamma_4| < 1/2$, so we have

$$\left|e^{\Gamma_4} - 1\right| = \left|\left(\sqrt{5}\right)^{k_i - k_j}\alpha^{k_j n_i - k_i n_j}(1 + \alpha^{m_i - n_i})^{k_j} - 1\right| \leq 2|\Gamma_4| < \frac{16n_2}{\alpha^{\nu}}. \tag{34}$$

In order to apply Theorem 2, first if $e^{\Gamma_4} = 1$, we obtain

$$(\sqrt{5})^{k_j - k_i} = \alpha^{k_j n_i - k_i n_j}(1 + \alpha^{-\lambda})^{k_j}. \tag{35}$$

Let us show that the above equation is impossible. Since the right–hand side is an algebraic integer (because α is a unit), it follows that $k_j > k_i$. We take norms (in $\mathbb{Q}(\sqrt{5})$) and absolute values in both sides of (35). We then get

$$5^{k_j-k_i} = ((1+\alpha^{\lambda})(1+\beta^{\lambda}))^{k_j} = \begin{cases} L_{\lambda}^{k_j} & \text{if} \quad \lambda \equiv 1 \pmod 2; \\ L_{\lambda/2}^{2k_j} & \text{if} \quad \lambda \equiv 0 \pmod 4; \\ (5F_{\lambda/2}^2)^{k_j} & \text{if} \quad \lambda \equiv 2 \pmod 4). \end{cases} \tag{36}$$

The above equation is impossible since the exponent of 5 in the left–hand side is positive and smaller than k_j, while in the right–hand side, ether it is at least k_j (if $\lambda \equiv 2 \pmod 4$) or is 0 (if $\lambda \not\equiv 2 \pmod 4$), because 5 never divides L_n for any positive integer n. Hence, $e^{\Gamma_4} \neq 1$. We apply Theorem 2 with the data:

$$t := 3, \quad \eta_1 := \sqrt{5}, \quad \eta_2 := \alpha, \quad \eta_3 := 1+\alpha^{-\lambda},$$

$$b_1 := k_i - k_j, \quad b_2 := k_j n_i - k_i n_j, \quad b_3 := k_j,$$

We take $D_{\mathbb{K}} := 2$, $A_1 := \log 5$, $A_2 := \log\alpha$, $A_3 := \lambda\log\alpha + 2\log 2 \leq 2\lambda\log\alpha$, and $B := 15n_2$. By Theorem 2, we get that

$$\begin{aligned} \log|e^{\Gamma_4} - 1| &> -1.4\times 30^6\times 3^{4.5}\times 2(1+\log 2)(1+\log(15n_2))(\log 5)(\log\alpha)(2\lambda\log\alpha) \\ &> -3.0\times 10^{12}\lambda\log(15n_2)\log\alpha. \end{aligned}$$

By comparing this with (34) together with the inequality (29), we get

$$\begin{aligned} &\nu\log\alpha - \log(16n_2) < 3.0\times 10^{12}\lambda\log(15n_2)\log\alpha, \\ &\nu := \min\{n_i, n_j - m_j\} < 3.2\times 10^{12}\lambda\log(15n_2) < 5.8\times 10^{22}(\log(15n_2))^2. \end{aligned} \tag{37}$$

Further, it also holds when the inequality (33) holds. So the above inequality holds in all cases. Note that the case $\{i,j\} = \{2,1\}$ leads to $n_1 - m_1 \leq n_1 \leq n_2 + 2$ whereas $\{i,j\} = \{1,2\}$ lead to $\nu = \min\{n_1, n_2 - m_2\}$. Hence, either the minimum is n_1, so

$$n_1 < 5.8\times 10^{22}(\log(15n_2))^2, \tag{38}$$

or the minimum is $n_j - m_j$ and from the inequality (29) we get that

$$\max_{1\leq j\leq 2}\{n_j - m_j\} < 5.8\times 10^{22}(\log(15n_2))^2. \tag{39}$$

Next, we assume that we are in the case (39). We evaluate (30) in $i = 1, 2$ and make a suitable cross product to eliminate the term involving $\log\delta$:

$$\begin{aligned} |\Gamma_5| \quad := \quad & \Big|(k_1 - k_2)\log(\sqrt{5}) + (k_2 n_1 - k_1 n_2)\log\alpha \\ & + k_2\log(1+\alpha^{m_1-n_1}) - k_1\log(1+\alpha^{m_2-n_2})\Big| \\ = \quad & \left|k_1\Gamma_1^{(2)} - k_2\Gamma_1^{(1)}\right| \leq k_1\left|\Gamma_1^{(2)}\right| + k_2\left|\Gamma_1^{(1)}\right| < \frac{8n_2}{\alpha^{n_1}}. \end{aligned} \tag{40}$$

In the above inequality we used the inequality (18) to conclude that $\min\{n_1, n_2\} \geq n_1 - 3$ as well as the fact that $n_i \geq 3$ for $i = 1, 2$. Next, we apply a linear form in four logarithms to obtain an upper bound to n_1. As in the previous calculations, we pass from (40) to

$$\left|e^{\Gamma_5} - 1\right| = \left|\left(\sqrt{5}\right)^{k_1-k_2}\alpha^{k_2n_1-k_1n_2}(1+\alpha^{m_1-n_1})^{k_2}(1+\alpha^{m_2-n_2})^{-k_1} - 1\right| < \frac{16n_2}{\alpha^{n_1}}, \tag{41}$$

which is implied by (40) except if n_1 is very small, say

$$n_1 \leq 3\log(16n_2). \tag{42}$$

Thus, we assume that (42) does not hold, therefore (41) holds. Then to apply Theorem 2, we first justify that $e^{\Gamma_5} \neq 1$. Otherwise,

$$(\sqrt{5})^{k_2-k_1} = \alpha^{k_2m_1-k_1m_2}(1+\alpha^{n_1-m_1})^{k_2}(1+\alpha^{n_2-m_2})^{-k_1}, \tag{43}$$

We need to check that the equation (43) has no positive integer solutions. We let $\mathbb{K} := \mathbb{Q}(\sqrt{5})$. We use, as we did in (36), that for any positive integer k,

$$N_{\mathbb{K}/\mathbb{Q}}(1+\alpha^k) = \begin{cases} L_k, & \text{if} \quad k \equiv 1 \pmod 2, \\ L_{k/2}^2, & \text{if} \quad k \equiv 0 \pmod 4, \\ 5F_{k/2}^2, & \text{if} \quad k \equiv 2 \pmod 4). \end{cases}$$

Now, we assume that (43) holds and take norms and absolute values on both sides to get

$$5^{k_2-k_1} = \left|N_{\mathbb{K}/\mathbb{Q}}\left(\sqrt{5}\right)\right|^{k_2-k_1} = \left|N_{\mathbb{K}/\mathbb{Q}}(\alpha)\right|^{k_2m_1-k_1m_2} \frac{|N_{\mathbb{K}/\mathbb{Q}}(1+\alpha^{n_1-m_1})|}{|N_{\mathbb{K}/\mathbb{Q}}(1+\alpha^{n_2-m_2})|} = \frac{E_{n_1-m_1}^{k_2}}{E_{n_2-m_2}^{k_1}},$$

where $E_k \in \left\{L_k, L_{k/2}^2, 5F_{k/2}^2\right\}$ according to the residue class of k modulo 4. Since 5 divides the left–hand side above which is an integer, 5 divides the numerator of the right–hand side. Since $5 \nmid L_m$ for any m, it follows that $E_{n_1-m_1} = 5F_{(n_1-m_1)/2}^2$. Then the exponent of 5 in the numerator of the right–hand side is at least $k_2 > k_2 - k_1$, we infer that 5 should also divide the denominator of the right–hand side meaning $E_{n_2-m_2} = 5F_{(n_2-m_2)/2}^2$. But then we get

$$F_{(n_1-m_1)/2}^{2k_2} = F_{(n_2-m_2)/2}^{2k_1}.$$

Since $k_2 > k_1$, we either have $F_{(n_1-m_1)/2} < F_{(n_2-m_2)/2}$ or both sides are 1. The only distinct Fibonacci numbers which are multiplicatively dependent are 2 and $8 = F_6$, but then $n_2 - m_2 = 12$, so $E_{n_2-m_2} = L_{(n_2-m_2)/2}^2$ (instead of $5F_{(n_2-m_2)/2}^2$), a contradiction. Hence, $F_{(n_1-m_1)/2} = F_{(n_2-m_2)/2} = 1$ and since $(n_i - m_i)/2$ is odd (in order for $E_{n_i-m_i} = 5F_{(n_i-m_i)/2}^2$ to hold), we get $n_i - m_i = 2$. Thus, $x_{k_i} = F_{n_i} + F_{n_i-2} = L_{n_i-1}$ for $i = 1, 2$.
Further, $1 + \alpha^{n_i-m_i} = 1 + \alpha^2 = \sqrt{5}\alpha$ for $i = 1, 2$ so (43) becomes

$$\sqrt{5}^{k_2-k_1} = \alpha^{k_2m_1-k_1m_2}(\sqrt{5}\alpha)^{k_2}(\sqrt{5}\alpha)^{-k_1} = \sqrt{5}^{k_2-k_1}\alpha^{k_2(m_1+1)-k_1(m_2+1)}.$$

We now get $k_1(n_2 - 1) = k_2(n_1 - 1)$ (because $n_i - 1 = m_i + 1$ for $i = 1, 2$) and since $\gcd(k_1, k_2) = 1$, we have $k_1 = (n_1 - 1)/\ell$ and $k_2 = (n_2 - 1)/\ell$ for some number ℓ. Thus, $n_1 - 1 = k_1\ell$ and $n_2 - 1 = k_2\ell$. So, we get

$$x_{k_i} = \delta^{k_i} + \sigma^{k_i} = (\alpha^\ell)^{k_i} + (\beta^\ell)^{k_i} \qquad \text{for} \qquad i = 1, 2.$$

Then

$$\delta^{k_i} - (\alpha^\ell)^{k_i} = (\delta - \alpha^\ell)(\delta^{k_i-1} + \cdots + (\alpha^\ell)^{k_i-1}) = -\sigma^{k_i} + (\beta^\ell)^{k_i}.$$

Assume now that $k_2 \geq 3$. We then get

$$\delta^2|\delta - \alpha^\ell| < |\delta - \alpha^\ell((\delta^{k_2-1} + \cdots + (\alpha^\ell)^{k_2-1}) \leq \delta^{-k_2} + (\alpha^\ell)^{-k_2}.$$

Now $\delta \geq \alpha$, so $\delta^{-k_2} + (\alpha^\ell)^{-k_2} \leq 2/\alpha^3$. Hence, $\delta^2|\delta - \alpha^\ell| < 2/\alpha^3$ giving $|\delta - \alpha^\ell| < 2/\alpha^5$. Thus, $\alpha^\ell \geq \delta - 2/\alpha^5$. Thus,

$$(\alpha^\ell)^{-k_2} \leq (\alpha^\ell)^{-3} \leq (\delta - 2/\alpha^5)^{-3} < 2\delta^{-3},$$

where we used the fact that

$$\left(\frac{\delta}{\delta - 2/\alpha^5}\right)^3 > 2,$$

which follows because $\delta > \alpha$ and $\alpha/(\alpha - 2/\alpha^5) < 2^{1/3}$. We thus get that

$$\delta^2|\delta - \alpha^\ell| \leq \delta^{-k_2} + (\alpha^\ell)^{-k_2} \leq 3\delta^{-3},$$

giving $|\delta - \alpha^\ell| < 3\delta^{-5}$. Assume $\delta \neq \alpha^\ell$. Then $\delta - \alpha^\ell$ is an algebraic integer of degree at most 4 and its conjugates are among $\sigma - \alpha^\ell$, $\sigma - \beta^\ell$ and $\delta - \beta^\ell$. These have absolute values at most $1/\delta + \alpha^\ell < \delta + 1/\delta + 3/\delta^5 \leq \delta + (1/\alpha + 3/\alpha^5) < \delta + 1,\ 2$ and $\delta + 1$ respectively. Computing the norm of the algebraic integer $\delta - \alpha^\ell$, we get

$$1 \leq |N_{\mathbb{K}/\mathbb{Q}}(\delta - \alpha^\ell)| \leq (3\delta^{-5})(2(\delta + 1)^2),$$

giving $\delta < 2.31$. Here, $\mathbb{K} = \mathbb{Q}(\sqrt{d}, \sqrt{5})$. The only value of $\delta < 2.31$ is α (the next value of δ is $1 + \sqrt{2} > 2.4$). This shows that $k_2 \geq 3$ is not possible. Thus, $k_1 = 1,\ k_2 = 2$, and so $n_1 - 1 = \ell$ and $n_2 - 1 = 2\ell$. So, we get

$$x_1 = L_\ell \qquad \text{and} \qquad x_2 = L_{2\ell}.$$

Now putting

$$x_1^2 - dy_1^2 = 4\epsilon,$$

it follows that $x_2 = x_1^2 - 2\epsilon$, so $L_{2\ell}^2 = L_\ell^2 - 2\epsilon$. Since in fact $L_{2\ell} = L_\ell^2 - 2(-1)^\ell$, it follows that $\epsilon = (-1)^\ell$. Thus,

$$L_\ell^2 - dy_1^2 = 4(-1)^\ell$$

and comparing it with the identity $L_\ell^2 - 5F_\ell^2 = 4(-1)^\ell$, we get $dy_1^2 = 5F_\ell^2$, so $d = 5u^2$ for some integer u (which in this case is F_ℓ/y_1), which is not the case. Thus, $e^{\Gamma_5} \neq 1$.

Thus, we apply Theorem 2 on the left-hand side of the inequalities (41) with the data

$$t := 4, \quad \eta_1 := \sqrt{5}, \quad \eta_2 := \alpha, \quad \eta_3 := 1 + \alpha^{m_1 - n_1}, \quad \eta_4 := 1 + \alpha^{m_2 - n_2},$$
$$b_1 := k_2 - k_1, \quad b_2 := k_2 n_1 - k_1 n_2, \quad b_3 := k_2, \quad b_4 := -k_1.$$

We take $D_{\mathbb{K}} := 2$, $a_1 := \log 5$, $A_2 := \log\alpha$, $A_3 := 2(n_1 - m_1)\log\alpha$, $A_3 := 2(n_2 - m_2)\log\alpha$, and $B := 15n_2$. By Theorem 2, we get

$$\begin{aligned}\log|e^{\Gamma_5} - 1| &> -1.4 \times 30^7 \times 4^{4.5} \times 2^2(1 + \log 2)(1 + \log(15n_2))(\log 5)(\log\alpha) \\ &\quad \times (2(n_1 - m_1)\log\alpha)(2(n_2 - m_2)\log\alpha) \\ &> -3.2 \times 10^{14}(n_1 - m_1)(n_2 - m_2)\log(15n_2)\log\alpha.\end{aligned}$$

By comparing this with (41) together with the inequalities (29) and (39), we get

$$\begin{aligned} n_1 &< 3.3 \times 10^{14}(n_1 - m_1)(n_2 - m_2)\log(15n_2) \\ &< 3.5 \times 10^{47}(\log(15n_2))^4. \end{aligned} \tag{44}$$

In the above we used the facts that

$$\min_{1\leq i\leq 2}\{n_i - m_i\} < 1.8 \times 10^{10}\log(15n_2) \quad \text{and} \quad \max_{1\leq i\leq 2}\{n_i - m_i\} < 5.8 \times 10^{22}(\log(15n_2))^2.$$

This was obtained under the assumption that the inequality (42) does not hold. If (42) holds, then so does (44). Thus, we have that inequality (44) holds provided that inequality (39) holds. Otherwise, inequality (38) holds which is a better bound than (44). Hence, conclude that (44) holds in all posibble cases.

We have,

$$\log\delta \leq k_1\log\delta \leq (n_1 + 1)\log\alpha < 1.7 \times 10^{49}(\log(15n_2))^4.$$

By substituting this into (25) we get $15n_2 < 3.6\times 10^{126}(\log(15n_2))^{10}$, and then, by Lemma 6, with the data $r := 10$, $H := 3.6\times 10^{126}$ and $L := 15n_2$, we get that $15n_2 < 1.6\times 10^{154}$. This immediately gives that $n_2 < 1.1\times 10^{153}$ and $n_1 < 5.6\times 10^{57}$.

We record what we have proved.

Lemma 8. *Let* (k_i, n_i, m_i) *be a solution to* $x_{k_i} = F_{n_i} + F_{m_i}$, *with* $0 \le m_i \le n_i$ *for* $i \in \{1,2\}$, $d \ne 5\square$ *and* $1 \le k_1 < k_2$, *then*

$$\max\{k_1, m_1\} \le n_1 < 10^{58} \quad \text{and} \quad \max\{k_2, m_2\} \le n_2 < 10^{154}.$$

5. Reducing the bounds for n_1 and n_2

In this section we reduce the bounds for n_1 and n_2 given in Lemma 8 to cases that can be computationally treated. For this, we return to the inequalities for Γ_3, Γ_4 and Γ_5.

5.1 The first reduction

We divide through both sides of the inequality (26) by $(k_2 - k_1)\log\alpha$. We get that

$$\left|\frac{\log(\sqrt{5})}{\log\alpha} - \frac{k_2 n_1 - k_1 n_2}{k_2 - k_1}\right| < \frac{8n_2}{\alpha^{\lambda}(k_2 - k_1)\log\alpha} \quad \text{with} \quad \lambda := \min_{1\le i\le 2}\{n_i - m_i\}. \tag{45}$$

We assume that $\lambda \ge 10$. Below we apply Lemma 3. We put $\tau := \frac{\log(\sqrt{5})}{\log\alpha}$, which is irrational and compute its continued fraction

$$[a_0; a_1, a_2, \ldots] = [1; 1, 2, 19, 2, 9, 1, 1, 3, 1, 9, 1, 2, 6, 1, 1, 1, 5, 1, 14, 29, 1, 2, 1, 4, 2, 1, \ldots]$$

and its convergents

$$\left[\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots\right] = \left[1, 2, \frac{5}{3}, \frac{97}{58}, \frac{199}{119}, \frac{1888}{1129}, \frac{2087}{1248}, \frac{3975}{2377}, \frac{14012}{8379}, \frac{17987}{10756}, \frac{175895}{105183}, \ldots\right].$$

Furthermore, we note that taking $M := 10^{154}$ (by Lemma 8), it follows that

$$q_{297} > M > n_2 > k_2 - k_1 \quad \text{and} \quad a(M) := \max\{a_i : 0 \le i \le 297\} = a_{170} = 330.$$

Thus, by Lemma 3, we have that

$$\left|\tau - \frac{k_2 n_1 - k_1 n_2}{k_2 - k_1}\right| > \frac{1}{332(k_2 - k_1)^2}. \tag{46}$$

Hence, combining the inequalities (45) and (46), we obtain

$$\alpha^{\lambda} < 5519 n_2 (k_2 - k_1) < 5.52\times 10^{311},$$

so $\lambda \le 1491$. This was obtained under the assumption that $\lambda \ge 10$, Otherwise, $\lambda < 10 < 1491$ holds as well.

Now, for each $n_i - m_i = \lambda \in [1, 1491]$ we estimate a lower bound $|\Gamma_4|$, with

$$\Gamma_4 \quad = \quad (k_i - k_j)\log(\sqrt{5}) + (k_j n_i - k_i n_j)\log\alpha + k_j\log(1 + \alpha^{m_i - n_i}) \tag{47}$$

given in the inequality (32), via the procedure described in Subsection 3.3 (LLL-algorithm). We recall that $\Gamma_4 \ne 0$.

We apply Lemma 5 with the data:

$$t := 3, \quad \tau_1 := \log(\sqrt{5}), \quad \tau_2 := \log\alpha, \quad \tau_3 := \log(1 + \alpha^{-\lambda}),$$

$$x_1 := k_i - k_j, \quad x_2 := k_j n_i - k_i n_j, \quad x_3 := k_j.$$

We set $X := 15\times 10^{154}$ as an upper bound to $|x_i| < 15n_2$ for all $i = 1, 2, 3$, and $C := (10X)^5$. A computer in *Mathematica* search allows us to conclude, together with the inequality (32), that

$$2\times 10^{-653} < \min_{1\le\lambda\le 1491}|\Gamma_4| < 8n_2\alpha^{-\nu}, \quad \text{with} \quad \nu := \min\{n_i, n_j - m_j\}$$

which leads to $\nu \le 3855$. As we have noted before, $\nu = n_1$ (so $n_1 \le 3855$) or $\nu = n_j - m_j$.

Next, we suppose that $n_j - m_j = \nu \le 3855$. Since $\lambda \le 1491$, we have

$$\lambda := \min_{1\le i\le 2}\{n_i - m_i\} \le 1491 \quad \text{and} \quad \chi := \max_{1\le i\le 2}\{n_i - m_i\} \le 3855.$$

Now, returning to the inequality (40) which involves

$$\begin{aligned}\Gamma_5 : &= (k_2 - k_1)\log(\sqrt{5}) + (k_2 n_1 - k_1 n_2)\log\alpha \\ &\quad + k_2\log(1 + \alpha^{m_1-n_1}) - k_1\log(1 + \alpha^{m_2-n_2}) \ne 0, \qquad (48)\end{aligned}$$

we use again the LLL-algorithm to estimate the lower bound for $|\Gamma_5|$ and thus, find a bound for n_1 that is better than the one given in Lemma 8.

We distinguish the cases $\lambda < \chi$ and $\lambda = \chi$.

5.2 The case $\lambda < \chi$.

We take $\lambda \in [1, 1491]$ and $\chi \in [\lambda + 1, 3855]$ and apply Lemma 5 with the data: $t := 4$,

$$\tau_1 := \log(\sqrt{5}), \quad \tau_2 := \log\alpha, \quad \tau_3 := \log(1 + \alpha^{m_1-n_1}), \quad \tau_4 := \log(1 + \alpha^{m_2-n_2}),$$

$$x_1 := k_2 - k_1, \quad x_2 := k_2 n_1 - k_1 n_2, \quad x_3 := k_2, \quad x_4 := -k_1.$$

We also put $X := 15 \times 10^{154}$ and $C := (20X)^9$. After a computer search in *Mathematica* together with the inequality (40), we can confirm that

$$10^{-1312} < \min_{\substack{1\le\lambda\le 1491 \\ \lambda+1\le\chi\le 3855}} |\Gamma_5| < 8n_2\alpha^{-n_1}. \qquad (49)$$

This leads to the inequality

$$\alpha^{n_1} < 8 \times 10^{1312} n_2. \qquad (50)$$

Subsitituting for the bound n_2 given in Lemma 8, we get that $n_1 \le 7019$.

5.3 The case $\lambda = \chi$.

In this case, we have

$$\Gamma_5 := (k_2 - k_1)\left(\log(1/\sqrt{5}) + \log(1 + \alpha^{m_1-n_1})\right) + (k_2 n_1 - k_1 n_2)\log\alpha \ne 0.$$

We divide through the inequality 40 by $(k_2 - k_1)\log\alpha$ to obtain

$$\left|\frac{\left|\log(1/\sqrt{5}) + \log(1 + \alpha^{m_1-n_1})\right|}{\log\alpha} - \frac{k_2 n_1 - k_1 n_2}{k_2 - k_1}\right| < \frac{8n_2}{\alpha^{n_1}(k_2 - k_1)\log\alpha}. \qquad (51)$$

We now put

$$\tau_\lambda := \frac{\left|\log(1/\sqrt{5}) + \log(1 + \alpha^{-\lambda})\right|}{\log\alpha}$$

and compute its continued fractions $[a_0^{(\lambda)}, a_1^{(\lambda)}, a_2^{(\lambda)}, \ldots]$ and its convergents $[p_0^{(\lambda)}/q_0^{(\lambda)}, p_1^{(\lambda)}/q_1^{(\lambda)}, p_2^{(\lambda)}/q_2^{(\lambda)}, \ldots]$ for each $\lambda \in [1, 1491]$. Furthermore, for each case we find an integer t_λ such that $q_{t_\lambda}^{(\lambda)} > M := 10^{154} > n_2 > k_2 - k_1$ and calculate

$$a(M) := \max_{1\le\lambda\le 1491}\left\{a_i^{(\lambda)} : 0 \le i \le t_\lambda\right\}.$$

A computer search in *Mathematica* reveals that for $\lambda = 61$, $t_\lambda = 276$ and $i = 224$, we have that $a(M) = a_{224}^{(61)} = 121895$. Hence, combining the conclusion of Lemma 3 and the inequality (51), we get

$$\alpha^{n_1} < 16.62 \times 121897 n_2(k_2 - k_1) < 2.02 \times 10^{314},$$

so $n_1 \le 1503$. Hence, we obtain that $n_1 \le 7019$ holds in all cases ($\nu = n_1$, $\lambda < \chi$ or $\lambda = \chi$). By the inequality (17), we have that

$$\log\delta \le k_1\log\delta \le (n_1 + 1)\log\alpha < 3378.$$

By considering the second inequality in (25), we can conclude that $n_2 \leq 1.0\times 10^{38}(\log n_2)^2$, which immediately yields $n_2 < 3.5\times 10^{40}$, by a simple application of Lemma 6. We summarise the first cycle of our reduction process as follows:

$$n_1 \leq 7019 \quad \text{and} \quad n_2 \leq 3.5\times 10^{40}. \tag{52}$$

From the above, we note that the upper bound on n_2 represents a very good reduction of the bound given in Lemma 8. Hence, we expect that if we restart our reduction cycle with the new bound on n_2, then we get a better bound on n_1. Thus, we return to the inequality (45) and take $M := 3.5\times 10^{40}$. A computer search in *Mathematica* reveals that

$$q_{86} > M > n_2 > k_2 - k_1 \quad \text{and} \quad a(M) := \max\{a_i : 0 \leq i \leq 86\} = a_{21} = 29,$$

from which it follows that $\lambda \leq 400$. We now return to (47) and we put $X := 5.25\times 10^{41}$ and $C := (10X)^5$ and then apply the LLL algorithm in Lemma 5 to $\lambda \in [1, 400]$. After a computer search, we get

$$1\times 10^{-172} < \min_{1\leq\lambda\leq 400} |\Gamma_4| < 16.62 n_2 \alpha^{-\nu},$$

then $\nu \leq 1022$. By continuing under the assumption that $n_j - m_j = \nu \leq 1022$, we return to (48) and put $X := 5.25\times 10^{41}$, $C := (10X)^9$ and $M := 3.5\times 10^{40}$ for the case $\lambda < \chi$ and $\lambda = \chi$. After a computer search, we confirm that

$$2\times 10^{-344} < \min_{\substack{1\leq\lambda\leq 400\\ \lambda+1\leq\chi\leq 1022}} |\Gamma_5| < 16.62 n_2 \alpha^{-n_1},$$

gives $n_1 \leq 1844$, and $a(M) = a_{55}^{(117)} = 30400$, leads to $n_1 \leq 415$. Hence, in both cases $n_1 \leq 1844$ holds. This gives $n_2 \leq 4.2\times 10^{38}$ by a similar procedure as before.

We record what we have proved.

Lemma 9. *Let (k_i, n_i, m_i) be a solution to $x_i = F_{n_i} + F_{m_i}$, with $0 \leq m_i \leq n_i$ for $i = 1, 2$ and $1 \leq k_1 < k_2$ and where $d \neq 5\square$, then*

$$\max\{k_1, m_1\} \leq n_1 \leq 1844 \quad \text{and} \quad \max\{k_2, m_2\} \leq n_2 \leq 4.2\times 10^{38}.$$

5.4 The final reduction

Returning back to (12) and (14) and using the fact that (x_1, y_1) is the smallest positive solution to the Pell equation (1), we obtain

$$\begin{aligned} x_k &= \delta^k + \sigma^k = \left(\frac{x_1 + y_1\sqrt{d}}{2}\right)^k + \left(\frac{x_1 - y_1\sqrt{d}}{2}\right)^k \\ &= \left(\frac{x_1 + \sqrt{x_1^2 \mp 4}}{2}\right)^k + \left(\frac{x_1 - \sqrt{x_1^2 \mp 4}}{2}\right)^k := P_k^{\pm}(x_1). \end{aligned}$$

Thus, we return to the Diophantine equation $x_{k_1} = P_{n_1} + P_{m_1}$ and consider the equations

$$P_{k_1}^{+}(x_1) = F_{n_1} + F_{m_1} \quad \text{and} \quad P_{k_1}^{-}(x_1) = F_{n_1} + F_{m_1}, \tag{53}$$

with $k_1 \in [1, 1844]$, $m_1 \in [0, 1844]$ and $n_1 \in [m_1 + 2, 1844]$.

Besides the trivial case $k_1 = 1$, with the help of a computer search in *Mathematica* on the above equations in (53), we list the only nontrivial solutions in Table 1. We also note that $3 + 2\sqrt{2} = (1 + \sqrt{2})^2$, so these solutions come from the same Pell equation when $d = 2$.

$P^+_{k_1}(x_1)$				
k_1	x_1	y_1	d	δ
2	6	4	2	$3+2\sqrt{2}$
2	4	2	3	$2+\sqrt{3}$
2	16	6	7	$8+3\sqrt{7}$
2	5	1	21	$(5+\sqrt{21})/2$
2	25	3	69	$(25+3\sqrt{69})/2$
2	40	2	399	$20+\sqrt{399}$

$P^-_{k_1}(x_1)$				
k_1	x_1	y_1	d	δ
2	2	2	2	$1+\sqrt{2}$
2	10	2	26	$5+\sqrt{26}$
2	12	2	37	$6+\sqrt{37}$
2	40	2	401	$20+\sqrt{401}$

Table 1. Solutions to $P^{\pm}_{k_1}(x_1) = F_{n_1} + F_{m_1}$

From the above tables, we set each $\delta := \delta_t$ for $t = 1, 2, \ldots 9$. We then work on the linear forms in logarithms Γ_1 and Γ_2, in order to reduce the bound on n_2 given in Lemma 9. From the inequality (23), for $(k, n, m) := (k_2, n_2, m_2)$, we write

$$\left| k_2 \frac{\log \delta_t}{\log \alpha} - n_2 + \frac{\log(\sqrt{5})}{\log \alpha} \right| < \left(\frac{4}{\log \alpha} \right) \alpha^{-(n_2 - m_2)}, \tag{54}$$

for $t = 1, 2, \ldots 9$.

We put

$$\tau_t := \frac{\log \delta_t}{\log \alpha}, \qquad \mu_t := \frac{\log(\sqrt{5})}{\log \alpha} \qquad \text{and} \quad (A_t, B_t) := \left(\frac{4}{\log \alpha}, \alpha \right).$$

We note that τ_t is transcendental by the Gelfond-Schneider's Theorem and thus, τ_t is irrational. We can rewrite the above inequality, (54) as

$$0 < |k_2 \tau_t - n_2 + \mu_t| < A_t B_t^{-(n_2 - m_2)}, \quad \text{for} \quad t = 1, 2, \ldots, 9. \tag{55}$$

We take $M := 4.2 \times 10^{38}$ which is the upper bound on n_2 according to Lemma 9 and apply Lemma 4 to the inequality (55). As before, for each τ_t with $t = 1, 2, \ldots, 9$, we compute its continued fraction $[a_0^{(t)}, a_1^{(t)}, a_2^{(t)}, \ldots]$ and its convergents $p_0^{(t)}/q_0^{(t)}, p_1^{(t)}/q_1^{(t)}, p_2^{(t)}/q_2^{(t)}, \ldots$. For each case, by means of a computer search in *Mathematica*, we find and integer s_t such that

$$q_{s_t}^{(t)} > 2.52 \times 10^{39} = 6M \qquad \text{and} \qquad \epsilon_t := \|\mu_t q^{(t)}\| - M\|\tau_t q^{(t)}| > 0.$$

We finally compute all the values of $b_t := \lfloor \log(A_t q_{s_t}^{(t)}/\epsilon_t)/\log B_t \rfloor$. The values of b_t correspond to the upper bounds on $n_2 - m_2$, for each $t = 1, 2, \ldots, 9$, according to Lemma 4. With the help of *Mathematica* we got that the maximum value of $n_2 - m_2$ is 201 for $t \in [1, 9]$. The results of the computation for each t are recorded in Table 2 below.

t	δ_t	s_t	q_{s_t}	$\varepsilon_t >$	b_t
1	$1+\sqrt{2}$	81	4.51994×10^{39}	0.388126	194
2	$2+\sqrt{3}$	72	8.76409×10^{40}	0.225348	201
3	$8+3\sqrt{7}$	76	1.32196×10^{40}	0.421692	196
4	$(5+\sqrt{21})/2$	80	6.12803×10^{39}	0.142135	197
5	$5+\sqrt{26}$	70	2.62621×10^{39}	0.158712	195
6	$6+\sqrt{37}$	89	3.06359×10^{39}	0.241184	194
7	$(25+3\sqrt{69})/2$	68	2.75772×10^{39}	0.048435	197
8	$20+\sqrt{399}$	84	2.84745×10^{39}	0.399493	193
9	$20+\sqrt{401}$	80	4.10314×10^{39}	0.125005	196

Table 2. First reduction computation results

By replacing $(k, n, m) := (k_2, n_2, m_2)$ in the inequality (20), we can write

$$\left| k_2 \frac{\log \delta_t}{\log \alpha} - n_2 + \frac{\log\left((\sqrt{5})/(1+\alpha^{-(n_2-m_2)})\right)}{\log \alpha} \right| < \left(\frac{4}{\log \alpha}\right)\alpha^{-n_2}, \text{ for } t = 1, 2, \ldots, 9. \quad (56)$$

We now put

$$\tau_t := \frac{\log \delta_t}{\log \alpha}, \quad \mu_{t,n_2-m_2} := \frac{\log\left((\sqrt{5})/(1+\alpha^{-(n_2-m_2)})\right)}{\log \alpha} \quad \text{and} \quad (A_t, B_t) := \left(\frac{4}{\log \alpha}, \alpha\right).$$

With the above notations, we can rewrite (56) as

$$0 < |k_2\tau_t - n_2 + \mu_{t,n_2-m_2}| < A_t B_t^{-n_2}, \quad \text{for} \quad t = 1, 2, \ldots 9. \quad (57)$$

We again apply Lemma 4 to the above inequality (57), for

$$t = 1, 2, \ldots, 9, \quad n_2 - m_2 = 1, 2, \ldots, b_t, \quad \text{with} \quad M := 4.2 \times 10^{38}.$$

We take

$$\varepsilon = \varepsilon_{t,n_2-m_2} := ||\mu_t q^{(t,n_2-m_2)}|| - M||\tau_t q^{(t,n_2-m_2)}|| > 0,$$

and

$$b = b_{t,n_2-m_2} := \lfloor \log(A_t q_{s_t}^{(t,n_2-m_2)}/\epsilon_{t,n_2-m_2})/\log B_t \rfloor.$$

With the help of Mathematica, we obtain the results in Table 3.

t	1	2	3	4	5	6	7	8	9
$\varepsilon >$	0.0019	0.0008	0.0006	0.0005	0.0017	0.0026	0.0016	0.0038	0.0071
b	207	215	211	213	204	205	209	204	202

Table 3. Final reduction computation results

$$\text{Therefore,} \quad \max\{b_{t,n_2-m_2} : t = 1, 2, \ldots, 9 \quad \text{and} \quad n_2 - m_2 = 1, 2, \ldots b_t\} \le 215.$$

Thus, by Lemma 4, we have that $n_2 \le 215$, for all $t = 1, 2, \ldots, 9$, and by the inequality (18) we have that $n_1 \le n_2 + 2$. From the fact that $\delta^k \le \alpha^{n+1}$, we can conclude that $k_1 < k_2 \le 104$. Collecting everything together, our problem is reduced to search for the solutions for (16) in the following range

$$1 \le k_1 \le k_2 \le 110, \quad 0 \le m_1 \le n_1 \le 220 \quad \text{and} \quad 0 \le m_2 \le n_2 \le 220.$$

After a computer search on the equation (16) on the above ranges, we obtained the following solutions, which are the only solutions for the exceptional d cases we have stated in Theorem 1:

For the +4 case:

$(d = 2)$ $x_1 = 6 = F_5 + F_2 = F_4 + F_4, \quad x_2 = 34 = F_9 + F_0 = F_8 + F_7;$

$(d = 3)$ $x_1 = 4 = F_4 + F_2 = F_3 + F_3, \quad x_2 = 14 = F_7 + F_2;$

$(d = 7)$ $x_1 = 16 = F_7 + F_4, \quad x_2 = 254 = F_{13} + F_8;$

$(d = 21)$ $x_1 = 5 = F_5 + F_0 = F_4 + F_3, \; x_2 = 23 = F_8 + F_3, \; x_3 = 110 = F_{11} + F_8.$

For the −4 case:

$(d = 2)$ $x_1 = 2 = F_3 + F_0 = F_2 + F_2, \quad x_2 = 6 = F_5 + F_2 = F_4 + F_4,$
$x_3 = 14 = F_7 + F_2, \quad x_4 = 34 = F_9 + F_0 = F_8 + F_7;$

$(d = 26)$ $x_1 = 10 = F_6 + F_3 = F_5 + F_5, \quad x_2 = 102 = F_{11} + F_7.$

This completes the proof of Theorem 1. □

Acknowledgements

M. D. was supported by the Austrian Science Fund (FWF) projects: F5510-N26 – Part of the special research program (SFB), "Quasi-Monte Carlo Methods: Theory and Applications" and W1230 –"Doctoral Program Discrete Mathematics". M. D. also thanks his supervisor Professor Robert Tichy for the encouragement and support during his PhD studies in Graz. F. L. was also supported by grant CPRR160325161141 from the NRF of South Africa, RTNUM18 from the CoEMaSS at Wits and by grant no. 17-02804S of the Czech Granting Agency.

References

[1] Bravo E F, Gómez-Ruiz C A, and Luca F, X-coordinates of Pell equations as sums of two tribonacci numbers, *Periodica Mathematica Hungarica*, **77** (2018) 175-190

[2] Gómez C A and Luca F, Zeckendorf representations with at most two terms to x-coordinates of Pell equations, *Science China Mathematics*, (2018) To appear

[3] Bugeaud Y, Mignotte M, and Siksek S, Classical Classical and modular approaches to exponential Diophantine equations I. Fibonacci and Lucas perfect powers. *Annals of Mathematics*, **163** (2006) 969-1018

[4] Cohen H, Number Theory. Volume I: Tools and Diophantine Equations *Graduate Texts in Mathematics* **239** (2017), (Springer) pp. 58-63

[5] Ddamulira M, On the x-coordinates of Pell equations which are sums of two Padovan numbers, Preprint (2019)

[6] Ddamulira M and Luca F, On the x-coordinates of Pell equations which are k-generalized Fibonacci numbers, *Journal of Number Theory*, **207** (2020) 156-195

[7] Dossavi-Yovo A, Luca F, and Togbé A, On the $X-$coordinates of Pell equations which are rep-digits, *Publicationes Mathematicae Debrecen*, **88** (2016) 381-399

[8] Dujella A and Pethő A, A generalization of a theorem of Baker and Davenport, *The Quarterly Journal of Mathematics*, **49** (1998) 291-306

[9] Faye B and Luca F, On the $X-$coordinates of Pell equations which are repdigits, *The Fibonacci Quarterly*, **56** (2018) 52-62

[10] Gúzman S S and Luca F, Linear combinations of factorials and s-units in a binary recurrence sequence, *Annales Mathemántiques du Québec*, **38** (2014) 169-188

[11] Kafle B, Luca F, and Togbé A, On the x-coordinates of Pell equations which are Fibonacci numbers II, *Colloquium Mathematicum*, **149** (2017) 75-85

[12] Kafle B, Luca F, and Togbé A, x-Coordinates of Pell equations which are Tribonacci numbers II, *Periodica Mathematica Hungarica*, **79** (2019) 157-167

[13] Kafle B, Luca F, and Togbé A, X-coordinates of Pell equations which are Lucas numbers, *Boletín de la Sociedad Matemática Mexicana*, **25** (2019) 481-493

[14] Luca F, Montejano A, Szalay L, and Togbé A, On the x-coordinates of Pell equations which are Tribonacci numbers, *Acta Arithmetica*, **179** (2017) 25-35

[15] Luca F and Togbé A, On the x-coordinates of Pell equations which are Fibonacci numbers, *Mathematica Scandinavica*, **122** (2018) 18-30

[16] Matveev, E M, An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers II, *Izv. Ross. Akad. Nauk Ser. Mat.*, **64** (2000) 125-180, in Russian: English translation in *Izv. Math.*, **64** (2000) 1217-1269

[17] Rihane S S, Hernane M O, and Togbé A, The x-coordinates of Pell equations and Padovan numbers, *Turkish Journal of Mathematics*, **43** (2019) 207-223