Dynamics of Cohomological Expanding Mappings I : First and Second Main Results

Armand Azonnahin
Institute of Mathematics and Statistics
University of São Paulo
armand@ime.usp.br

April 23, 2020

ABSTRACT

1 Let $f : \mathcal{V} \longrightarrow \mathcal{V}$ be a Cohomological Expanding Mapping¹ of a smooth complex compact homogeneous manifold with $\dim_{\mathbb{C}}(\mathcal{V}) = k \geq 1$ and Kodaira Dimension ≤ 0 . We
2 study the dynamics of such mapping from a probabilistic point of view, that is, we describe the asymptotic behavior of the orbit $O_f(x) = \{f^n(x), n \in \mathbb{N} \text{ or } \mathbb{Z}\}$ of a
3 generic point. Using pluripotential methods, we construct a natural invariant canonical probability measure of maximum Cohomological Entropy μ_f such that $\chi_{2l}^{-m}(f^m)^*\Omega \rightarrow$
4 μ_f as $m \rightarrow \infty$ for each smooth probability measure Ω on $\mathcal{V}$. Then we study the main stochastic properties of μ_f and show that μ_f is a measure of equilibrium, smooth, ergodic, mixing, K-mixing, exponential-mixing and the unique measure with maximum Cohomological Entropy. We also conjectured that $\mu_f := T_l^+ \wedge T_{k-l}^-$, $\dim_{\mathcal{H}}(\mu_f) = \Psi h_{\chi}(f)$
5 and $\dim_{\mathcal{H}}(\text{Supp} T_l^+) \geq 2(k-l) + \frac{\log \chi_{2l}}{\psi_l}$.
6
7
8
9
10
11

12 **Keywords** Complex Dynamics · Cohomological Expanding Mapping · Cohomological Degree · Cohomological Entropy · Cohomological Quotient .
13

14 **Classification – MSC2020** 37C45; 37D05; 37D20; 37D25; 37D35; 37F10; 37F15; 37F80

15 Contents

16	1 Introduction	2
17	2 First Main Result	5
18	3 Second Main Result	13
19	4 Conjectures	17

¹cf Definition 1.6.

1 Introduction

Let μ be the equilibrium measure associated with an endomorphism f . If φ is an observable, $(\varphi \circ f^n)_{n \geq 0}$ can be seen as a sequence of dependent random variables. As the measure is invariant, these variables are distributed in an identical way, that is, the Borel sets $\{\varphi \circ f^n < t\}$ have the same measure μ for any fixed constant t . We recall some general facts of ergodic theory and probability theory. We refer to [47, 49] for the general theory.

Consider a dynamic system associated with a map $g : X \rightarrow X$, measurable against a σ -algebra $\mathcal{F}$ on X . The direct image of a probability measure ν by g is the measure of probability $g_*(\nu)$ defined by

$$g_*(\nu)(A) := \nu(g^{-1}(A))$$

for each measurable set A . Likewise, for any positive measurable function φ , we have

$$\langle g_*(\nu), \varphi \rangle := \langle \nu, \varphi \circ g \rangle.$$

The measure ν is **invariant** if $g_*(\nu) = \nu$. When X is a compact metric space and g is continuous, the set $\mathcal{M}(g)$ of invariant probability measures is convex, compact and not empty: for any sequence of probability measures ν_N , the limit values of

$$\frac{1}{N} \sum_{j=0}^{N-1} (g^j)_*(\nu_N)$$

are invariant probability measures.

A measurable set A is **totally invariant** if $\nu(A \setminus g^{-1}(A)) = \nu(g^{-1}(A) \setminus A) = 0$. An invariant probability measure ν is **ergodic** if any totally invariant set is of measure ν zero or complete. It is easy to show that ν is ergodic if and only if $\varphi \circ g = \varphi$, for $\varphi \in L^1(\nu)$, then φ is constant. Here, we can replace $L^1(\nu)$ by $L^p(\nu)$ with $1 \leq p \leq +\infty$. The **ergodicity** of ν is also equivalent to the fact that it is extremal on $\mathcal{M}(g)$. We remember Birkhoff's ergodic theorem, which is the analogue of the law of large numbers for independent random variables [49].

Theorem 1.1 (Birkhoff). *Let $g : X \rightarrow X$ be a measurable map as above. Suppose that ν is an invariant ergodic probability measure. Let φ be a function on $L^1(\nu)$. Then*

$$\frac{1}{N} \sum_{n=0}^{N-1} \varphi(g^n(x)) \rightarrow \langle \nu, \varphi \rangle$$

almost everywhere in relation to ν .

When X is a compact metric space, we can apply Birkhoff's theorem to continuous functions φ and deduce that for ν almost all x

$$\frac{1}{N} \sum_{n=0}^{N-1} \delta_{g^n(x)} \rightarrow \nu,$$

where δ_x indicates the mass of Dirac at x . The sum

$$\text{St}_N(\varphi) := \sum_{n=0}^{N-1} \varphi \circ g^n$$

is called **Birkhoff sum**. Therefore, Birkhoff's theorem describes the behavior of $\frac{1}{N} \text{St}_N(\varphi)$ for an observable φ .

A stronger notion than **ergodicity** is the notion of **mixing**. An invariant probability measure ν is **mixing** if for each measurable set A, B

$$\lim_{n \rightarrow \infty} \nu(g^{-n}(A) \cap B) = \nu(A)\nu(B).$$

Clearly, **mixing** implies **ergodicity**. It is not difficult to see that ν is mixing if, and only if, for any test functions φ, ψ on $L^\infty(\nu)$ or on $L^2(\nu)$, we have

$$\lim_{n \rightarrow \infty} \langle \nu, (\varphi \circ g^n)\psi \rangle = \langle \nu, \varphi \rangle \langle \nu, \psi \rangle.$$

The Quantity

$$I_n(\varphi, \psi) := |\langle \nu, (\varphi \circ g^n)\psi \rangle - \langle \nu, \varphi \rangle \langle \nu, \psi \rangle|$$

is called **the correlation on time n** of φ and ψ . Thus, **mixing** is equivalent to the convergence of $I_n(\varphi, \psi)$ to 0. We say that ν is **K-mixing** if for each $\psi \in L^2(\nu)$

$$\sup_{\|\varphi\|_{L^2(\nu)} \leq 1} I_n(\varphi, \psi) \rightarrow 0.$$

1 Note that **K-mixing** is equivalent to the fact that the σ -algebra $\mathcal{F}_\infty := \cap g^{-n}(\mathcal{F})$ contains only sets zero
2 and complete measures. This is the strongest form of mixing for observables on $L^2(\nu)$. However, it is of
3 interest to obtain quantitative information about the mixing speed for more regular observables, such as
4 smooth functions or Hölder continuous.

Now consider an endomorphism f of degree $d \geq 2$ of $\mathbb{P}^k$ as above and its equilibrium measure μ . We know that μ is totally invariant: $f^*(\mu) = d^k \mu$. If φ is a continuous function, so

$$\langle \mu, \varphi \circ f \rangle = \langle d^{-k} f^*(\mu), \varphi \circ f \rangle = \langle \mu, d^{-k} f_*(\varphi \circ f) \rangle = \langle \mu, \varphi \rangle.$$

5 We use the obvious fact that $f_*(\varphi \circ f) = d^k \varphi$. Thus, μ is invariant.

6 Mixing for measure μ was proved in [45].

7 **Theorem 1.2.** *Let f be an endomorphism of degree $d \geq 2$ of $\mathbb{P}^k$. So its measure of Green μ is K-mixing.*

8 The equilibrium measure μ satisfies remarkable stochastic properties that are quite difficult to obtain in
9 the real dynamic systems scenario. Pluripotential methods replace the delicate estimates used in some real
10 dynamic systems.

11 Consider a dynamic system $g : (X, \mathcal{F}, \nu) \rightarrow (X, \mathcal{F}, \nu)$ as above, where ν is an invariant probability measure.
12 Therefore, g^* defines a linear operator of norm 1 on $L^2(\nu)$. We say that g has *the Jacobian limited* if there
13 is a constant $\kappa > 0$ such that $\nu(g(A)) \leq \kappa \nu(A)$ for each $A \in \mathcal{F}$.

14 When X is a complex manifold, it is necessarily orientable .

15 Let $\mathcal{V}$ be a smooth complex compact homogeneous manifold with $\dim_{\mathbb{C}}(\mathcal{V}) = k \geq 1$ and Kodaira dimension ≤ 0 and $f : \mathcal{V} \rightarrow \mathcal{V}$ be a dominant surjective meromorphic endomorphism, that is, whose Jacobian
16 is not identically null in any local chart. Let ω be a $(1, 1)$ -strictly positive Hermitian form on $\mathcal{V}$. Let ℓ be a
17 prime number.

18 **Definition 1.3.** The i -th **Cohomological Degree** $\chi_i(f)$ of f is defined as the spectral radius of the pullback
19 action f^* in the cohomology group ℓ -adic étale $H_{\text{ét}}^i(\mathcal{V}, \mathbb{Q}_\ell)$ independent of ℓ by: (cf [3] [6] [1] [4] [5] for
20 more details)

$$\chi_i(f) = \rho(f^*|_{H_{\text{ét}}^i(\mathcal{V}, \mathbb{Q}_\ell)}).$$

22 **Definition 1.4.** We define the (l, n) -th **Cohomological Quotient** $\xi_l^n(f)$ of f as follows:

$$\xi_l^n(f) = \left[\frac{\chi_{2l-1}(f)}{\chi_{2l}(f)} \right]^n$$

23 **Definition 1.5.** The **Cohomological Entropy** of f is defined by

$$h_\chi(f) = \max_i \log \chi_{2i}(f).$$

24 **Definition 1.6.** We say that f is a **Cohomological Expanding Mapping** when f is dynamically compatible
25 (that is $(f^n)^* = (f^*)^n$) and there is $l \in \{1, \dots, k\}$ such that :

$$\xi_l^{-1}(f) > 1.$$

26 We will write χ_i for $\chi_i(f)$ and ξ_l^n for $\xi_l^n(f)$ if there is no confusion.

27

Let $(M, \mathcal{F}, m)$ be a probability space and $g : M \rightarrow M$ be a measurable map that preserves m , that is, m is
 g_* -invariant: $g_* m = m$. The measure m is **ergodic** if for any measurable set A such that $g^{-1}(A) = A$,
we have $m(A) = 0$ or $m(A) = 1$. This is equivalent to the property that m is extremal on the convex set
of invariant probability measures (if m is mixing, so it is ergodic). When m is ergodic, Birkhoff's theorem
implies that if ψ is an observable on $L^1(m)$ then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left[\psi(x) + \psi(g(x)) + \dots + \psi(g^{n-1}(x)) \right] = \langle m, \psi \rangle$$

28 for m -almost all x .

Suppose now that $\langle m, \psi \rangle = 0$. Then, the previous limit is equal to 0. The theorem of limit central (TLC), when it occurs, provides the speed of this convergence. We say that ψ satisfies the TLC if there is a constant $\sigma > 0$ such that

$$\frac{1}{\sqrt{n}} \left[\psi(x) + \psi(g(x)) + \cdots + \psi(g^{n-1}(x)) \right]$$

converges in distribution for the Gaussian random variable $\mathcal{N}(0, \sigma)$ of mean 0 and variance σ . Remember that ψ is a *coboundary* whether there is a function ψ' on $L^2(\mu)$ such that $\psi = \psi' - \psi' \circ g$. In that case, it is easy to see that

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left[\psi(x) + \psi(g(x)) + \cdots + \psi(g^{n-1}(x)) \right] = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left[\psi'(x) - \psi'(g^n(x)) \right] = 0$$

in distribution. Therefore, ψ does not satisfy the TLC (sometimes it is said that ψ satisfies the TLC by $\sigma = 0$).

The TLC can be deduced from strong mixing, see [11, 46, 48]. In the following result, $Et(\psi|\mathcal{F}_n)$ indicates the expectation of ψ in relation to $\mathcal{F}_n$, that is, $\psi \mapsto Et(\psi|\mathcal{F}_n)$ is the orthogonal projection of $L^2(m)$ in the subspace generated by the measurable functions $\mathcal{F}_n$.

Theorem 1.7 (Gordin). *Consider the decreasing sequence $\mathcal{F}_n := g^{-n}(\mathcal{F})$, $n \geq 0$, of algebras. Let ψ be a function with real value on $L^2(m)$ such that $\langle m, \psi \rangle = 0$. Suppose that*

$$\sum_{n \geq 0} \|Et(\psi|\mathcal{F}_n)\|_{L^2(m)} < \infty.$$

So, the positive number σ defined by

$$\sigma^2 := \langle m, \psi^2 \rangle + 2 \sum_{n \geq 1} \langle m, \psi(\psi \circ g^n) \rangle$$

is finite. It vanishes if and only if ψ is a coboundary. Furthermore, when $\sigma \neq 0$, then ψ satisfies the TLC with variance σ .

Note that σ is equal to the limit of $n^{-1/2} \|\psi + \cdots + \psi \circ g^{n-1}\|_{L^2(m)}$. The last expression is equal to $\|\psi\|_{L^2(m)}$ if the family $(\psi \circ g^n)_{n \geq 0}$ is orthogonal on $L^2(m)$.

We refer to [47, 49] for the notion of Lyapunov exponent.

Definition 1.8. An invariant positive measure is *hyperbolic* if its Lyapunov exponents are non-zero.

A function *quasi-p.s.h.* on $\mathcal{V}$ is a function of $\mathcal{V}$ on $[-\infty, \infty)$, which is locally the sum of a plurisubharmonic function and a smooth function. For a given $(1, 1)$ -continuous form η , denote by $\text{PSH}_0(\eta)$ the set of quasi-p.s.h. functions φ such that $dd^c \varphi + \eta \geq 0$ and $\sup_{\mathcal{V}} \varphi = 0$. Equip $\text{PSH}_0(\eta)$ with induced distance of $L^1(\mathcal{V})$ using natural inclusion $\text{PSH}_0(\eta) \subset L^1(\mathcal{V})$.

Remember from [22] that a complex measure μ on $\mathcal{V}$ is considered *PC* if each quasi-p.s.h. function is μ -integrable and for each sequence $(\varphi_n)_{n \in \mathbb{N}}$ of quasi-p.s.h. functions converging to φ on L^1 , so that $dd^c \varphi_n + \eta \geq 0$ for some smooth form η independent of n , we have $\langle \mu, \varphi_n \rangle \rightarrow \langle \mu, \varphi \rangle$.

A *pluripolar set* on $\mathcal{V}$ is a subset of $\mathcal{V}$ contained on $\{\varphi = -\infty\}$ for some quasi-p.s.h. function φ . By [29], any locally pluripolar set on $\mathcal{V}$ is pluripolar. This result implies in particular that there are abundantly quasi-p.s.h singular functions on $\mathcal{V}$. Note that every PC measure has no mass on pluripolar sets.

Next, we will consider the dynamics of f with $\xi_t^{-1}(f) > 1$.

Here is the first Main Result.

Theorem 1.9. *Let $\mathcal{V}$ be a smooth compact complex homogeneous manifold with $\dim_{\mathbb{C}}(\mathcal{V}) = k \geq 1$ and Kodaira dimension ≤ 0 and $f : \mathcal{V} \rightarrow \mathcal{V}$ a **Cohomological Expanding Mapping**. Let ν be a complex measure with density L^{2k+1} on $\mathcal{V}$ such that $\nu(\mathcal{V}) = 1$. Let ω be a $(1, 1)$ -strictly positive Hermitian form on $\mathcal{V}$. So the sequence $\frac{1}{\chi_{2l}^m}(f^m)^* \nu$ converges weakly to a measure of probability PC μ_f with **Cohomological***

Entropy $\geq \log \chi_{2l}^{-1}$ independent of ν as $m \rightarrow \infty$ so that $\chi_{2l}^{-1} f^* \mu_f = \mu_f = f_* \mu_f$ and if f is holomorphic, then for each Hermitian metric ω on $\mathcal{V}$, μ_f is Hölder continuous on $\text{PSH}_0(\omega)$.

The Hölder continuity of μ_f on $\text{PSH}_0(\omega)$ for f holomorphic implies that μ_f is *moderate* in the sense that there are constants $\varepsilon, M > 0$ such that for each $\varphi \in \text{PSH}_0(\omega)$, we have

$$\int_{\mathcal{V}} e^{-\varepsilon \varphi} d\mu_f \leq M.$$

We remember a new class of functions called *weakly d.s.h.* that replace the role of d.s.h functions (differences of two functions quasi-psh) in case of Kähler. These functions enjoy a compactness property similar to that of the d.s.h functions and the pull-backs of d.s.h functions by meromorphic maps are weakly d.s.h. We obtain the property of **exponential mixing** of μ_f

Here is the second Main Result.

Theorem 1.10. *Let $\mathcal{V}, f, \chi_{2l}, \mu_f$ be as in Theorem 1.9. So μ_f is **exponential mixing** in the sense that for each constant $0 < \alpha \leq 1$, there is a constant A_α such that*

$$|\langle \mu_f, (\psi \circ f^m) \varphi \rangle - \langle \mu_f, \psi \rangle \langle \mu_f, \varphi \rangle| \leq A_\alpha \xi_l^{\frac{m\alpha}{2}} \|\psi\|_\infty \|\varphi\|_{C^\alpha}$$

for each $m \geq 0$, each $\psi \in L^\infty(\mathcal{V})$ and every function Hölder continuous φ of order α . In particular, μ_f is **K-mixing**.

If a real function Hölder continuous φ is not a **coboundary**, i.e, there is not $\psi \in L^2(\mathcal{V})$ with $\varphi = \psi \circ f - \psi$, and satisfies $\langle \mu, \varphi \rangle = 0$, then μ_f satisfies the **central limit theorem**, which means that there is a constant $\sigma > 0$ such that for each interval $I \subset \mathbb{R}$, we have

$$\lim_{p \rightarrow \infty} \mu_f \left\{ \frac{1}{\sqrt{p}} \sum_{j=0}^{p-1} \varphi \circ f^j \in I \right\} = \frac{1}{\sqrt{2\pi}\sigma} \int_I e^{-x^2/(2\sigma^2)} dx.$$

The expression $\langle \mu_f, (\psi \circ f^m) \varphi \rangle - \langle \mu_f, \psi \rangle \langle \mu_f, \varphi \rangle$ is called **the Correlation of order m** between the observables φ and ψ . The measure μ_f is said **mixing** if this correlation converges to 0, when m tends to infinity, for smooth observables (or equivalently, observables continuous, limited or $L^2(\mu_f)$).

Remember that $f_*\varphi$ is defined by

$$f_*\varphi(x) := \sum_{y \in f^{-1}(x)} \varphi(y)$$

where the points on $f^{-1}(x)$ are counted with multiplicities (there are exactly χ_{2k} points). Also define the **Perron-Frobenius Operator** by

$$\Lambda\varphi := \chi_{2k}^{-1} f_*\varphi.$$

As μ_f is totally invariant, this is the adjoint operator of f^* on $L^2(\mu_f)$.

2 First Main Result

In this section, we will prove the Theorem 1.9. For a current T of order 0 defined in a manifold $\mathcal{V}$, we denote by $\|T\|_{\mathcal{V}}$ the mass of T on $\mathcal{V}$. Let's write $\lesssim$ (resp. $\gtrsim$) for $\leq$ (resp. $\geq$) module a multiplicative constant independent of involving terms in inequality.

Theorem 2.1 (Theorem 1.9 " **First Main Result** "). *Let $\mathcal{V}$ be a smooth compact complex homogeneous manifold with $\dim_{\mathbb{C}}(\mathcal{V}) = k \geq 1$ and Kodaira dimension ≤ 0 and $f : \mathcal{V} \rightarrow \mathcal{V}$ a **Cohomological Expanding Mapping**. Let ν be a complex measure with density L^{2k+1} on $\mathcal{V}$ such that $\nu(\mathcal{V}) = 1$. Let ω be a $(1,1)$ -strictly positive Hermitian form on $\mathcal{V}$. So the sequence $\frac{1}{\chi_{2l}^m} (f^m)^*\nu$ converges weakly to a measure of probability PC μ_f with **Cohomological Entropy** $\geq \log \chi_{2l}$ independent of ν as $m \rightarrow \infty$ so that $\chi_{2l}^{-1} f^* \mu_f = \mu_f = f_* \mu_f$ and if f is holomorphic, then for each Hermitian metric ω on $\mathcal{V}$, μ_f is Hölder continuous on $\text{PSH}_0(\omega)$.*

Let $\mathcal{B}_r$ be the ball centered on 0 of radius r of $\mathbb{C}^k$, where $r \in \mathbb{R}^+$. For $r := 1$ we put $\mathcal{B} := \mathcal{B}_1$. The following result is very important.

Lemma 2.2. (Classical) *Let $r \in (0, 1)$. So, for each $(1,1)$ -closed real current R of order 0 defined on $\mathcal{B}$, there is a function U_R on $L^{1+1/(2k)}(\mathcal{B}_r)$ so that the following three properties are verified:*

(i)

$$R = dd^c U_R$$

on $\mathcal{B}_r$,

(ii)

$$\|U_R\|_{L^{1+1/(2k)}(\mathcal{B}_r)} \leq c_r \|R\|_{\mathcal{B}}$$

for some constant c_r independent of R ,

1 (iii) if $(R_n)_{n \in \mathbb{N}}$ is a sequence of $(1, 1)$ -closed real currents of order 0 of uniformly limited mass, converging
 2 weakly to R on $\mathcal{B}$ so $U_{R_n} \rightarrow U_R$ on $L^{1+1/(2k)}(\mathcal{B}_r)$.

3 *Proof.* The new point is the estimate for the norm $L^{1+1/(2k)}$ of the potential U_R and its continuity on R .
 4 These properties will be obtained by carefully examining the steps in the usual construction of U_R , cf [15, p.
 5 135] [5], [3], [8], [7] for example.

Let R be a $(1, 1)$ -real current closed on $\mathcal{B}$. Let $x \in \mathbb{C}^k$ be the canonical coordinate system. Let ρ be a smooth function supported compactly on $\mathcal{B}$ and $\int_{\mathcal{B}} \rho dx = 1$. For $y \in \mathcal{B}$, let $A_y : \mathcal{B} \rightarrow \mathcal{B}$ be the diffeomorphism defined by

$$A_y(x) := x + \frac{1}{2}(1 - \|x\|)y$$

for $x \in \mathcal{B}$. Since A_y is homotopic to $A_0 := \text{id}$ through homotopy $H_y : [0, 1] \times \mathcal{B} \rightarrow \mathcal{B}$ defined by $H_y(t, x) := A_{ty}(x)$ for $t \in [0, 1]$, the average

$$R' := \int_{\mathcal{B}} (A_y^* R) \rho(y) dy$$

is a smooth closed form that is cohomologous to R . Precisely, by the formula of homotopy, we have

$$R - R' = dL_1, \quad \text{where } L_1 = L_1(R) := \int_{\mathcal{B}} (H_y)_*([0, 1] \otimes R) \rho(y) dy.$$

6 Note that

$$\|R'\|_{L^\infty(\mathcal{B})} \lesssim \|R\|_{\mathcal{B}}, \quad \|L_1\|_{\mathcal{B}} \lesssim \|R\|_{\mathcal{B}}. \quad (2.1)$$

Since R' is a smooth closed form on $\mathcal{B}$, we can use an explicit formula (cf [15, p. 13]) to define a smooth form $L_2 = L_2(R')$ on $\mathcal{B}$ such that

$$R' = dL_2, \quad \|L_2\|_{L^\infty(\mathcal{B})} \lesssim \|R'\|_{L^\infty(\mathcal{B})}.$$

7 This combined with (2.1) shows that for $L_3 := L_1 + L_2$, we have

$$R = dL_3, \quad \|L_3\|_{\mathcal{B}} \lesssim \|R\|_{\mathcal{B}} \quad (2.2)$$

8 and L_3 continuously depends on R . So if $(R_n)_{n \in \mathbb{N}}$ is a sequence of $(1, 1)$ -currents of order 0 with uniformly
 9 limited mass, converging towards R so $L_3(R_n)$ is also of uniformly limited mass and converges to $L_3(R)$.

10 Since R is a $(1, 1)$ -real form, L_3 is a 1-real form. We decompose L_3 in the sum of one $(1, 0)$ -form and a
 11 $(0, 1)$ -form as

$$L_3 = L_3^{(1,0)} + L_3^{(0,1)} \quad (2.3)$$

12 such that $L_3^{(1,0)} = \overline{L_3^{(0,1)}}$ and $L_3^{(1,0)}, L_3^{(0,1)}$ are currents of order 0. We deduce from (2.2) that

$$\|L_3^{(0,1)}\|_{\mathcal{B}} \lesssim \|R\|_{\mathcal{B}} \quad (2.4)$$

13 For a bidirectional reason and the fact that $R = dL_3$, we have $\bar{\partial} L_3^{(0,1)} = 0$. It is known that there is a
 14 distribution v defined in an open neighborhood of $\bar{\mathcal{B}}_r$ with $\bar{\partial} v = L_3^{(0,1)}$. We will briefly remember how to
 15 build such a v as a function of $L_3^{(0,1)}$. The reference is [15, p. 28].

16 Let ρ be the function as above. We can assume $\rho \equiv 1$ on an open neighborhood of $\bar{\mathcal{B}}_r$. By the Koppelman
 17 formula, we have

$$\rho L_3^{(0,1)}(x) = \bar{\partial} \int_{\mathcal{B}} K_1(x, y) \wedge \rho(y) L_3^{(0,1)}(y) + \int_{\mathcal{B}} K_2(x, y) \wedge \bar{\partial} \rho(y) \wedge L_3^{(0,1)}(y). \quad (2.5)$$

18 We do not give explicit formulas here for K_1, K_2 but we emphasize only that K_1, K_2 are the products of
 19 $\|x - y\|^{-2k+1}$ with smooth forms on $\mathbb{C}^k$.

Denote by I_1, I_2 the first and second integrals, respectively, on the right side of (2.5). We have

$$\bar{\partial}I_1 + I_2 = \rho L_3^{(0,1)}$$

1 which is equal to $L_3^{(0,1)}$ on $\mathcal{B}_r$.

2 By the type of singularity of K_1 and the fact that $L_3^{(0,1)}$ is of order 0, we see that I_1 is a form with coefficients
3 in $L^{1+1/(2k)}(\mathcal{B})$ with

$$\|I_1\|_{L^{1+1/(2k)}(\mathcal{B})} \lesssim \|L_3^{(0,1)}\|_{\mathcal{B}} \lesssim \|R\|_{\mathcal{B}} \quad (2.6)$$

4 by (2.4). On the other hand, as $\bar{\partial}\rho \equiv 0$ on an open neighborhood of $\bar{\mathcal{B}}_r$, the current I_2 is smooth on $\mathcal{B}_{r'}$ for
5 some $r' > r$. Following exactly the arguments in [15, p. 29], we get a smooth function I_3 on $\mathcal{B}_{r'}$ for some
6 $r' > r$ such that $I_2 = \bar{\partial}I_3$ on $\mathcal{B}_r$ and

$$\|I_3\|_{L^\infty(\mathcal{B}_r)} \leq \|L_3^{(0,1)}\|_{\mathcal{B}} \lesssim \|R\|_{\mathcal{B}} \quad (2.7)$$

by (2.4) and $I_3 : R \mapsto I_3(R) \in L^\infty(\mathcal{B}_r)$ is continuous. So if $v := (I_1 + I_3)$ then

$$L_3^{(0,1)} = \bar{\partial}v$$

on $\mathcal{B}_r$. This together with (2.3) gives

$$L_3 = \bar{\partial}v + \partial\bar{v}.$$

We deduce from this and (2.2) that

$$R = dL_3 = \partial\bar{\partial}(v - \bar{v}).$$

Consequently $U_R := 2\pi \operatorname{Im} v$ satisfies $R = dd^c U_R$ (remember that $dd^c = (i/\pi)\partial\bar{\partial}$) and

$$\|U_R\|_{L^{1+1/(2k)}(\mathcal{B}_r)} \lesssim \|I_1\|_{L^{1+1/(2k)}(\mathcal{B}_r)} + \|I_3\|_{L^{1+1/(2k)}(\mathcal{B}_r)} \lesssim \|R\|_{\mathcal{B}}$$

7 by (2.6) and (2.7).

It remains to prove the property of continuity of U_R . We saw that I_3, L_3 are continuous on R . We just need
to check this property to I_1 . Let (R_n) be the sequence as defined above. Let's show that $I_1(R_n) \rightarrow I_1(R)$
on $L^{1+1/(2k)}(\mathcal{B})$. For the continuity property above of L_3 , we have that $S_n := \rho L_3^{(0,1)}(R_n)$ is of uniformly
limited mass and converges to $S := \rho L_3^{(0,1)}(R)$ when $n \rightarrow \infty$. Write

$$K_1(x, y) = \|x - y\|^{-2k+1} K_1'(x, y),$$

where $K_1'(x, y)$ is a smooth form. For every small constant $\varepsilon > 0$, let

$$K_{1,\varepsilon}(x, y) := \max\{\|x - y\|, \varepsilon\}^{-2k+1} K_1'(x, y)$$

which is a continuous form. Since $\varepsilon \rightarrow 0$, we have $K_{1,\varepsilon}(\cdot, y) \rightarrow K_1(\cdot, y)$ on $L^{1+1/(2k)}(\mathcal{B})$ uniformly on
 $y \in \mathcal{B}$. So when $n \rightarrow \infty$,

$$\int_{\{y \in \mathcal{B}\}} (K_{1,\varepsilon}(x, y) - K_1(x, y)) \wedge (S_n(y) - S(y)) \rightarrow 0$$

on $L^{1+1/(2k)}(\mathcal{B})$ because the mass of S_n is uniformly limited. On the other hand,

$$\int_{\{y \in \mathcal{B}\}} K_{1,\varepsilon}(x, y) \wedge (S_n(y) - S(y))$$

8 converges uniformly to 0 as ε is fixed because $K_{1,\varepsilon}$ is continuous. We deduce that $I_1(R_n) \rightarrow I_1(R)$ on
9 $L^{1+1/(2k)}(\mathcal{B})$. This completes the proof. $\blacksquare$

10 **Definition 2.3.** Let $\mathcal{V}$ be a complex manifold. A function of $\mathcal{V}$ to $[-\infty, \infty)$ is said *function quasi-p.s.h.*
11 if it can be written locally as the sum of a plurisubharmonic function (p.s.h.) and other smooth. For each
12 $(1, 1)$ -continuous form η , a function quasi-p.s.h. φ is η -p.s.h. if $dd^c\varphi + \eta \geq 0$. Through the partition of the
13 unit, each function quasi-p.s.h. is η -p.s.h. for some smooth form η . For a given form η , denote by $\operatorname{PSH}(\eta)$
14 the set of functions quasi-p.s.h. φ for which $dd^c\varphi + \eta \geq 0$.

15 **Definition 2.4.** A locally integrable function φ on $\mathcal{V}$ is said *weakly d.s.h.* if $dd^c\varphi$ is a current of order 0 on
16 $\mathcal{V}$. Let $\mathcal{W}$ be the complex vector space of all functions weakly d.s.h. on $\mathcal{V}$.

Definition 2.5. Every function quasi-p.s.h is weakly d.s.h.. A subset of $\mathcal{V}$ is a *pluripolar set* if it is contained on $\{\varphi = -\infty\}$ for some function quasi-p.s.h. φ . If $\mathcal{V}$ is compact, each locally pluripolar set is pluripolar by [29]. We use a specific case of this result: each analytic proper subset of a compact manifold $\mathcal{V}$ is pluripolar, cf Lemma 2.11 above.

Now consider that $\mathcal{V}$ is *compact*. Let μ_0 be a smooth probability measure on $\mathcal{V}$. We use this measure to define norms L^p on $\mathcal{V}$. For $\varphi \in \mathcal{W}$, put

$$\|\varphi\|_{\mathcal{W}} := \left| \int_{\mathcal{V}} \varphi d\mu_0 \right| + \|dd^c \varphi\|_{\mathcal{V}}, \quad (2.8)$$

where $\|\cdot\|_{\mathcal{V}}$ is the mass of a current on $\mathcal{V}$. Let's write from now $\|\cdot\|$ instead of $\|\cdot\|_{\mathcal{V}}$ if there is no confusion. The function $\|\cdot\|_{\mathcal{W}}$ is a norm on $\mathcal{W}$ because if $dd^c \varphi = 0$ then φ must be a constant. The norm $\|\cdot\|_{\mathcal{W}}$ is similar to the norm of the space of functions d.s.h. in case of Kähler introduced by Dinh-Sibony [22]. However, we do not know whether these two norms are equivalent in this case.

We introduce *the topology* on $\mathcal{W}$ in the following way: we say that $\varphi_n \in \mathcal{W}$ converges to $\varphi \in \mathcal{W}$ when $n \rightarrow \infty$ if $\varphi_n \rightarrow \varphi$ as current and $\|\varphi_n\|_{\mathcal{W}}$ is uniformly limited.

We have the following compactness result.

Lemma 2.6. *Let $\mathcal{V}$ be a compact complex manifold. There is a constant c so that for each function weakly d.s.h φ on $\mathcal{V}$ with $\int_{\mathcal{V}} \varphi d\mu_0 = 0$, we have*

$$\|\varphi\|_{L^{1+1/(2k)}(\mathcal{V})} \leq c \|dd^c \varphi\|_{\mathcal{V}}. \quad (2.9)$$

Furthermore, given a positive constant A , the set $\mathcal{W}_0$ of functions weakly quasi-p.s.h. φ with $\int_{\mathcal{V}} \varphi d\mu_0 = 0$ such that $\|dd^c \varphi\| \leq A$ is compact on $L^{1+1/(2k)}(\mathcal{V})$.

A direct consequence of Lemma 2.6 is that if $\varphi_n \rightarrow \varphi$ on $\mathcal{W}$ then $\varphi_n \rightarrow \varphi$ on $L^{1+1/(2k)}$. In case of Kähler, a similar version of inequality (2.9) for functions d.s.h. with norm L^p in place of norm $L^{1+1/(2k)}$ and $\|\cdot\|_*$ in place of $\|\cdot\|_{\mathcal{V}}$ was proven on [22] using cohomological tools for functions d.s.h. . His proof uses cohomological arguments that are not applicable to prove (2.9) for weakly functions quasi-p.s.h. .

Proof. Consider a function weakly quasi-p.s.h. φ with $\|dd^c \varphi\| \leq A$. Let (W_j) be an open (finite) cover of $\mathcal{V}$ where the W_j are local charts of $\mathcal{V}$ biholomorph to the unit ball of $\mathbb{C}^k$. Since $\|dd^c \varphi\| \leq A$, by Lemma 2.2, we have $\tau_j \in L^{1+1/(2k)}(W_j)$ for which $dd^c \tau_j = dd^c \varphi$ on W_j and

$$\|\tau_j\|_{L^{1+1/(2k)}(W_j)} \lesssim A. \quad (2.10)$$

Therefore, $\varphi - \tau_j$ can be represented by a pluriharmonic function on W_j . For simplicity, we identified this function with $(\varphi - \tau_j)$. We deduce that $\varphi \in L^{1+1/(2k)}(\mathcal{V})$.

We now assume, on the contrary, that (2.9) is not valid, it means that there is a sequence of non-null functions weakly quasi-p.s.h. φ_n with $\int_{\mathcal{V}} \varphi_n d\mu_0 = 0$ and

$$\infty > \|\varphi_n\|_{L^{1+1/(2k)}(\mathcal{V})} \geq n \|dd^c \varphi_n\|_{\mathcal{V}}.$$

Multiplying φ_n by a positive constant, we can assume that

$$\|\varphi_n\|_{L^{1+1/(2k)}(\mathcal{V})} = 1. \quad (2.11)$$

So we have

$$\|dd^c \varphi_n\| \leq 1/n. \quad (2.12)$$

Note that we still have $\int_{\mathcal{V}} \varphi_n d\mu_0 = 0$. Let τ_j^n be the function τ_j for φ_n in place of φ . Put $T_n := dd^c \varphi_n$. These currents of order 0 are of uniformly limited mass and converge to 0 by (2.12). The Lemma 2.2 tells us that τ_j^n converges to 0 on $L^{1+1/(2k)}(W_j')$, for each $W_j' \Subset W_j$. We can also provide that (W_j') continue to be a cover of $\mathcal{V}$. For simplicity, we can assume that $W_j' = W_j$ for each j .

Now remember that $\varphi_n - \tau_j^n$ is pluriharmonic on W_j . The last function is of $L^{1+1/(2k)}$ -norm limited on W_j because of (2.10) and (2.11). The average equality for pluriharmonic functions implies that $(\varphi_n - \tau_j^n)$ is of $\mathcal{C}^l$ -norm uniformly limited on compact subsets of W_j on $n \in \mathbb{N}$ for each $l \in \mathbb{N}$. We deduce that, extracting a subsequence, we can assume that $\varphi_n - \tau_j^n$ converging uniformly to a pluriharmonic function τ_j^∞ on compact subsets of W_j when $n \rightarrow \infty$. Since $\|\tau_j^n\|_{L^{1+1/(2k)}(W_j)} \rightarrow 0$, we get that

$$\varphi_n \rightarrow \tau_j^\infty \quad \text{em } L^{1+1/(2k)}(W_j).$$

1 This produces this function $\tau^\infty := \tau_j^\infty$ on W_j for each j is a well-defined pluriharmonic function on $\mathcal{V}$.
2 Since $\mathcal{V}$ is compact, τ^∞ is a constant. This combined with $\int_{\mathcal{V}} \varphi_n d\mu_0 = 0$ gives $\tau^\infty = 0$. We proved that
3 $\varphi_n \rightarrow 0$ on $L^{1+1/(2k)}(\mathcal{V})$, consequently $\|\varphi_n\|_{L^{1+1/(2k)}} \rightarrow 0$, a contradiction. Therefore, (2.9) is verified.

4 To prove the second desired statement, we again use the function τ_j above. We have that $\varphi - \tau_j$ is pluri-
5 harmonic on W_j and by (2.9), the $L^{1+1/(2k)}$ -norm of φ is also $\lesssim A$. Then the $L^{1+1/(2k)}$ -norm of the
6 pluri-harmonic function $(\varphi - \tau_j)$ is $\lesssim A$. It follows that its $\mathcal{C}^l$ -norm is also $\lesssim A$. Therefore, we can extract
7 a convergent subsequence of $(\varphi - \tau_j)$ for $\varphi \in \mathcal{W}$ on $\mathcal{C}^l$. This combined with the $L^{1+1/(2k)}$ continuity of τ_j
8 on T implies the desired statement. This completes the proof. ■

10 We equip the vector space $\mathcal{B}$ of Borel's measurable functions on $\mathcal{V}$ with the pointwise convergence topology:
11 $h_n \rightarrow h$ if h_n converges pointwise to h at almost all points (with respect to the Lebesgue measure). Let P
12 be a continuous linear endomorphism of the last vector space. Define $\mathcal{W}_P$ to be the set of $\varphi \in \mathcal{W}$ for which
13 $P\varphi \in \mathcal{W}$.

14 **Lemma 2.7.** *There is a constant c such that*

$$\|P\varphi\|_{L^{1+1/(2k)}} \leq c(\|\varphi\|_{\mathcal{W}} + \|dd^c(P\varphi)\|), \quad (2.13)$$

15 *for any $\varphi \in \mathcal{W}_P$. In particular, there is a constant c' such that*

$$\|P\varphi\|_{L^{1+1/(2k)}} \leq c(\|dd^c\varphi\| + \|dd^c(P\varphi)\|) \quad (2.14)$$

16 *for each $\varphi \in \mathcal{W}_P \cap \mathcal{W}_0$. Furthermore, if $\varphi_n \in \mathcal{W}_P \cap \mathcal{W}_0 \rightarrow \varphi$ as currents when $n \rightarrow \infty$ such that*
17 *$(\|dd^c\varphi_n\| + \|dd^c(P\varphi_n)\|)$ are uniformly bounded, then $P\varphi_n \rightarrow P\varphi$ on $L^{1+1/(2k)}$.*

18 *Proof.* The Inequality (2.14) is a direct consequence of (2.13) and of Lemma 2.6. Now suppose there is a
19 sequence $(\varphi_n) \subset \mathcal{W}_P$ for which

$$\|P\varphi_n\|_{L^{1+1/(2k)}} = 1, \quad \|\varphi_n\|_{\mathcal{W}} + \|dd^c(P\varphi_n)\| \leq 1/n. \quad (2.15)$$

20 Applying compactness property in Lema 2.6 for the sequence $(P\varphi_n)_{n \in \mathbb{N}}$, we see that by extracting a sub-
21 sequence from φ_n if necessary, the sequence $P\varphi_n$ converges on $L^{1+1/(2k)}$ for a function weakly d.s.h φ'_∞ .
22 Consequently,

$$\|\varphi'_\infty\|_{L^{1+1/(2k)}} = 1, \quad \|dd^c\varphi'_\infty\| = 0. \quad (2.16)$$

23 Therefore φ'_∞ is a constant. As the convergence on L^1 implies the convergence almost always of a subse-
24 quence, we can also assume that $P\varphi_n$ converges almost always to φ'_∞ .

25 On the other hand, the inequality of (2.15) allows us to use the compactness property in the Lemma 2.6
26 again for (φ_n) . Therefore, we can extract a subsequence of (φ_n) converging to $\varphi_\infty := 0$ on $L^{1+1/(2k)}$ and
27 almost always. Thus $P\varphi_n$ converges almost always to $P\varphi_\infty$ because of the continuity of P . It follows that
28 $\varphi'_\infty = P\varphi_\infty = 0$, note here $P(0) = 0$ by the linearity of P . This is a contradiction because of (2.16). Thus
29 (2.13) follows. The last desired statement follows directly from the arguments above. This completes the
30 proof. ■

Let $a \in \mathbb{C}^*$, r be a constant on $(0, |a|)$ and $\delta > 0$ a constant. Assume that $P(1) = a$, where 1 is the constant
function equal to 1 on $\mathcal{V}$. Define $\mathcal{W}_{P,r,\delta}^\infty$ to be the set of all $\varphi \in \mathcal{B}$ such that $P^n\varphi \in \mathcal{W}$ for each $n \geq 0$ and

$$\|dd^c(P^n\varphi)\| \leq \delta r^n$$

for each $n \geq 0$, here P^0 denotes the identity map. By the linearity of P , every constant function belongs to
 $\mathcal{W}_{P,r,\delta}^\infty$. We equip $\mathcal{W}_{P,r,\delta}^\infty$ with the topology induced from there on $\mathcal{W}$. Note that $\mathcal{W}_{P,r,\delta}^\infty$ is closed on $\mathcal{W}$ and

$$r^{-m}P^m(\mathcal{W}_{P,r,\delta}^\infty) \subset \mathcal{W}_{P,r,\delta}^\infty$$

32 for every positive integer m . So $\mathcal{W}_{P,r,\delta}^\infty \cap \mathcal{W}_0$ is compact and $P^m(\mathcal{W}_{P,r,\delta}^\infty)$ is contained in the complex
33 vector subspace $\widetilde{\mathcal{W}}_{P,r,\delta}^\infty$ of $\mathcal{W}$ generated by $\mathcal{W}_{P,r,\delta}^\infty$.

34 **Proposition 2.8.** *There is a continuous linear functional function $\mu_P : \widetilde{\mathcal{W}}_{P,r,\delta}^\infty \rightarrow \mathbb{C}$ such that for each*
35 *complex measure ν with density L^{2k+1} on $\mathcal{V}$, $\nu(\mathcal{V}) = 1$ and for each $\varphi \in \widetilde{\mathcal{W}}_{P,r,\delta}$, we have*

$$\langle a^{-n}(P^n)_*\nu, \varphi \rangle \rightarrow \langle \mu_P, \varphi \rangle. \quad (2.17)$$

1 Here for $Q : \mathcal{B} \rightarrow \mathcal{B}$, by definition, $\langle Q_*\nu, \varphi \rangle := \langle \nu, Q\varphi \rangle$ for $\varphi \in \mathcal{B}$ such that $Q\varphi$ is ν -integrable.

2 *Proof.* Remember that μ_0 is a form of smooth probability volume on $\mathcal{V}$. We just need to construct μ_P on
3 $\mathcal{W}_{P,r,\delta}^\infty$ and prove (2.17) for $\varphi \in \mathcal{W}_{P,r,\delta}^\infty$. The extension of μ_P to $\widetilde{\mathcal{W}}_{P,r,\delta}^\infty$ is done automatically using the
4 linearity of $(P^n)_*\nu$ and (2.17).

Let $\varphi \in \mathcal{W}_{P,r,\delta}^\infty$. Put $b_0 := \int_X \varphi d\mu_0$ and $\varphi_0 := \varphi - b_0$. We define two sequences φ_n, b_n as follows. Put

$$b_n = b_n(\varphi) := \int_X (P\varphi_{n-1}) d\mu_0, \quad \varphi_n := P\varphi_{n-1} - b_n$$

5 for $n \geq 1$. We have $r^{-n}\varphi_n \in \mathcal{W}_0 \cap \mathcal{W}_{P,r,\delta}^\infty$ and $dd^c(P^m\varphi_n) = dd^c(P^{m+n}\varphi)$ for each n, m . By Lemma
6 2.7, we have

$$\|\varphi_n\|_{L^{1+1/(2k)}} \leq c(\|dd^c(P\varphi_{n-1})\| + \|dd^c\varphi_{n-1}\|), \quad |b_n| \leq c(\|dd^c(P\varphi_{n-1})\| + \|dd^c\varphi_{n-1}\|) \quad (2.18)$$

7 for some constant c independent of n, φ . It follows that

$$\|\varphi_n\|_{L^{1+1/(2k)}} \leq c(\|dd^c(P^n\varphi)\| + \|dd^c(P^{n-1}\varphi)\|) \leq c\delta(r+1)r^{n-1}, \quad |b_n| \leq c\delta(r+1)r^{n-1} \quad (2.19)$$

8 for $n \geq 1$. Since $P(1) = a$ we have $P(b_n) = ab_n$ for each n . Using this, it gives

$$a^{-n}P^n\varphi = b_0 + a^{-n}P^n\varphi_0 = b_0 + a^{-n}P^{n-1}(P\varphi_0) = b_0 + a^{-1}b_1 + a^{-n}P^{n-1}\varphi_1 \quad (2.20)$$

$$= \dots = b_0 + a^{-1}b_1 + \dots a^{-n}b_n + a^{-n}\varphi_n. \quad (2.21)$$

Put $b'_n = b'_n(\varphi) := b_0 + a^{-1}b_1 + \dots a^{-n}b_n$ that converges to a number b'_∞ (depending on φ) by (2.19) and
the fact that $|a| > r$. We deduce from (2.20) that

$$|a^{-n}P^n\varphi - b'_n| \leq |a|^{-n}|\varphi_n|.$$

9 This combined with the first inequality of (2.19) implies that $a^{-n}P\varphi$ converges to b'_∞ on $L^{1+1/(2k)}$. Pre-
10 cisely, we have

$$\|a^{-n}P^n\varphi - b'_n\|_{L^{1+1/(2k)}} \lesssim \delta|a|^{-n}r^n. \quad (2.22)$$

Since $\nu(X) = 1$, we have

$$\langle a^{-n}(P^n)_*\nu, \varphi \rangle - b'_n = \langle \nu, a^{-n}P^n\varphi - b'_n \rangle.$$

11 Using this, (2.22) and Hölder's inequality imply that $\langle a^{-n}(P^n)_*\nu, \varphi \rangle$ converges to $b'_\infty = b'_\infty(\varphi)$ because
12 ν has L^{2k+1} density. Define $\langle \mu_P, \varphi \rangle := b'_\infty(\varphi)$ that is independent of ν . Then, we obtain the desired
13 convergence for μ_P .

14 Consider a sequence $\tilde{\varphi}_m \rightarrow \varphi$ on $\mathcal{W}_{P,r,\delta}^\infty$. Let $\tilde{b}_{nm}, \tilde{\varphi}_{nm}$ respectively the b_n and φ_n for $\tilde{\varphi}_m$ in place of φ .
15 By the last statement of the Lemma 2.7, $\tilde{b}_{nm} \rightarrow b_n$ when $m \rightarrow \infty$ for each n and (2.19) still applies to
16 $\tilde{b}_{nm}, \tilde{\varphi}_{nm}$ in place of b_n, φ_n . We infer that $\tilde{b}'_{nm} \rightarrow b'_n$ and $a^{-n}\tilde{\varphi}_{nm} \rightarrow 0$ on $L^{1+1/(2k)}$ when $m \rightarrow \infty$.
17 Thus, $\langle \mu_P, \tilde{\varphi}_m \rangle \rightarrow \langle \mu_P, \varphi \rangle$ when $m \rightarrow \infty$. In other words, μ_P is continuous. This completes the proof. ■

19 Let $\mathcal{V}$ be a complex compact manifold and f be a meromorphic self-map on $\mathcal{V}$. Denote by Γ the graph of f
20 on $\mathcal{V} \times \mathcal{V}$ and π_1, π_2 the restrictions to Γ of natural projections of $\mathcal{V} \times \mathcal{V}$ for the first and second components
21 respectively.

22 Let Φ be a form with measurable coefficients on $\mathcal{V}$. We say that $\Phi \in L^1$ if its coefficients are L^1 functions
23 (in relation to the Lebesgue measure on $\mathcal{V}$). If Ω is a dense open subset of Zariski of $\mathcal{V}$ such that π_2 is a
24 unrestricted cover on Ω , the form $f_*\Phi := (\pi_2|_{\pi_2^{-1}(\Omega)})_*(\pi_1^*\Phi)$ is a measurable form on Ω . Consequently $f_*\Phi$
25 is a measurable form on $\mathcal{V}$ independent of Ω . We can verify that $f_* : \mathcal{B} \rightarrow \mathcal{B}$ is continuous. Consequently,
26 f_* is an example of the map P considered above.

27 If $f_*\Phi \in L^1$, then we can define $f_*\Phi$ to be a current of order 0 induced by $f_*\Phi$ on $\mathcal{V}$. This definition is
28 independent of the choice of Ω . Note that the pull-back by f of smooth functions or smooth forms is always
29 on L^1 . The following is similar to the results on [9, 23].

1 **Lemma 2.9.** For each quasi-p.s.h. function φ on $\mathcal{V}$, we have $f_*\varphi \in L^1$ and if $dd^c\varphi + \eta \geq 0$ for some
2 $(1, 1)$ -continuous form $\eta > 0$, then $dd^c(f_*\varphi) + f_*\eta \geq 0$. In particular,

$$(f^n)_*\varphi \in \mathcal{W}_{f_*} \cap \mathcal{W}. \quad (2.23)$$

3 The inclusion (2.23) explains the crucial roles of $\mathcal{W}_{f_*}, \mathcal{W}$ in this study.

Proof. Let $\sigma : \Gamma' \rightarrow \Gamma$ be a desingularization of Γ . Let Ω be as above. Put $\pi'_j := \pi_j \circ \sigma$ for $j = 1, 2$. Since φ is quasi-p.s.h., $\varphi \circ \pi'_1$ is also. Thus, $\varphi \circ \pi_1 = \sigma_*(\varphi \circ \pi'_1)$ is on $L^1(\Gamma_f)$. Since

$$\|f_*\varphi\|_{L^1(\Omega)} = \|(\pi_2)_*(\varphi \circ \pi_1)\|_{L^1(\Omega)} \lesssim \|\varphi \circ \pi_1\|_{L^1(\Gamma)},$$

4 we get the first desired statement.

By [2], [4] and the fact that $\eta > 0$, there is a decreasing sequence of smooth functions quasi-p.s.h φ_n converging pointwise to φ such that $dd^c\varphi_n + \eta \geq 0$ for each n . By Lebesgue's dominated convergence theorem, the sequence $\varphi_n \circ \pi'_1$ converges on L^1 to $\varphi \circ \pi'_1$. It follows that the sequence of positive smooth forms $dd^c(\varphi_n \circ \pi'_1) + \pi'^{*}_1\eta$ converges weakly to $dd^c(\varphi \circ \pi'_1) + \pi'^{*}_1\eta$. Thus, the last current is also positive. Now note that

$$(\pi'_2)_*(dd^c(\varphi \circ \pi'_1) + \pi'^{*}_1\eta) = dd^c((\pi'_2)_*\pi'^{*}_1\varphi) + (\pi'_2)_*\pi'^{*}_1\eta = dd^c((\pi_2)_*\pi'^{*}_1\varphi) + (\pi_2)_*\pi'^{*}_1\eta$$

5 because $\pi'^{*}_1\varphi$ and $\pi'^{*}_1\eta$ have no mass in zero Lebesgue measure sets. Therefore $dd^c(f_*\varphi) + f_*\eta \geq 0$.

6 Note that $f_*\eta$ has finite mass on $\mathcal{V}$. We infer that $f_*\varphi \in \mathcal{W}$. In other words, $\varphi \in \mathcal{W}_{f_*} \cap \mathcal{W}$. Applying this to
7 f^n instead of f and using the formula that $(f^n)_*\varphi = f_*(f^{n-1})_*\varphi$ as functions in some suitable open dense
8 subset of $\mathcal{V}$, we get (2.23). This completes the proof. ■

9
10 **Lemma 2.10.** Let $\mathcal{V}$ be a compact complex manifold of dimension k and $f : \mathcal{V} \rightarrow \mathcal{V}$ be a **Cohomological**
11 **Expanding Mapping**. Let φ be a function quasi-p.s.h. on $\mathcal{V}$ with $dd^c\varphi + \eta \geq 0$ for some $(1, 1)$ -continuous
12 form η . So there is a constant A independent of φ, η for which

$$\|dd^c(f^n)_*\varphi\| \leq A\chi_{2l-1}^n\|\eta\|_{L^\infty} \quad (2.24)$$

13 for each $n \geq 1$.

Proof. Replacing η by a strictly positive smooth form that dominates it, we can assume that $\eta > 0$. Let ω be a metric of Gauduchon on $\mathcal{V}$, this means that ω is a Hermitian metric and $dd^c\omega^{k-1} = 0$, cf [?]. Let Γ_n be the graph of f^n and $\pi_{1,n}, \pi_{2,n}$ the natural maps of Γ_n for the first and second components of $\mathcal{V} \times \mathcal{V}$. By Lemma 2.9, the current $dd^c(f^n)_*\varphi + (f^n)_*\eta$ is positive. So, using $dd^c\omega^{k-1} = 0$ gives

$$\|dd^c(f^n)_*\varphi + (f^n)_*\eta\| \lesssim \langle dd^c(f^n)_*\varphi + (f^n)_*\eta, \omega^{k-1} \rangle = \langle (f^n)_*\eta, \omega^{k-1} \rangle \lesssim \langle (f^n)_*\eta, \omega^{k-1} \rangle$$

This combined with the definition of $\chi_{2l-1}(f)$ gives

$$\|dd^c(f^n)_*\varphi + (f^n)_*\eta\| \leq A(\chi_{2l-1})^n\|\eta\|_{L^\infty}.$$

14 The desired inequality follows immediately. This completes the proof. ■

15
16 We come now to the end of the proof of the first main result.

End of Proof of Theorem 1.9. $\xi_l^{-1}(f) > 1$. Put

$$P := f_*, \quad a := \chi_{2l}, \quad r := \chi_{2l-1}, \quad \delta := A,$$

17 where A is the constant on Lemma 2.10. Let φ be a function quasi-p.s.h. with $dd^c\varphi + \eta \geq 0$ for some
18 $(1, 1)$ -continuous form $\eta > 0$ such that $\|\eta\|_{L^\infty} \leq 1$. We have $P(1) = a$ and $\varphi \in \mathcal{W}_{P,r,\delta}^\infty$ by Lemma 2.10.

19 Every function quasi-p.s.h. is on $\widetilde{\mathcal{W}}_{P,r,\delta}^\infty$. Since ν does not have mass in proper analytical subsets of $\mathcal{V}$, Note
20 that

$$\langle (f^m)^*\nu, \varphi \rangle = \langle \nu, (f^m)_*\varphi \rangle = \langle \nu, P^m\varphi \rangle \quad (2.25)$$

because we only need to consider integrals on a dense open subset of Zariski of $\mathcal{V}$. Applying Proposition 2.8 for P , we get a continuous functional μ_P on $\widetilde{\mathcal{W}}_{P,r,\delta}^\infty$ such that

$$\langle \chi_{2l}^{-m}(f^m)^*\nu, \varphi \rangle \rightarrow \langle \mu_P, \varphi \rangle,$$

1 for each $\varphi \in \widetilde{\mathcal{W}}_{P,r,\delta}^\infty$. Choosing $\nu \geq 0$, we see that $\langle \mu_P, \varphi \rangle \geq 0$ if $\varphi \geq 0$. Let μ_f be the probability measure
2 on $\mathcal{V}$ defined by $\langle \mu_f, \varphi \rangle := \langle \mu_P, \varphi \rangle$ for each smooth function φ . Remember here that smooth functions are
3 quasi-p.s.h. on $\mathcal{V}$. Let's prove that $\mu_f = \mu_P$ for each function quasi-p.s.h. φ .

4 Consider a sequence of smooth functions quasi-p.s.h. φ'_m with $dd^c \varphi'_m + \eta \geq 0$ decreasing to φ , we have
5 $\langle \mu_f, \varphi'_m \rangle = \langle \mu_P, \varphi'_m \rangle$ and $\langle \mu_f, \varphi'_m \rangle \rightarrow \langle \mu_f, \varphi \rangle$ by Lebesgue's monotonous convergence theorem. This
6 combined with the continuity of μ_P gives $\langle \mu_f, \varphi \rangle = \langle \mu_P, \varphi \rangle$. So we have

$$\lim_{m \rightarrow \infty} \langle \chi_{2l}^{-m}(f^m)^*\nu - \mu_f, \varphi \rangle = 0 \quad (2.26)$$

7 for each function quasi-p.s.h. φ on $\mathcal{V}$.

8 As the functions quasi-p.s.h. are μ_f -integrable, μ_f has no mass on pluripolar sets. By Lemma 2.11 below,
9 proper analytic subsets of $\mathcal{V}$ are pluripolar. This implies that μ_f has no mass on proper analytic subsets of
10 $\mathcal{V}$. We deduce that the pull-back $f^*\mu_f$ is well defined. Here we just take the pull-back of μ_f on an open
11 subset of Zariski Ω of $\mathcal{V}$ where π_2 is a non-branched cover. It can be seen that this definition is independent
12 of the choice of Ω and if $(\Phi_m)_{m \in \mathbb{N}}$ is a sequence of positive measures without mass on the proper analytical
13 subsets of $\mathcal{V}$ and converging to μ_f , then $f^*\Phi_m$ converges to $f^*\mu_f$ because the mass of $f^*\Phi_m$ converges to
14 that of $f^*\mu_f$, cf for example [28, Lema 3.6]. The Equality

$$\chi_{2l}^{-1} f^* \mu_f = \mu_f \quad (2.27)$$

15 is obtained by applying the pull-back f^* for convergence $\chi_{2l}^{-m}(f^m)^*\nu \rightarrow \mu_f$, where ν is a smooth measure
16 of probability. Once we have $f_* f^* = \chi_{2l}$ on Borel's measurable functions, we get $f_* \mu_f = \mu_f$, in other
17 words, μ_f is invariant by f .

18 Let I_f be the indeterminacy set of f . Put $Z := \cup_{m \in \mathbb{Z}} f^m(I_f)$. The measure μ_f has no mass on Z . The
19 cohomological entropy of μ_f is by definition $1_{\mathcal{V} \setminus Z} \mu_f$ in relation to $f|_{\mathcal{V} \setminus Z}$. For Parry's inequality [24, 27],
20 using $f^* \mu_f = \chi_{2l} \mu_f$, we deduce that the cohomological entropy of μ_f is at least $\log \chi_{2l}$.

21 Suppose now that f is holomorphic. To prove that μ_f is Hölder continuous on $\text{PSH}(\omega)$, we use a known
22 idea of [24]. Without loss of generality, we can assume that $\|\omega\|_{L^\infty} \leq 1$. Let φ, ψ be two functions quasi-
23 p.s.h. on $\text{PSH}(\omega)$. Remember that they are on $\mathcal{W}_{P,r,\delta}^\infty$.

Let $b_n(\varphi), b_n(\psi)$ be as in the proof of the proposition 2.8. Let J_f be the Jacobian of f . We have

$$\|f_* \varphi - f_* \psi\|_{L^1} = \sup_{\|h\|_{L^\infty} \leq 1} |\langle f_* \varphi - f_* \psi, h \mu_0 \rangle| = \sup_{\|h\|_{L^\infty} \leq 1} |\langle \varphi - \psi, (h \circ f) f^* \mu_0 \rangle|$$

what is

$$\leq \|J_f\|_{L^\infty} \|\varphi - \psi\|_{L^1}.$$

Applying the latest inequality to f^n in place of f gives

$$|b_n(\varphi) - b_n(\psi)| \leq 2^n \|J_f\|_{L^\infty}^n \|\varphi - \psi\|_{L^1}.$$

Put

$$A_1 := \sum_{n=0}^{M+1} \chi_{2l}^{-n} [b_n(\varphi) - b_n(\psi)], \quad A_2 := \sum_{n=M+1}^{\infty} \chi_{2l}^{-n} [b_n(\varphi) - b_n(\psi)].$$

Using (2.20) gives

$$\langle \mu_f, \varphi - \psi \rangle = A_1 + A_2, \quad |A_1| \leq \sum_{n=0}^M \chi_{2l}^{-n} 2^n \|J_f\|_{L^\infty}^n \|\varphi - \psi\|_{L^1}, \quad |A_2| \lesssim (\chi_{2l-1})^M \chi_{2l}^{-M}.$$

Consider the case where $2\|J_f\|_{L^\infty} \leq \chi_{2l}$. We have $|A_1| \leq M\|\varphi - \psi\|_{L^1}$. Choosing M to be the smallest
integer for which $M \geq -\log \|\varphi - \psi\|_{L^1} / \log \tau$, where $\tau := \chi_{2l} / (\chi_{2l-1})$, we get that

$$|\langle \mu_f, \varphi - \psi \rangle| \leq |A_1| + |A_2| \lesssim \|\varphi - \psi\|_{L^1}^{1-\varepsilon}$$

which implies that μ_f is Hölder continuous in that case. It remains to treat the case $2\|J_f\|_{L^\infty} \geq \chi_{2l}$. We
have

$$|A_1| \leq M 2^M \chi_{2l}^{-M} \|J_f\|_{L^\infty}^M \|\varphi - \psi\|_{L^1} + \tau^{-M}.$$

Choose $M := -\log \|\varphi - \psi\|_{L^1} / \log(2\chi_{2l}^{-1} \tau \|J_f\|_{L^\infty})$. We see that

$$|A_1| + |A_2| \lesssim -\log \|\varphi - \psi\|_{L^1} \|\varphi - \psi\|_{L^1}^{\log \tau / \log(2\chi_{2l}^{-1} \tau \|J_f\|_{L^\infty})}.$$

24 Consequently, μ_f is also Hölder continuous in this case. This completes the proof. ■

Now we would like to say something about Theorem 1.10. If we try to imitate the arguments in the proof of [22, Teorema. 1.3] to prove Theorem 1.10, we are led to estimate $|\langle \mu_f, |\varphi_n| \rangle|$. The measure μ_f still satisfies the property that for each ω -p.s.h. function φ with $\sup_{\mathcal{V}} \varphi = 0$ is of $L^1(\mu_f)$ -norm uniformly limited, cf [22, Proposition 2.3]. But unlike the case of Kähler, we don't know if φ_n is the difference of two ω -p.s.h functions. So this explains why we cannot directly apply the approach in [22] to obtain a correlation decay for μ_f .

Lemma 2.11. *Any proper analytic subset V of a complex compact manifold $\mathcal{V}$ is a pluripolar set on $\mathcal{V}$.*

Proof. We use here the idea in [22] where the authors prove the same result when $\mathcal{V}$ is Kähler. Suppose now that V is smooth and $\text{codim} V \geq 2$ (otherwise the problem is trivial). Let $\sigma : \widehat{\mathcal{V}} \rightarrow \mathcal{V}$ be the explosion of $\mathcal{V}$ along V . Denote by $\widehat{V}$ the exceptional hypersurface.

Let ω be a positive-defined Hermitian form on $\mathcal{V}$. Let $\widehat{\omega}_h$ be a form of Chern of $\mathcal{O}(-\widehat{V})$ whose restriction to each fiber of $\widehat{\mathcal{V}} \approx \mathbb{P}(E)$ is strictly positive. Choosing ω if necessary, we can assume that $\widehat{\omega} := \sigma^* \omega + \widehat{\omega}_h > 0$. Since $\sigma_* \widehat{\omega}_h = \sigma_* \widehat{\omega} - \omega$, the closed current $\sigma_* \widehat{\omega}_h$ is quasi positive. Thus, there is a function quasi-p.s.h. φ on $\widehat{\mathcal{V}}$ such that

$$\sigma_* \widehat{\omega}_h = dd^c \varphi + \eta \quad (2.28)$$

for some smooth closed form η . Multiplying $\widehat{\omega}_h$ by a strictly positive constant, we have $\sigma^* \sigma_* \widehat{\omega}_h = \widehat{\omega}_h + [\widehat{V}]$. Thus $|\varphi \circ \sigma(\widehat{x}) - \log \text{dist}(\widehat{x}, \widehat{V})|$ is a limited function on $\widehat{\mathcal{V}}$. As a consequence,

$$|\varphi(x) - \log \text{dist}(x, V)| \lesssim 1 \quad (2.29)$$

on compact subsets of $\mathcal{V}$. Consequently, V is contained in $\{\varphi = -\infty\}$. Thus V is pluripolar in this case.

By the construction above, we can build a Hermitian metric in the explosion $\widehat{\mathcal{V}}$ of $\mathcal{V}$ along V as the sum of a pull-back of a Hermitian on $\mathcal{V}$ and an appropriate form of Chern of $\mathcal{O}(-\widehat{V})$. Thus, if $\sigma' : \widehat{\mathcal{V}}' \rightarrow \mathcal{V}$ is a composition of explosions along smooth submanifolds, so there's a form $(1, 1)$ closed and smooth η' on $\widehat{\mathcal{V}}'$ and a Hermitian metric ω on $\mathcal{V}$ such that $\widehat{\omega}' = \sigma'^* \omega + \eta'$ is a Hermitian metric on $\widehat{\mathcal{V}}'$.

Now consider the general situation where V is an analytical subset of $\mathcal{V}$. As a finite union of pluripolar sets is again pluripolar, it is enough to prove that the regular part $\text{Reg} V$ of V is a pluripolar set because we can write V as a finite union of the regular parts of suitable analytical subsets of $\mathcal{V}$. By Hironaka's desingularization, there is a composition $\sigma' : \widehat{\mathcal{V}}' \rightarrow \mathcal{V}$ of explosions along smooth submanifolds that do not cross $\text{Reg} V$ (or their inverse images) so that the strict transformation $\widehat{\mathcal{V}}'$ of V is smooth.

Let $\widehat{\omega}', \omega, \eta$ be as above. For the above arguments, $\widehat{\mathcal{V}}' \subset \{\widehat{\varphi}' = -\infty\}$ for some function quasi-p.s.h. $\widehat{\varphi}'$ on $\widehat{\mathcal{V}}'$ and $dd^c \widehat{\varphi}' + \widehat{\omega}' \geq 0$. Put $S := \sigma'_*(dd^c \widehat{\varphi}' + \eta')$ which is a $(1, 1)$ -current closed on $\mathcal{V}$ and $S + \omega \geq 0$. We can write

$$S = dd^c \varphi_S + \eta_S, \quad \sigma_* \eta' = dd^c \psi + \eta$$

for some smooth closed forms η_S, η . We have

$$dd^c \varphi_S + \eta_S + \omega \geq 0, \quad dd^c \psi + \eta + \omega \geq 0.$$

Thus φ_S, ψ are quasi-p.s.h. functions on $\mathcal{V}$. In addition, we also have

$$\varphi_S = \sigma'_*(\widehat{\varphi}') + \psi + \text{a smooth function}$$

on an open neighborhood of $\text{Reg} V$ in which σ' is biholomorph. Consequently, $\text{Reg} V \subset \{\varphi'_S = -\infty\}$. This completes the proof. ■

3 Second Main Result

In this section, we prove the Theorem 1.10. Our idea is to consider suitable test functions in the Sobolev space $W^{1,2}$. This approach is inspired by [21].

Fix a smooth volume form μ_0 on $\mathcal{V}$ and we use this form to define the norm in space $L^2(\mathcal{V})$. Let $W^{1,2}$ be the function space with real value $\varphi \in L^2(\mathcal{V})$ such that $d\varphi$ has L^2 coefficients. Remember the following inequality of Poincaré-Sobolev: for $\varphi \in W^{1,2}$ with $\int_{\mathcal{V}} \varphi d\mu_0 = 0$, we have

$$\|\varphi\|_{L^2} \leq c \|d\varphi\|_{L^2}, \quad (3.1)$$

for some constant c independent of φ , cf for example [26] or [25]. Note that the term $\|d\varphi\|_{L^2}^2$ is comparable to the mass of the positive current $i\partial\varphi \wedge \bar{\partial}\varphi$. We have the following lemma.

Lemma 3.1. ([21, Pro. 3.1]) Let I be a compact subset of $\mathcal{V}$ ($2k-1$)- Hausdorff's zero dimensional measure. Let φ be a function with real value $L^1_{loc}(\mathcal{V} \setminus I)$. Suppose that the coefficients of $d\varphi$ are in $L^2(\mathcal{V} \setminus I)$. Then $\varphi \in W^{1,2}$ and there is a compact subset M of $\mathcal{V} \setminus I$ and a constant $c > 0$ both independent of φ such that

$$\|\varphi\|_{L^1(\mathcal{V})} \leq c(\|\varphi\|_{L^1(M)} + \|d\varphi\|_{L^1(\mathcal{V})}).$$

Definition 3.2. Let $W^{1,2}_{*,f}$ be the subset of $W^{1,2}$ consisting of φ such that there are $m_1 \in \mathbb{N}$, a $(1,1)$ -continuous form η and a function η -p.s.h. ψ satisfying

$$i\partial\varphi \wedge \bar{\partial}\varphi \leq dd^c((f^{m_1})_*\psi) + (f^{m_1})_*\eta \quad (3.2)$$

as currents. A size representative of φ is $\mathbf{m} := (m_0, m_1)$, where m_0 is an upper limit of $\|\eta\|_{L^\infty}$.

If $\mathcal{V}$ is Kähler, $W^{1,2}_{*,f}$ coincides with the space $W^{1,2}_*$ considered in [21] that is independent of f . In this context, the space $W^{1,2}_*$ is studied in detail in [10] and used in [16] for the study of correspondences on Riemann surfaces with two equal dynamic degrees. Let $\xi_l^{-1}(f) > 1$. We have the following observation.

Lemma 3.3. Let $\varphi \in W^{1,2}_{*,f}$ and $\mathbf{m} = (m_0, m_1)$ be a size representative of φ . So we have

$$\|d\varphi\|_{L^2} \leq Am_0^{1/2}(\chi_{2l-1})^{m_1/2}$$

for some constant A independent of φ .

Proof. Let η be as on (3.2). Let ω be a Hermitian metric on $\mathcal{V}$ with $dd^c\omega^{k-1} = 0$. Testing $dd^c((f^{m_1})_*\psi) + (f^{m_1})_*\eta$ with this form, we see that the norm of $dd^c((f^{m_1})_*\psi) + (f^{m_1})_*\eta$ is equal to $\int_{\mathcal{V}} (f^{m_1})_*\eta \wedge \omega^{k-1}$ which is limited by $Am_0(\chi_{2l-1})^{m_1}$ for some constant A independent of η, m_0, m_1 . The desired inequality then follows. This completes the proof. ■

Let $\varphi \in W^{1,2}_{*,f}$. Define $\varphi^+ := \max\{\varphi, 0\}$ a $\varphi^- := \max\{-\varphi, 0\}$. Consider a Lipschitz function $\chi : \mathbb{R} \rightarrow \mathbb{R}$. We have $\partial(\chi \circ \varphi) = (\chi' \circ \varphi)\partial\varphi$. This can be seen using a sequence of smooth functions, converging to φ on $W^{1,2}$. We deduce that

$$i\partial(\chi \circ \varphi) \wedge \bar{\partial}(\chi \circ \varphi) = (\chi' \circ \varphi)^2 i\partial\varphi \wedge \bar{\partial}\varphi.$$

Consequently, $\chi \circ \varphi \in W^{1,2}_{*,f}$. In particular, let $\chi(t) := |t|, \max\{t, 0\}$ or $\max\{-t, 0\}$ for $t \in \mathbb{R}$, we get the following crucial property.

Lemma 3.4. For each $\varphi \in W^{1,2}_{*,f}$, if $\mathbf{m} = (m_0, m_1)$ is a representative of size of φ , then $\mathbf{m}$ is also a size representative of $|\varphi|, \varphi^+$ and φ^- .

We already know that the pushforward of a function quasi-p.s.h. by f is a function weakly d.s.h. The following result, which explains the role of $W^{1,2}_*$ in this study, provides a more accurate description in the case of functions quasi-p.s.h limited.

Lemma 3.5. Each function quasi-p.s.h limited is on $W^{1,2}_{*,f}$ and f_* preserves $W^{1,2}_{*,f}$. In addition, for each $\varphi \in W^{1,2}_{*,f}$, if $\mathbf{m} = (m_0, m_1)$ is a size representative of φ , then $\mathbf{m}' := (d_k m_0, m_1 + 1)$ is a size representative of $f_*\varphi$ and

$$\|f_*\varphi\|_{L^2} \leq c(\|\varphi\|_{L^1} + \|d(f_*\varphi)\|_{L^2}) \quad (3.3)$$

for some constant c independent of φ .

Proof. Let φ be a function quasi-p.s.h limited and $f : \mathcal{V} \rightarrow \mathcal{V}$ a dominant meromorphic map. Using the identity

$$2i\partial\varphi \wedge \bar{\partial}\varphi = i\partial\bar{\partial}\varphi^2 - 2\varphi i\partial\bar{\partial}\varphi$$

we see that there is a $(1,1)$ -continuous form η and a function η -p.s.h. ψ for which $i\partial\varphi \wedge \bar{\partial}\varphi \leq dd^c\psi + \eta$. Consequently $\varphi \in W^{1,2}_{*,f}$.

Now let φ be an arbitrary element of $W^{1,2}_{*,f}$. Let η and ψ be such that (3.2) holds. Fix a dense open subset of Zariski Ω of $\mathcal{V}$ in which $f_*\varphi, (f^{m_1})_*\psi, (f^{m_1})_*\eta$ are well-defined functions or forms and π_1 is an unbranched cover on $f^{-1}(\Omega)$. We have $f_*\varphi \in L^1_{loc}(\Omega)$ and

$$\|f_*\varphi\|_{L^1(K)} \leq c\|\varphi\|_{L^1}, \quad (3.4)$$

for any compact K on Ω and some constant c independent of φ . Note that $\mathcal{V} \setminus \Omega$ is a proper analytical subset of $\mathcal{V}$, Thus, is of Hausdorff $(2k - 1)$ -dimensional and zero measure. On Ω , by Cauchy-Schwarz inequality, we have

$$\begin{aligned} i\partial(f_*\varphi) \wedge \bar{\partial}(f_*\varphi) &\leq \chi_{2l} f_* (i\partial\varphi \wedge \bar{\partial}\varphi) \leq \chi_{2l} f_* [dd^c((f^{m_1})_*\psi) + (f^{m_1})_*\eta] \\ &= \chi_{2l} [dd^c((f^{m_1+1})_*\psi) + (f^{m_1+1})_*\eta]. \end{aligned}$$

It follows that $d(f_*\varphi) \in L^2(\Omega)$. For this and by Lemma 3.1, we get $f_*\varphi \in W^{1,2}$. Thus, $i\partial(f_*\varphi) \wedge \bar{\partial}(f_*\varphi)$ has no mass on $\mathcal{V} \setminus \Omega$. It follows that

$$i\partial(f_*\varphi) \wedge \bar{\partial}(f_*\varphi) \leq \chi_{2l} \mathbf{1}_\Omega [dd^c((f^{m_1+1})_*\psi) + (f^{m_1+1})_*\eta] \leq \chi_{2l} [dd^c((f^{m_1+1})_*\psi) + (f^{m_1+1})_*\eta]$$

- 1 because the last current is positive by Lemma 2.9. Combining this with (3.1) and (3.4) gives (3.3). The
2 desired statement then follows. This completes the proof. $\blacksquare$

Let $\varphi \in W_{*,f}^{1,2}$ and $\mathbf{m} = (m_0, m_1)$ be a size representative of φ . Consider f_* acting on Borel's measurable functions. Remember that f_* preserves the set of constant functions. As in the last section, let $b_0 := \int_{\mathcal{V}} \varphi d\mu_0$, and $\varphi_0 := \varphi - b_0$. We define two sequences φ_n, b_n as follows. Put

$$b_n = b_n(\varphi) := \int_{\mathcal{V}} (f_*\varphi_{n-1}) d\mu_0, \quad \varphi_n := f_*\varphi_{n-1} - b_n$$

- 3 for $n \geq 1$. Note that φ_n differs from $((f^n)_*\varphi)$ by a constant. Lemma 3.5 implies that $\mathbf{m}_n := (\chi_{2l}^n m_0, m_1 +$
4 $n)$ is a size representative of φ_n . This together with Lemma 3.4 imply that
5 **Lemma 3.6.** $\mathbf{m}_n := (\chi_{2l}^n m_0, m_1 + n)$ is also a size representative of $|\varphi_n|, \varphi_n^+$ and φ_n^- .
6 By Lemma 3.3, we get

$$\|d\varphi_n\|_{L^2} A m_0^{1/2} \chi_{2l}^{n/2} (\chi_{2l-1})^{(n+m_1)/2} \quad (3.5)$$

- 7 Using (3.5), (3.1) and (3.3) give

$$\|\varphi_n\|_{L^2} \leq A m_0^{1/2} \chi_{2l}^{n/2} (\chi_{2l-1})^{(n+m_1)/2}, \quad |b_n| \leq A m_0^{1/2} \chi_{2l}^{n/2} (\chi_{2l-1})^{(n+m_1)/2} \quad (3.6)$$

for $n \geq 1$ and some possible different constant A . Now we are in a situation very similar to the one in the last section. Using arguments similar to those in the last section, we can show that $\lim_{n \rightarrow \infty} \langle \chi_{2l}^{-n} (f^n)^* \omega^k, \varphi \rangle$ exists and denote by $b'_\infty(\varphi)$ its limit. In fact, we have

$$b'_\infty = \sum_{j=0}^{\infty} \chi_{2l}^{-j} b_j.$$

- 8 It follows that

$$|b'_\infty(\varphi)| \leq \|\varphi\|_{L^1} + A m_0^{1/2} (\chi_{2l-1})^{m_1/2} \quad (3.7)$$

for some constant A independent of φ . Clearly, if φ is a function quasi-p.s.h limited, b'_∞ is equal to the same number defined in the last section. So we have

$$\langle \mu_f, \varphi \rangle = b'_\infty(\varphi)$$

- 9 for function quasi-p.s.h limited φ . Let $W_{**,f}^{1,2}$ the subset of $W_{*,f}^{1,2}$ consisting of functions that are continuous
10 outside a closed pluripolar set. Note that f_* preserves $W_{**,f}^{1,2}$ because f is a covering outside an analytical
11 subset of $\mathcal{V}$. We now affirm that
12 **Lemma 3.7.** For $\varphi \in W_{**,f}^{1,2}$, we have $\langle \mu_f, \varphi \rangle = b'_\infty(\varphi)$.

Proof. The proof is similar to that on [21, Lemma 5.5]. We proved first that φ is μ_f -integrable. We assume for a moment that $\varphi \geq 0$. Let V be a closed pluripolar set such that φ is continuous outside of V . Remember that μ_f has no mass on pluripolar sets, therefore, on V . Since $\chi_{2l}^{-n} (f^n)^* \omega^k$ converges to μ_f as positive measures and $\mathcal{V} \setminus V$ is open, we have

$$\langle \mu_f, \varphi \rangle \leq \liminf_{n \rightarrow \infty} \langle \chi_{2l}^{-n} (f^n)^* \omega^k, \varphi \rangle = \lim_{n \rightarrow \infty} \sum_{j=0}^n \chi_{2l}^{-j} b_j + \liminf_{n \rightarrow \infty} \langle \omega^k, \chi_{2l}^{-n} \varphi_n \rangle$$

1 which is equal to $b'_\infty(\varphi)$. Thus φ is μ_f -integrable if $\varphi \geq 0$. In general, write $\varphi = \varphi^+ - \varphi^-$ and applying
2 the last property, show that φ is μ_f -integrable. If $\mathbf{m} = (m_0, m_1)$ is a size representative of φ , then we also
3 get that

$$|\langle \mu_f, \varphi \rangle| \leq |b'_\infty(\varphi^+)| + |b'_\infty(\varphi^-)| \leq A(\|\varphi\|_{L^1} + m_0^{1/2}(\chi_{2l-1})^{m_1/2}), \quad (3.8)$$

4 for some constant c independent of φ . Now using $f^*\mu_f = \chi_{2l}\mu_f$ gives

$$|\langle \mu_f, \varphi \rangle - b'_\infty(\varphi)| = |\langle \mu_f, \chi_{2l}^{-n}(f^n)_*\varphi - b'_\infty(\varphi) \rangle| \leq |c_n| + |\langle \mu_f, \chi_{2l}^{-n}\varphi_n \rangle|,$$

where $c_n := -\sum_{j \geq n+1} \chi_{2l}^{-j}|b_j|$. Note that the first term on the right side of the last inequality tends to 0
because of (3.6). On the other hand, by (3.8) and Lemma 3.6, the second term is limited by

$$A\chi_{2l}^{-n}(\|\varphi_n\|_{L^1} + m_0^{1/2}\chi_{2l}^{n/2}(\chi_{2l-1})^{(m_1+n)/2})$$

5 which tends to 0 when $n \rightarrow \infty$. This produces the desired equality. This completes the proof. ■

6

Theorem 3.8. Let $\mathcal{V}, f, \chi_{2l}, \chi_{2l-1}$ be as above with $\xi_l^{-1}(f) > 1$. So there is a constant $A > 0$ such that

$$I_n(\psi, \varphi) := |\langle \mu_f, (\psi \circ f^n)\varphi \rangle - \langle \mu_f, \psi \rangle \langle \mu_f, \varphi \rangle| \leq A\|\psi\|_\infty A_n(\varphi),$$

where

$$A_n(\varphi) := [\|\varphi\|_{L^1} + m_0^{1/2}(\chi_{2l-1})^{m_1/2}]\chi_{2l}^{-n/2}(\chi_{2l-1})^{n/2},$$

7 for each $\psi \in L^\infty(\mu_f)$, $\varphi \in W_{**f}^{1,2}$ and (m_0, m_1) a size representative of φ ,

8 Note that if φ is a function η -p.s.h. limited for some $(1, 1)$ -continuous form η of L^∞ -norm ≤ 1 , then there
9 is a constant $\widetilde{m}_0$ independent of φ such that $(\widetilde{m}_0, 1)$ is a size representative of φ . Therefore, the above
10 theorem gives a uniform correlation decay for each φ .

11 *Proof.* Let the annotations be as above. $I_n(\psi, \varphi + c) = I_n(\psi, \varphi)$ for each constant c because of the
12 invariance of μ_f . We can assume that $\langle \mu_f, \varphi \rangle = 0$. By Lemma 3.7, we get $b'_\infty(\varphi) = 0$. Consequently,
13 $\chi_{2l}^{-n}(f^n)_*(\varphi) = c_n + \chi_{2l}^{-n}\varphi_n$. Using $f^*\mu_f = \chi_{2l}\mu_f$ gives

$$I_n(\psi, \varphi) = \chi_{2l}^{-n}|\langle \mu_f, \psi(f^n)_*(\varphi) \rangle| = |\langle \mu_f, \psi(c_n + \chi_{2l}^{-n}\varphi_n) \rangle| \leq |c_n| + \chi_{2l}^{-n}|\langle \mu_f, |\varphi_n| \rangle|. \quad (3.9)$$

Note that, as before, we have

$$|c_n| \leq AA_n(\varphi)$$

for some constant A independent of φ . On the other hand, f_* preserves $W_{**f}^{1,2}$, thus $\varphi_n \in W_{**f}^{1,2}$ and so is
 $|\varphi_n|$. By Lemma 3.6, $(\chi_{2l}^n m_0, m_1 + n)$ is a size representative of $|\varphi_n|$ if (m_0, m_1) is a size representative
of φ . Arguing as in the proof of Lemma 3.7 gives that

$$\chi_{2l}^{-n}|\langle \mu_f, |\varphi_n| \rangle| \leq AA_n(\varphi)$$

14 for some constant A independent of φ . Hence the desired inequality follows. This completes the proof. ■

15

End of Proof of Theorem 1.10. The central limit theorem for μ_f is a direct consequence of its correlation
decay as shown in [21]. Therefore, it remains to prove the property of the correlation decay. By Theorem
3.8, for each C^1 function φ on $\mathcal{V}$, we have

$$I(\psi, \varphi) \leq A\|\psi\|_\infty \|\varphi\|_{C^1} \chi_{2l}^{-n/2}(\chi_{2l-1})^{n/2}.$$

16 This combined with the interpolation inequality for functional in Banach spaces C^1, C^0 provides the desired
17 correlation decay for μ_f , cf [21].

18 Remember that μ_f is K-mixing if for each $\varphi \in L^2(\mu_f)$, we have

$$\sup_{\psi \in L^2(\mu_f)} I_n(\psi, \varphi) \rightarrow 0. \quad (3.10)$$

19 Note that the operator $\chi_{2l}^{-1}f_*$ can be extended to be a continuous linear operator on $L^2(\mu_f)$ because $|f_*\varphi|^2 \leq$
20 $\chi_{2l}f_*(|\varphi|^2)$. As above, to prove (3.10), we can assume that $\langle \mu_f, \varphi \rangle = 0$. Using (3.9) we have

$$I(\psi, \varphi) \leq \|\chi_{2l}^{-n}(f^n)_*\varphi\|_{L^2(\mu_f)}. \quad (3.11)$$

Consider now φ to be a limited function on $W_{**f}^{1,2}$. The set of these functions is dense on $L^2(\mu_f)$. We have

$$\|\chi_{2l}^{-n}(f^n)_*\varphi\|_{L^2(\mu_f)} \leq \|\varphi\|_\infty \|\chi_{2l}^{-n}(f^n)_*\varphi\|_{L^1(\mu_f)}$$

21 that tends to 0 by proof of theorem 3.8. This combined with (3.11) gives (3.10). The proof is completed. ■

1 *Remark 1.* By inequality (3.6), we see that for each complex measure ν with density L^2 and $\nu(X) = 1$,
2 $\chi_{2l}^{-n}(f^n)^*\nu$ converges weakly to μ_f .

3 4 Conjectures

4 Here is the First Conjecture.

Conjecture 4.1. Let $\mathcal{V}, f, \mu_f$ as in Theorem 1.9. Let $\psi_1, \dots, \psi_k$ be the Lyapunov exponents of μ_f and $\Psi = \sum_i \frac{1}{\psi_i}$ its inverse sum. So the Hausdorff dimension of μ_f satisfies

$$\dim_{\mathcal{H}}(\mu_f) = \Psi h_{\chi}(f).$$

5 Here is the Second Conjecture.

Conjecture 4.2. Let $\mathcal{V}, f, \mu_f$ be as in Theorem 1.9. So there are T_l^+ and T_{k-l}^- such that μ_f is defined by :

$$\mu_f := T_l^+ \wedge T_{k-l}^- ,$$

where T_l^+ is a positive invariant closed current of bidegree (l, l) , i.e.

$$\frac{1}{\chi_{2l}^m}(f^m)^*\omega^l \longrightarrow T_l^+$$

and T_{k-l}^- designates a positive invariant closed current of $(k-l, k-l)$, i.e.

$$\frac{1}{\chi_{2(k-l)}^m}(f^m)^*\omega^{k-l} \longrightarrow T_{k-l}^-.$$

6 Here is the Third Conjecture.

Conjecture 4.3. Let $\mathcal{V}, f, \mu_f$ be as in Theorem 1.9 and T_l^+ as in Conjecture 4.2. Let $\psi_1, \dots, \psi_k$ be the
7 Lyapunov exponents of μ_f with $\psi_l = \max_{1 \leq i \leq k} \psi_i$. So the Hausdorff dimension of the **Support** of T_l^+
8 satisfies
9

$$\dim_{\mathcal{H}}(\text{Supp} T_l^+) \geq 2(k-l) + \frac{\log \chi_{2l}}{\psi_l}.$$

10 References

- 11 [1] ARMAND AZONNAHIN. Dinâmica de aplicações cohomologicamente hiperbólicas.
- 12 [2] ARMAND AZONNAHIN. Conceitos fundamentais e métodos pluripotenciais para aplicações
13 cohomologicamente expansíveis. *researchgate.net from: DOI: 10.13140/RG.2.2.31156.81281*
14 [https://www.researchgate.net/publication/334694628_Conceitos_Fundamentais_](https://www.researchgate.net/publication/334694628_Conceitos_Fundamentais_e_Metodos_Pluripotenciais_para_Aplicacoes_Cohomologicamente_Expansiveis)
15 [e_Metodos_Pluripotenciais_para_Aplicacoes_Cohomologicamente_Expansiveis,](https://www.researchgate.net/publication/334694628_Conceitos_Fundamentais_e_Metodos_Pluripotenciais_para_Aplicacoes_Cohomologicamente_Expansiveis)
16 [https://www.researchgate.net/publication/334694628_Conceitos_Fundamentais_e_Metodos_Pluripotenciais_para_Aplicacoes_Cohomologicamente_Expansiveis,](https://www.researchgate.net/publication/334694628_Conceitos_Fundamentais_e_Metodos_Pluripotenciais_para_Aplicacoes_Cohomologicamente_Expansiveis) 2019.
- 17 [3] ARMAND AZONNAHIN. On cohomological stochastic dynamics of higher dimensions. 2019.
- 18 [4] ARMAND AZONNAHIN. On cohomological stochastic dynamics of higher dimen-
19 sions. *32º Colóquio Brasileiro de Matemática - IMPA, Rio de Janeiro* <https://hal.archives-ouvertes.fr/hal-02280087> https://www.researchgate.net/profile/Armand_Azonnahin/publication/335618281_On_Cohomological_Stochastic_Dynamics_of_Higher_Dimensions/links/5d707f09a6fdcc9961af983d/On-Cohomological-Stochastic-Dynamics-of-Higher-Dimensions.pdf, 2019.
- 20 https://www.researchgate.net/profile/Armand_Azonnahin/publication/335618281_On_Cohomological_Stochastic_Dynamics_of_Higher_Dimensions/links/5d707f09a6fdcc9961af983d/On-Cohomological-Stochastic-Dynamics-of-Higher-Dimensions.pdf, 2019.
- 21 [5] ARMAND AZONNAHIN. *On Cohomological Stochastic Dynamics of Higher Dimensions*. Septem-
22 ber 2019.
- 23 [6] ARMAND AZONNAHIN. Dinâmica de aplicações cohomologicamente hiperbólicas. <https://hal.archives-ouvertes.fr/hal-02281122/>, 2019.
- 24 [7] J.P. Demailly. Complex analytic geometry. *available at* www.fourier.ujf-grenoble.fr/demailly.
- 25 [8] J.P. Demailly. Monge ampère operators, lelong numbers and intersection theory in complex analysis
26 and geometry. *Plemum Press*, pages 115–193, 1993.
- 27 [9] M. Méo. Image inverse d'un courant positif fermé par une application surjective. *C.R.A.S.*, (322),
28 1996.
- 29 [10] G. Vigny. Dirichlet like space and capacity in complex analysis in several variables. *J. Funct. Anal.*,
30 (1):247–277, 2007.

References

- [1] Azonnahin, Armand, Dinâmica de Aplicações Cohomologicamente Expansíveis: Treze Novos Resultados, *10.13140/RG.2.2.24202.82880*, <https://www.researchgate.net/publication/338825994> (2020), 1-177.
- [2] H. Alexander, Projective capacity, *Ann. Math. Studies*, **100** (1981), 3-27.
- [3] A. Blanchard, Sur les variétés analytiques complexes, *Ann. Sci. Ecole Norm. Sup.* (3), **73** (1956), 157–202.
- [4] J.Y. Briend et J. Duval, Deux caractérisations de la mesure d'équilibre d'un endomorphisme de $\mathbb{P}^k(\mathbb{C})$, *IHES Publ. Math.*, **93** (2001), 145-159.
- [5] L. Clozel et E. Ullmo, Correspondances modulaires et mesures invariantes, *prépublication*, (2002).
- [6] J.P. Demailly, Monge-Ampère Operators, Lelong numbers and Intersection theory in Complex Analysis and Geometry, *Pleum Press* (1993), 115-193, (V. Ancona and A. Silva editors).
- [7] J.P. Demailly, Regularization of closed positive currents and intersection theory, *J. Algebraic Geometry*, **1** (1992), 361-409.
- [8] T.C. Dinh, Distribution des préimages et des points périodiques d'une correspondance polynomiale, *prépublication* (2003).
- [9] T.C. Dinh et N. Sibony, Dynamique des applications d'allure polynomiale, à paraître dans *J.M.P.A.*
- [10] T.C. Dinh et N. Sibony, Une borne supérieure pour l'entropie topologique d'une application rationnelle, *prépublication* (2003).
- [11] Liverani C., Central limit theorem for deterministic systems. International Conference on Dynamical Systems (Montevideo, 1995), 56-75, Pitman Res. Notes Math. Ser., 362, Longman, Harlow, 1996.
- [12] ARMAND AZONNAHIN. *Graus Dinâmicos e Cohomológicos em Variedades Abelianas*. Year 2019. Available at <https://www.researchgate.net/publication/336511093>.
- [13] Friedland S., Entropy of algebraic maps, Proceedings of the Conference in Honor of Jean-Pierre Kahane (Orsay, 1993), *J. Fourier Anal. Appl.*, 1995, Special Issue, 215-228.
- [14] H. DE THÉLIN AND G. VIGNY, *Entropy of meromorphic maps and dynamics of birational maps*, *Mém. Soc. Math. Fr. (N.S.)*, (2010), pp. vi+98.
- [15] J.-P. DEMAILLY, *Complex analytic and differential geometry*. <http://www.fourier.ujf-grenoble.fr/~demailly>.
- [16] T.-C. DINH, L. KAUFMANN, AND H. WU, *Dynamics of holomorphic correspondences on riemann surfaces*. *arxiv:1808.10130*, 2018.
- [17] T.-C. DINH AND V.-A. NGUYÊN, *Characterization of Monge-Ampère measures with Hölder continuous potentials*, *J. Funct. Anal.*, 266 (2014), pp. 67–84.
- [18] T.-C. DINH AND N. SIBONY, *Dynamique des applications d'allure polynomiale*, *J. Math. Pures Appl.* (9), 82 (2003).
- [19] ———, *Regularization of currents and entropy*, *Ann. Sci. École Norm. Sup.* (4), 37 (2004), pp. 959–971.
- [20] ———, *Une borne supérieure pour l'entropie topologique d'une application rationnelle*, *Ann. of Math.* (2), 161 (2005), pp. 1637–1644.
- [21] ———, *Decay of correlations and the central limit theorem for meromorphic maps*, *Comm. Pure Appl. Math.*, 59 (2006), pp. 754–768.
- [22] ———, *Distribution des valeurs de transformations méromorphes et applications*, *Comment. Math. Helv.*, 81 (2006), pp. 221–258.
- [23] ———, *Pull-back of currents by holomorphic maps*, *Manuscripta Math.*, 123 (2007), pp. 357–371.
- [24] ———, *Dynamics in several complex variables: endomorphisms of projective spaces and polynomial-like mappings*, in *Holomorphic dynamical systems*, vol. 1998 of *Lecture Notes in Math.*, Springer, Berlin, 2010, pp. 165–294.
- [25] L. C. EVANS, *Partial differential equations*, vol. 19 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI, second ed., 2010.
- [26] E. HEBEY, *Sobolev spaces on Riemannian manifolds*, vol. 1635 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, 1996.

- 1 [27] W. PARRY, *Entropy and generators in ergodic theory*, W. A. Benjamin, Inc., New York-Amsterdam,
2 1969.
- 3 [28] D.-V. VU, *Intersection of positive closed currents of higher bidegrees*, Michigan Math. J., 65 (2016),
4 pp. 863–872.
- 5 [29] ———, *Locally pluripolar sets are pluripolar*. arxiv:1812.00465, 2018.
- 6 [30] Abate M., The residual index and the dynamics of holomorphic maps tangent to the identity, *Duke*
7 *Math. J.*, **107** (2001), no. 1, 173-207.
- 8 [31] Alexander H., Projective capacity. Recent developments in several complex variables, *Ann. of Math.*
9 *Stud.* **100**, Princeton Univ. Press, Princeton, N.J., 1981, 3-27.
- 10 [32] Amerik E., Campana F., Exceptional points of an endomorphism of the projective plane, *Math. Z.*, **249**
11 (2005), no. 4, 741-754.
- 12 [33] ———, Deux caractérisations de la mesure d'équilibre d'un endomorphisme de $\mathbb{P}^k(\mathbb{C})$, *Publ. Math.*
13 *Inst. Hautes Études Sci.*, **93** (2001), 145-159.
- 14 [34] ———, Personal communication.
- 15 [35] Brin M., Katok A., On local entropy, *Geometric dynamics Lecture Notes in Math.*, **1007**, Springer
16 Berlin (1983), 30-38.
- 17 [36] Brolin H., Invariant sets under iteration of rational functions, *Ark. Mat.*, **6** (1965), 103-144.
- 18 [37] Carleson L., Gamelin T.W., *Complex dynamics*, Universitext: Tracts in Mathematics, Springer-Verlag,
19 New York, 1993.
- 20 [38] Cerveau D., Lins Neto A., Hypersurfaces exceptionnelles des endomorphismes de $\mathbb{C}P(n)$, *Bol. Soc.*
21 *Brasil. Mat. (N.S.)*, **31** (2000), no. 2, 155-161.
- 22 [39] Chemin J.-Y., *Théorie des distributions et analyse de Fourier*, cours à l'école polytechnique de Paris,
23 2003.
- 24 [40] Chern S.S., Levine H.I., Nirenberg L., Intrinsic norms on a complex manifold, in Global Analysis
25 (Papers in Honor of K. Kodaira), Univ. Tokyo Press, 1969, 119-139.
- 26 [41] Demailly J.-P., Monge-Ampère Operators, Lelong numbers and Intersection theory in Complex Anal-
27 ysis and Geometry, *Pleum Press* (1993), 115-193, (V. Ancona and A. Silva editors).
- 28 [42] ———, *Complex analytic geometry*, available at
29 www.fourier.ujf-grenoble.fr/~demailly.
- 30 [43] DeMarco L., Dynamics of rational maps: Lyapunov exponents, bifurcations, and capacity, *Math. Ann.*,
31 **326** (2003), no. 1, 43-73.
- 32 [44] Dembo A., Zeitouni O., *Large deviations techniques and applications*, Second edition, Applications
33 of Mathematics (New York), **38**, Springer-Verlag, New York, 1998.
- 34 [45] ———, Complex dynamics in higher dimensions. Notes partially written by Estela A. Gavosto, *NATO*
35 *Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, **439**, Complex potential theory (Montreal, PQ, 1993), 131-186,
36 *Kluwer Acad. Publ.*, Dordrecht, 1994.
- 37 [46] Gordin M.I., The central limit theorem for stationary processes, *Dokl. Akad. Nauk SSSR*, **188** (1969),
38 739-741.
- 39 [47] Katok A., Hasselblatt B., *Introduction to the modern theory of dynamical systems*, Encyclopedia of
40 Mathematics and its Applications, **54**, Cambridge University Press, Cambridge, 1995.
- 41 [48] Viana M., Stochastic dynamics of deterministic systems, vol. **21**, IMPA (1997).
- 42 [49] Walters P., *An introduction to ergodic theory*, Graduate Texts in Mathematics, **79**, Springer-Verlag,
43 New York-Berlin, 1982.

44 ARMAND AZONNAHIN

45 Department of Mathematics , IME-USP, São Paulo, Brazil

46 Lattes iD: <http://lattes.cnpq.br/4425081054732430>

47 E-mail: armand@ime.usp.br

48 <https://orcid.org/0000-0001-9574-1722>

49 <https://www.researchgate.net/publication/335218998>