# "Riemann's Hypothesis, a Second_Three_Pages_Proof"

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*Abstract*
After the very short and simple *first* proof of the RC (short for "Riemann Conjecture"), using the series expansion about z=1/2 of the Xi function $\xi(z)$ and the Hilbert Spaces Theory, now the author provides a new, *second*, proof, using the same series expansion about z=1/2 of the Xi function $\xi(z)$ and Algebra. This new proof is so simple that the author wonders why a great mathematician like Riemann did not see it; therefore F. Galetto thinks that somewhere in the purported proof there should be an error.

## 1. Introduction

This paper is a preprint of a future paper…

On April 14th 2019 the author provided a (first) "one page proof [5] based on the fact (found in a Wolfram MathWorld page [4]) that the entire function $\xi(z)$ can be expressed as $\xi(z)=\sum_{n=0}^{\infty} a_n (z-0.5)^{2n}$ a "series expansion about z=1/2" with suitable coefficients $a_n$, shown later. Working on this fact the author showed a very simple (*first*) proof of the Riemann Conjecture.

To get our *first* proof in [5] the author modified the expansion from $\xi(z)=\sum_{n=0}^{\infty} a_n (z-0.5)^{2n}$ to $\xi(z)=\sum_{n=0}^{\infty} b_n (z-0.5)^{2n}/(2n)!$, where $(z-0.5)^{2n}/(2n)!$ were the entries of the expansion of the analytic function $\cosh(z-0.5)$, so that he derived the relationship $\xi(z)=4\int_1^{\infty}\frac{d[x^{3/2}\psi'(x)]}{dx}\cosh[(z-0.5)\,0.5\ln(x)]\ \ x^{-0.25}dx$. Then he considered the two infinite dimensional vectors (the 1st real and the 2nd complex) $b=[b_0, b_2, \ldots, b_{2n}, \ldots, \ldots, b_\infty]$ and $c(z)=[1, (z-0.5)^2/2!, (z-0.5)^4/4!, \ldots, (z-0.5)^{2n}/(2n)!, \ldots.]$, such that $\xi(z)$ was the inner product [or scalar product] $c(z)b'=\xi(z)$ with norms $||b||<\infty$ and $||c(z)||<\infty$, because $c(z)c'(\bar{z})<\cosh[(x-0.5)^2+y^2]<\infty$. Letting $z_1=x_1+y$ and $z_2=x_2+y$ be two zeros symmetric to the Critical Line, $\xi(z_1)=\xi(z_2)=0$, that is $c(z_1)b'=c(1-\bar{z_1})b'=0$, which means orthogonality in an $l^2$ Hilbert space (on the field C of complex numbers), the two vectors $c(z_1)$ and $c(1-\bar{z_1})$ were ***both orthogonal to b*** so proving that ***RH is true.***

Now, using the same "series expansion about z=1/2" $\xi(z)=\sum_{n=0}^{\infty} a_n (z-0.5)^{2n}$, the author shows a new *second* proof of the Riemann Hypothesis (Conjecture).

It is well known that for over a century mathematicians have been trying to prove the Riemann Conjecture (also known as Hypothesis), RC for short, a conjecture claimed by Riemann [professor at University of Gottingen in Germany], near 1859 in a 8-page paper "*On the number of primes less than a given magnitude*" shown at Berlin Academy, and dated/published in 1859; at that time, Bernhard Riemann conjectured that all the zeros of the zeta function should have their real part $\sigma$ equal to 0.5; unfortunately he left the problem aside by writing "*... it is very probable that all roots are real. Without doubt it would be desirable to have a rigorous proof of this proposition; however I have left this research aside for the time being after some quick unsuccessful attempts, because it appears to be unnecessary for the immediate goal of my study...*". The comment was related to the real function $\Xi(t)$ [named $\xi(t)$ by B. Riemann] obtained from the zeta function $\zeta(1/2+it)$ with $\sigma$=0.5, (the line $\sigma$=Re(z)=1/2 is named *Critical Line*).

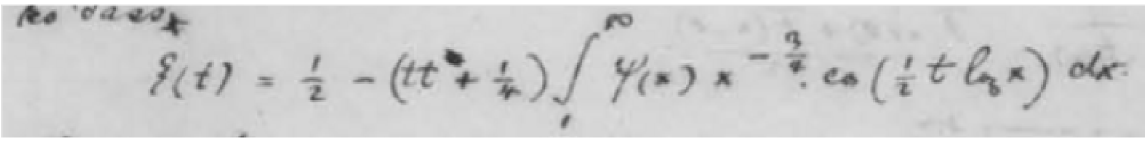

Figure 1 From the original B. Riemann manuscript

## 2. The proof of RC

To get our result we use the entire function ξ(z) which has the functional equation $\xi(\mathrm{z}) = \xi(1-\mathrm{z})$ and is analytic, defined by $\xi(\mathrm{z}) = \frac{1}{2}\mathrm{z}(\mathrm{z}-1)\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\zeta(\mathrm{z})$ [no poles].

The nontrivial zeros of the Riemann Zeta Function ζ(z) exactly correspond to those of ξ(z); putting z=1/2+it (i.e. for real t, the z points are on the Critical Line) the roots of ξ(1/2+it) are the same as those of ζ(1/2+it); moreover ξ(1/2+it)=Ξ(t) is a purely real function, with (see fig. 1, with different notation) $\Xi(\mathrm{t}) = -\frac{1}{2}(t^2+\frac{1}{4})\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)\pi^{-\frac{1}{4}+\frac{it}{2}}\zeta\left(\frac{1}{2}-\mathrm{it}\right)$.

The coefficients $a_n$, in Wolfram MathWorld [4], are given by the formula, which depends only on 2n, $a_{2n} = 4\int_1^{\infty}\frac{d[x^{3/2}\psi'(x)]}{dx}\frac{[0.5\ln(x)]^{2n}}{(2n)!}x^{-0.25}dx$; once computed $a_{2n}$ we have only to compute ξ(z).

To get our *second* proof we write explicitly some items of the expansion

$$\xi(s) = \sum_{n=0}^{\infty} a_{2n}(s-0.5)^{2n} \qquad (1)$$

as follows

$$\xi(s) = a_0 + \left(s-\frac{1}{2}\right)^2 a_2 + \left(s-\frac{1}{2}\right)^4 a_4 + \left(s-\frac{1}{2}\right)^6 a_6 + \cdots \qquad (1')$$

To go on we assume that RH is false: there are at least four points (zeros), out of the *Critical Line*, CL; at these points the Xi function is zero. They are shown in figure 2, where the *Critical Line*, CL, intercepts the real axe in the point P(0.5, 0), the red dot.

The 4 points are

- Symmetric to the Critical Line
- Symmetric to the real axe
- Symmetric to the point P(0.5, 0), interception of the Critical Line with the real axe

The 4 points on the s plane are the black dots $s_1$, $\overline{s_1}$ [complex conjugate of $\mathrm{s}_1$], $1-\mathrm{s}_1$, $\overline{1-s_1}$ [complex conjugate of $1-\mathrm{s}_1$].

In figure 2 it is “only indicated” the function $\xi(s)$, near the black dots; the function cannot be shown because we would need a 4-dimensional space [Re(s), Im(s), Re $\xi(s)$, Im $\xi(s)$].

If we set a=($s_1$-0.5) and b=($\overline{s_1}$-0.5), we have $\xi(s_1)=0$ and $\xi(\overline{s_1})=0$; therefore

$$0 = \xi(s_1) - \xi(\overline{s_1}) = \sum_1^{\infty}[a^{2n}-b^{2n}]a_{2n} \qquad (2)$$

Any power difference can be splitted into the product

$$a^{2n}-b^{2n} = [a^n-b^n][a^n+b^n] \qquad (3)$$

IF n is odd the splitting provides the product

$$a^n-b^n = [a-b]P_1(a,b)\{a+b\}P_2(a,b) \qquad (4)$$

with $P_1$(a, b) and $P_2$(a, b) suitable polynomials.

IF, on the contrary, *n is even and a suitable power of 2*, say $2^k$, then we can continue the splitting at its end

$$a^n-b^n = [a-b]\{a+b\}P_3(a,b)P_4(a,b)\ldots P_{k-1}(a,b) \qquad (5)$$

with $P_3(a, b)$ and $P_4(a, b)$ … suitable polynomials.

IF *none of the previous two cases happens*, sooner or later, there will be an exponent odd, say m, for which we can apply (4).

The end of the story is that we have, for any couple of points a and b

$$\xi(a) - \xi(b) = [a - b]\{a + b\}\Phi(a, b) \qquad (6)$$

We use $\Phi(a, b)$ for any suitable couple of zeros…

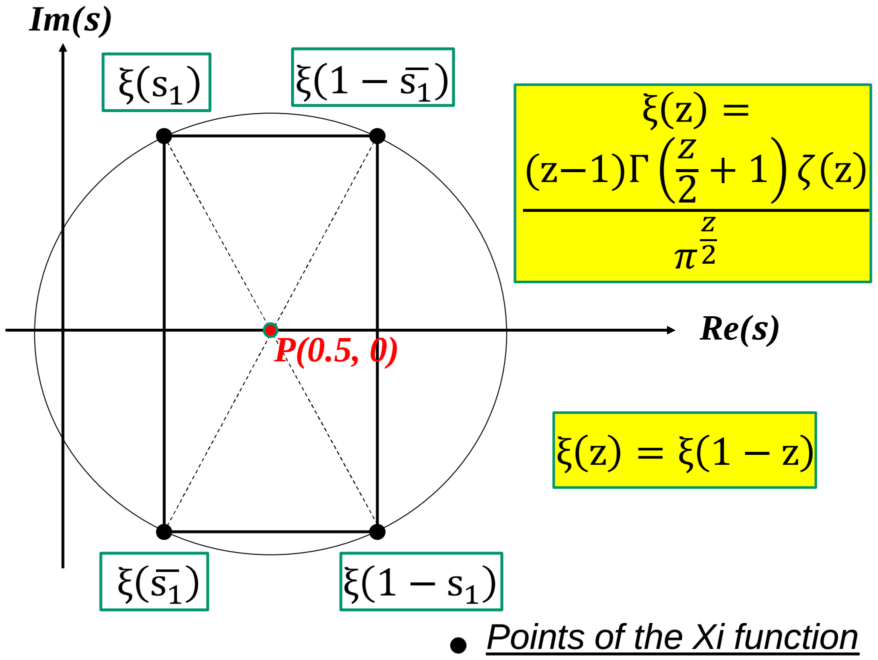


Figure 2 Four points satisfying ξ(s)=0

Now we use (6) for various couples of zeros, as depicted in figure 2.

Let's consider first the <u>two zeros on the left</u> of the CL

$$s_1 = \alpha_1 + i\beta_1 \quad \textbf{and} \quad \overline{s_1} = \alpha_1 - i\beta_1 \qquad (7a)$$

By our hypothesis we have

$$\xi(s_1) = 0 \quad \textbf{and} \quad \xi(\overline{s_1}) = 0 \qquad (8a)$$

and therefore (here we provide any transformation! We will not do later)

$$0 = \xi(s_1) - \xi(\overline{s_1}) = [s_1 - 0.5 - (\overline{s_1} - 0.5)]\{s_1 - 0.5 + \overline{s_1} - 0.5\}\Phi(a, b) = [s_1 - \overline{s_1}]\{s_1 + \overline{s_1} - 1\}\Phi(a, b) = [2i\beta_1]\{2\alpha_1 - 1\}\Phi(a, b)$$

(9a)

from which we get

$$\{2\alpha_i - 1\} = 0 \quad \Rightarrow \quad \alpha_i = 0.5 \qquad (10a)$$

Let's consider secondly the <u>two zeros on the right</u> of the CL

$$1 - s_1 = 1 - \alpha_1 - i\beta_1 \quad \textbf{and} \quad 1 - \overline{s_1} = 1 - \alpha_1 + i\beta_1 \qquad (7b)$$

By our hypothesis we have

$$\xi(1 - s_1) = 0 \quad \textbf{and} \quad \xi(1 - \overline{s_1}) = 0 \qquad (8b)$$

and therefore (*here we do not provide any transformation, as said before…*)

$$0 = \xi(1 - s_1) - \xi(1 - \overline{s_1}) = \xi(s_1) - \xi(\overline{s_1}) = [1 - s_1 - 0.5 - (1 - \overline{s_1} - 0.5)]\{1 - s_1 - 0.5 + 1 - \overline{s_1} - 0.5\}\Phi(a, b) = [-s_1 + \overline{s_1}]\{-s_1 - \overline{s_1} + 1\}\Phi(a, b) = [-2i\beta_1]\{-2\alpha_1 + 1\}\Phi(a, b) \qquad (9b)$$

from which we get

$$\{-2\alpha_i + 1\} = 0 \quad \Rightarrow \quad \alpha_i = 0.5 \qquad (10b)$$

Let's consider thirdly the two zeros above the real axe

$$s_1 = \alpha_1 + i\beta_1 \quad \textbf{\textit{and}} \quad 1 - \bar{s_1} = 1 - \alpha_1 + i\beta_1 \qquad (7c)$$

By our hypothesis we have

$$\xi(s_1) = 0 \quad \textbf{\textit{and}} \quad \xi(1 - \bar{s_1}) = 0 \qquad (8c)$$

and therefore (*here we do not provide any transformation, as said before…*)

$$0 = \xi(s_1) - \xi(1 - \bar{s_1}) = [s_1 - 0.5 - (1 - \bar{s_1} - 0.5)]\{s_1 - 0.5 + 1 - \bar{s_1} - 0.5\}\Phi(a,b) = [s_1 - 1 + \bar{s_1}]\{s_1 - \bar{s_1}\}\Phi(a,b) = [2\alpha_1 - 1]\{2i\beta_1\}\Phi(a,b) \qquad (9c)$$

from which we get

$$[2\alpha_i - 1] = 0 \quad \Rightarrow \quad \alpha_i = 0.5 \qquad (10c)$$

Let's consider fourthly the two zeros below the real axe

$$\bar{s_1} = \alpha_1 - i\beta_1 \quad \textbf{\textit{and}} \quad 1 - s_1 = 1 - \alpha_1 - i\beta_1 \qquad (7d)$$

By our hypothesis we have

$$\xi(\bar{s_1}) = 0 \quad \textbf{\textit{and}} \quad \xi(1 - s_1) = 0 \qquad (8d)$$

and therefore (*here we do not provide any transformation, as said before…*)

$$0 = \xi(\bar{s_1}) - \xi(1 - s_1) = \xi(s_1) - \xi(1 - \bar{s_1}) = [\bar{s_1} - 0.5 - (1 - s_1 - 0.5)]\{\bar{s_1} - 0.5 + 1 - s_1 - 0.5\}\Phi(a,b) = [\bar{s_1} - 1 + s_1]\{\bar{s_1} - s_1\}\Phi(a,b) = [2\alpha_1 - 1]\{-2i\beta_1\}\Phi(a,b) \qquad (9d)$$

from which we get

$$[2\alpha_i - 1] = 0 \quad \Rightarrow \quad \alpha_i = 0.5 \qquad (10d)$$

There no need to verify the other two couples due to the functional relationship

$$\xi(1 - s) = \xi(s)$$

We can repeat the same arguments for any zero of the function $\xi(s)$: the 4 assumed zeros are actually "two" zeros (complex conjugated) on the *Critical Line*: x=1/2, which *contradicts ''our previous hypothesis that RH was false' '*. Then the Riemann's Hypothesis (Riemann's Conjecture) **is TRUE**, because all the zeros of the function $\xi(s)$ are on the Critical Line.

## 3. Conclusion

We assumed that RC was false (see figure 2). Using the series expansion of $\xi(s)$ about s=0.5 [formula 1] we found that actually $\alpha_m$=0.5 for any zero $s_m$, $\xi(s_m) = 0$, out of the *Critical Line*: x=1/2. Since it was proved (Titchmarsh 1986) that there are infinite zeros of the *Riemann zeta function* ζ(s) in the Critical Strip, there are infinite values $s_k=\alpha_k+i\beta_k$ such that $\zeta(s_k)=0=\xi(s_k)$, $[0 < \alpha_k < 1]$. For any $\beta_k$ such that $\zeta(s_k)=\xi(s_k)=0$, either there is only one zero with $\alpha_k=1/2$ (on the Critical Line) or four zeros, two couples symmetric to the critical line and two couples symmetric to the real axe, with different real parts $\alpha_k$ and $1-\alpha_k$, that we proved, above, impossible for a particular value $\beta_k$ such that $\zeta(s_k)=0=\xi(s_k)$. Since we can repeat the same argument for any quadruple of zeros assumed symmetric [for any $i\beta_k$ and $-i\beta_k$, such that $\xi(s_k)=0$, for any nontrivial zero $s_k$] to the *Critical Line* and to the real axe. any zero has α=1/2 and therefore ***RH is true.***

## References


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