

# A NEW CONTINUED FRACTION OF GOLDEN RATIO AND SOME GENERALIZATIONS :

## A VISUAL APPROACH

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**Abstract.** In this article we made an attempt to connect Geometry and Number Theory in a very much interesting and beautiful way. The most irrational number turns out to be a number already known in geometry. It is Golden ratio i.e. Phi . The continued fraction representation of an irrational is unique. This article introduces Palash's fraction ,which is a new continued fraction of Phi. Palash's fraction is equal to Phi upto 12 decimal places. Here, we use a recursive formula to show the convergence. The diagram gives a visualisation of convergence and shows that how Palash's fraction is going to be an approximation of Golden ratio. Here, we suggest a generalized form of a special type of recursive continued fraction to visualise it perfectly. In this paper we show how Palash's Fraction links with Fibonacci numbers and Fibonacci sequence.

**Keywords.** Continued fraction ; Golden ratio ; Visualisation ; Generalizations ; Fibonacci numbers.

### 1. New Continued Fraction (Palash's Fraction)

$$\phi \approx 1 + \frac{1}{1.00001 + \frac{1}{1.00001 + \frac{1}{1.00001 + \dots \infty}}}$$

### 2. Introduction

The representation of real numbers by continued fractions dates back to Bombelli in the 16<sup>th</sup> century. In 17<sup>th</sup> century Huygens used continued fraction in constructing a model of solar system. Several books and papers are devoted entirely to the subject of continued fraction. Irwin M.C.(1989) presented a paper on the geometry of continued fraction, which got a good recognition. Recently, Alan F.& Ian (2014) described a geometric representation of real and complex continued fraction by chain of horocycles and horospheres in hyperbolic space. The relationship between Golden ratio and Continued fraction is commonly known about throughout the mathematical World. The continued fraction representation of an irrational is unique. In mathematics , regular continued fraction play's an important role in representing real numbers and have a rich general theory touching on a variety of topics in Number Theory. Although, the proofs of the continued fractions are not difficult ,but during reading them, I have a tendency to lose sight of where they are leading. That's why I always looking for geometry in everything. Here, we use a recursive formula ,

$$X_{n+1} = 1 + \frac{1}{(0.00001 + X_n)} \quad (1)$$

. We start from  $X=1$  and after an infinite cyclic process we end up with  $X = \phi$  . Hope, you will be focused on the logic of explaining the process of convergence through the below diagram , rather than numerical estimations. Though the following geometrical treatment is of the Palash's Fraction, but it maybe of some help to readers, who ,like myself, need a picture of what is going

on. This geometrical representation will help the readers to visualize a particular types of continued fraction ,later discussed in this article.

### 3. Visual representation

The below picture (Figure 1) shows that, how the convergence happens starting with  $X = 1$  and ending with  $X = \phi$  (Golden Ratio). And this is the vital part of this paper. Now, it is easy to see that how does the process of convergence happens here though our diagram. Palash's fraction is equal to Golden Ratio upto 12 decimal places.

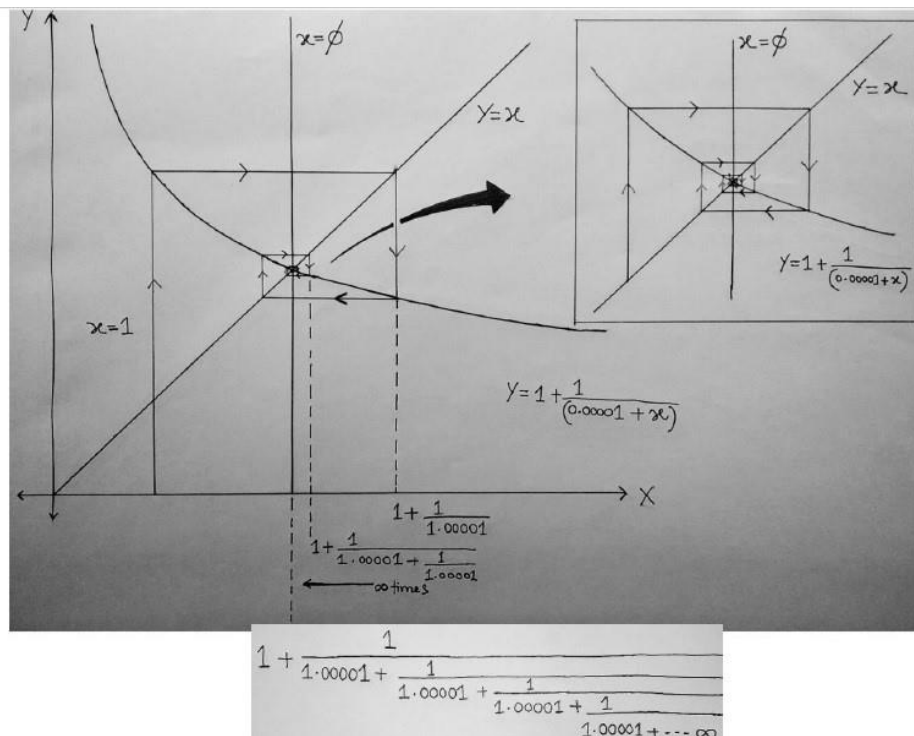


Figure 1. convergence to  $\phi$

### 4. Process

Here, we are dealing with only the proof with visual representation, not in light of rigorous Mathematics. Now, let's see, how the process happens. Here, we proposed the above continued fraction. And equation (1) is the recursion of the proposed continued fraction. Here,  $(\phi, \phi)$  is the intersecting point of the line  $Y = X$  and  $Y = 1 + \frac{1}{0.00001 + X}$ , we can always show it with our rigorous Mathematics, which implies  $(\phi, \phi)$  is solution of those two equations. Thus, Palash's fraction is approximate to Golden ratio. But here we will focus on the idea of the above diagram i.e. the visual representation. Here, we start with the line  $X = 1$  and follow the arrow. At first it cuts  $Y = X$ , then cuts  $Y = 1 + \frac{1}{0.00001 + X}$ , therefore cuts  $Y = X$  and then, cuts  $Y = 1 + \frac{1}{0.00001 + X}$  again. Now, here, we get our first iteration i.e. the last intersecting point. Following the arrow, again it cuts these line and curve respectively in same way, in which it had done previously. And we get our 2nd iteration. Now, here the interesting fact is, let's assume we are doing nth cycle, and in the nth cycle when it cuts the line  $Y = 1 + \frac{1}{0.00001 + X}$  2nd times, then we will be able to get our nth iteration. Thus, proceeding with this, after infinite Cycles we will be able to get Palash's fraction i.e. the new continued fraction representation of Golden ratio.

## 5. Generalization of visualisation

Now, note that, any continued fraction of the recursion form  $X_{n+1} = 1 + \frac{1}{(X_n+K)}$  such that  $K$  belongs to  $\mathbb{R}$ , can be visualised as like the above diagram of Visualisation to Palash's fraction. Let,  $X_{n+1} = 1 + \frac{1}{(X_n+K)}$  ... (2) is the recursive formula of the continued fraction of a real number  $P$ . Then we take,  $Y = X$  ... (3),  $Y = 1 + \frac{1}{X_n+K}$  ... (4). and proceedings as like the above diagram i.e. we starts form  $X = 1$  and after Infinite cyclic process we ends up with  $X = P$ . So, if we take this sequence,  $\{X_{n+1}\} = \{1 + 1/(X_n + K)\}$ . Then, this sequence is always convergent to a real number  $P$ . Here, the terms of the sequence are defined by the below set.

$$\left\{ 1 + \frac{1}{x+k}, 1 + \frac{1}{k+1+\frac{1}{x+k}}, 1 + \frac{1}{k+1+\frac{1}{k+1+\frac{1}{x+k}}}, \dots, x_\infty \right\}$$

To observe the convergence, we need to observe a diagram as like the Figure-1 and try to get the process. And also need to see below table including inputs and outputs through the recursion. Here,  $x_\infty$  is just a notation, using to show the the infinite continued fraction.

Input	Output
$x_1 = x$	$x_2 = 1 + \frac{1}{x+k}$
$x_2 = 1 + \frac{1}{x+k}$	$x_3 = 1 + \frac{1}{k+1+\frac{1}{x+k}}$
.	.
.	.
.	.
input of $x_\infty$	$x_\infty = 1 + \frac{1}{(k+1)+\frac{1}{(k+1)+\frac{1}{\dots \infty}}}$

Table 1. Recursive formula  $x_{n+1} = 1 + \frac{1}{k+x_n}$

$x_{n+1}$  is the recursion of the continued fraction of the real number  $P$  and  $x_\infty = P$ . Hope, now everything is clear to you.

**Example:** One of the best example is the continued fraction of  $\sqrt{2}$ ;

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots \infty}}}$$

Now, the recursive formula of  $\sqrt{2}$  is  $X_{n+1} = 1 + \frac{1}{(X_n+1)}$ , here  $K = 1$  So, it is easy to realise/see that, if any real number  $P$  can be expressed as continued fraction with the recursion  $X_{n+1} = 1 + \frac{1}{(X_n+K)}$  such that  $K \in \mathbb{R}$ , then we can easily visualize the convergence of the recursion to  $P$  and we will be able to get our required continued fraction.

## 6. Links to Fibonacci Numbers

The celebrated Fibonacci sequence,  $F := \{f_n\}$  is given by the inductive definition,

$$F_1 = 1, F_2 = 1, f_{n+1} = f_{n-1} + f_n \forall n \geq 2$$

. Thus, each term past the second is the sum of it's two immediate predecessors. The first few terms ( terms in the sence of Fibonacci numbers) of Fibonacci sequence, we mentioned here,  $\{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 154, \dots\}$  Now, we construct a sequence  $\{Y_n\}$ , which is defined by,  $\{Y_n\} := \{f_{n+1}/f_n\}$ . Here,  $\{Y_n\}$  is a Contractive sequence, since  $|Y_{n+1} - Y_n| \leq \frac{4}{9} |Y_n - Y_{n-1}|$

**Theorem:** Every contractive sequence is a Cauchy sequence, and therefore it is convergent.

So,  $\{Y_n\}$  is a convergent sequence. [ I will not discuss the proof of the limit here.]

Now we have,  $\lim\{Y_n\} = \frac{\sqrt{5}+1}{2} = \text{Golden ratio}$ . Now, we know that Palash's fraction is equal to Golden ratio upto 12 decimal places (Basically an approximation). [By visual representational proof]. so, we can conclude that,  $\lim\{Y_n\} = \lim(f_{n+1}/f_n) \approx \text{Palash's fraction}$ .

[ Golden ratio (  $\approx$  Palash's Fraction ) coincides with this limit not because it is the root with maximum modulus and multiplicity of the characteristic polynomial, but, from a more general point of view, because it is the root with maximum modulus and multiplicity of a restricted set of roots, which in this special case coincides with the two roots of the characteristic polynomial.] Thus, Palash's fraction links with Fibonacci numbers.

## 7. conclusion

So, we estimate here a new continued fraction form of Golden ratio , which I named Palash's Fraction. And gave an idea how to visualise continued fraction with a particular form i.e.

$$X_{n+1} = 1 + \frac{1}{X_n + K}$$

[ General form , which I suggest]. Lastly , Palash's Fraction links with Fibonacci numbers in some limiting sense.

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