A Visual Proof: $e \leq A \leq B \Rightarrow A^B > B^A$.

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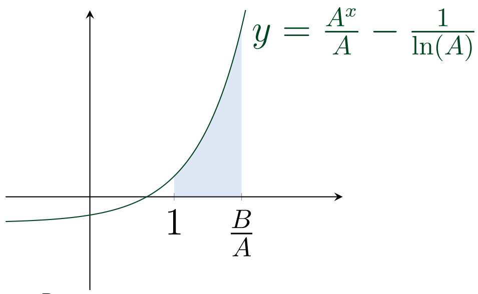
Fascination with the constants e and π has encouraged numerous visual proofs of the inequality $\pi^e < e^\pi$. Nakhli [4] used the fact that $\frac{1}{e}$ is a global maximum for $y = \frac{\ln(x)}{x}$ to conclude the relation, and Nelsen [5] used the fact that $y = e^{x/e}$ lies above the line $y = x$. More recently, Chakraborty [1] used Napier's inequality (see [6] for a general visual proof of this inequality), and then Chakraborty and Mukherjee together [3] utilized the fact that the line $y = x - 1$ lies above the curve $y = \ln(x)$ when $x > 1$.Also I [7] have submitted an article in Intelligencer journal on $e^A > A^e$ and it is accepted.

Gallant [2] provided the most general proof for which this inequality is a consequence, showing that when $e \leq A < B$, we have $A^B > B^A$; he used slopes of secant lines connecting the origin to points on the curve $y = \ln(x)$. We provide an alternate visual proof for this general inequality using an area argument.

Theorem. *For all real numbers A and B with $e \leq A < B$, $A^B > B^A$.*

Proof.



$$0 < \int_1^{\frac{B}{A}} \left(\frac{A^x}{A} - \frac{1}{\ln(A)} \right) dx = \frac{A^{\frac{B}{A}}}{A\ln(A)} - \frac{B}{A\ln(A)}$$

$$\Longrightarrow B < A^{\frac{B}{A}} \Longrightarrow B^A < A^B. \qquad \square$$

Letting $A = e$ and $B = \pi$, the pictured equation becomes $y = e^{x-1} - 1$ and we conclude $\pi^e < e^\pi$.

References

[1] Chakraborty, B. (2019). A visual proof that $\pi^e < e^\pi$. Math. Intelligencer, 41(1): 56.

[2] Gallant, C. (1991). $A^B > B^A$ for $e \leq A \leq B$. Math. Mag. 64(1): 31.

[3] Mukherjee, A., Chakraborty, B. (2019). Yet Another Visual Proof that $\pi^e < e^\pi$. Math. Intelligencer, 41(2): 60.

[4] Fouad Nakhli. (1987). $e^\pi > \pi^e$. Math. Mag. 60(3): 165.

[5] Nelsen, R.B. (2009) Proof Without Words: Steiner's Problem on the Number e. Math. Mag. 82(2): 102.

[6] Nelsen, R.B. (1993). Napier's Inequality (two proofs). College Math. J. 24(2): 165.

[7] Nazrul Haque,(2019) A visual proof that $e < A$ implies $e^A > A^e$, Math. Intelligencer, DOI: 10.1007/s00283-019-09964-x