To the issue of the physical meaning of the Laplace – Runge – Lenz vector

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Abstract

In the paper there is presented the Laplace – Runge – Lenz vector as physical force parameter in the regard of its dimension.

Based on the expression of the LRL vector the vector equation is generated where each term has force dimension. In this case, the LRL vector is the determinant of the sum of gravitation forces and fictitious producing no work forces.

Failing the gravitation forces or other real ones, the body motion can be considered as the constant motion in the compensated vector field of producing no work forces.

Such an approach can be justified by the viewpoint of the Newton's laws, the body motion while the forces are absent and the body motion in the compensated field of forces are equivalent and similar to each other.

Key words: Laplace–Runge–Lenz vector, forces, Newton's laws, vector fields.

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Introduction

It is known that when a body moves under the influence of the central force which is inversely proportional to the square of distance the Laplace-Runge-Lenz vector **A** is conserved together with the angular momentum and total energy [1-2]. Vector **A** is in the orbit plane and its direction coincides with the line of apsides – the line connecting the force field center and pericenter of the mechanical trajectory of the body. It is also defined as the eccentricity vector. From this viewpoint, Laplace-Runge-Lenz vector is better considered as a geometric parameter used for the orbit shape and orientation description [2-4].

The Laplace-Runge-Lenz vector (LRL vector) has been known for about 300 years. However, unlike the angular momentum and energy it is less frequently used at the motion description [1-3].

In this paper, the possibility to represent the Laplace – Runge – Lenz vector according to its dimension as the force parameter is considered.

1. The Laplace – Runge – Lenz vector dimension and its physical interpretation

For a body moving in the central field (Newtonian or Coulomb fields) the LRL vector can be noted as follows [2-6]:

$$\boldsymbol{A} = \boldsymbol{v} \times \boldsymbol{L} - \frac{\alpha \cdot \boldsymbol{r}}{r} \tag{1}$$

where $\boldsymbol{v}$ - is the velocity of the body motion, $\boldsymbol{L}$ – is the angular momentum, r - is the distance, α – is the parameter determining the central force value.

The vector $\boldsymbol{A}$ has the same dimension as the parameter α used in the formula for the Newtonian or Coulomb forces. In case the expression (1) is divided on r^2, we obtain a vector equation, each member of which has the dimension of force:

$$\frac{\boldsymbol{A}}{r^2} = \frac{\boldsymbol{v} \times \boldsymbol{L}}{r^2} - \frac{\alpha \cdot \boldsymbol{r}}{r^3} \tag{2}$$

$\boldsymbol{F}_1 = -\frac{\alpha \times \boldsymbol{r}}{r^3}$ – is the attraction or repulsion force depending on the sign of α.

$\boldsymbol{F}_2 = \frac{\boldsymbol{v} \times \boldsymbol{L}}{r^2}$ – is the component, considered as a fictitious force that does not work, since is always directed perpendicular to the velocity vector.

The action of the forces above is depicted in figure 1.

Examination of the additional force $\boldsymbol{F}_2$ is not regarded here as redundant or synthetic since there are some interesting coincidences.

If to resolve the velocity vector into the radial component $\boldsymbol{v_r} = \dot{\boldsymbol{r}}$ and that perpendicular to the radius vector $\boldsymbol{v}_\varphi = \dot{\varphi} \times \boldsymbol{r}$, then one can obtain the following equivalent components of the force $\boldsymbol{F}_2$ ($\boldsymbol{F}_{II}$ and $\boldsymbol{F}_\perp$).

$$\boldsymbol{F}_2 = \boldsymbol{F}_{II} + \boldsymbol{F}_\perp = \frac{\boldsymbol{v}_\varphi \times \boldsymbol{L}}{r^2} + \frac{\boldsymbol{v}_\mathrm{r} \times \boldsymbol{L}}{r^2} \tag{3}$$

The first member of the right part equation ($\boldsymbol{F}_{II}$) coincides with the centrifugal force of momentum by the amount and direction under the assumption that the centrifugal force equals mv_φ^2/r under rotary motion. The second member ($\boldsymbol{F}_\perp$) can be considered equal to a half the

Coriolis force. Consequently, the equation (3) can be used to note down in a vectorial form such forces as the centrifugal force, centripetal force and Coriolis force. The given notation may be convenient to solve some problems.

The potential of the components F_{II} and $F_{\perp}$ is found as:

$$V_{II} = \int_r^\infty \boldsymbol{F}_{\text{II}} \cdot d\,\boldsymbol{r} = \frac{L^2}{2mr^2}\,, \qquad V_{\perp} = \int_r^\infty \mathbf{F}_{\perp} \cdot d\,\boldsymbol{r} = -\frac{L^2}{2mr^2} \tag{4}$$

From the expressions (4) follows, as has been said above that the force $\boldsymbol{F}_2$ does not work when the body movement (component of the total work is equal to zero), and in this respect it is not real.

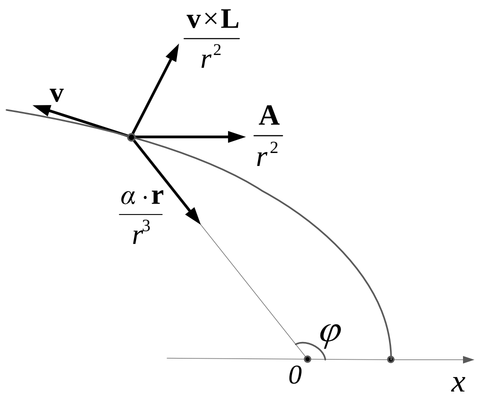

Fig. 1. The forces considered in motion in a gravitational field.

Vector $\frac{\boldsymbol{A}}{r^2}$ is parallel to the $0x$ axis.

Taking into account the conservation of the vector $\boldsymbol{A}$ the expression (2) can be considered as a condition of the directional operation of forces when the total component of gravitational forces and $\boldsymbol{F}_2$ changes in magnitude only (inversely proportional to the distance squared) and remains constant in its direction.

Having the equation (2) been written in projections onto polar axes one obtains the known equations to solve the task of the body motion in the central field (Kepler-Rutherford problem).

$$\begin{aligned} A\cos\varphi &= r\dot{\varphi}L - \alpha \\ A\sin\varphi &= \dot{r}L \end{aligned} \tag{5}$$

Let's transfer α in the first equation from the right to the left side and divide the second equation by the first. Then one can obtain:

$$\frac{dr}{rd\varphi} = \frac{A\sin\varphi}{\alpha + A\cos\varphi}; \quad \Rightarrow \quad \int_{r_0}^{r} \frac{dr}{r} = \int_{\varphi_0}^{\varphi} \frac{A\sin\varphi\, d\varphi}{\alpha + A\cos\varphi};$$

$$\ln r\Big|_{r_0}^{r} = -\ln(\alpha + A\cos\varphi)\Big|_{\varphi_0}^{\varphi}$$

Thus, for the mechanical trajectory one can obtain the known equation of conic in polar coordinates:

$$r(\alpha + A\cos\varphi) = r_0(\alpha + A\cos\varphi_0) = \text{const} \qquad (6)$$

It turns out that the physical magnitude $r(\alpha + A\cos\varphi)$ is the integral of motion.

2. On the body motion in the space considered as the equilibrated field of vectors

As stated above the expression of the Laplace-Runge-Lenz vector (1) is used to describe a body motion in the central force field. Still failing the force field the vector $\boldsymbol{A}$ can be in use to describe the body motion in the space. In this case, the parameter α is adopted to be equal to zero and the equation is formulated as:

$$\boldsymbol{A} = \boldsymbol{v} \times \boldsymbol{L} \qquad (7)$$

By analogy with (3) decompose the velocity vector in radial and rotary components and derive the equations:

$$\begin{aligned} A\cos\varphi &= r\dot{\varphi}L \\ A\sin\varphi &= \dot{r}L \end{aligned} \qquad (8)$$

Divide the second equation in the first one, integrate it and formulate the expression of the body mechanical trajectory:

$$r\cos\varphi = const \qquad (9)$$

Thus, the mechanical trajectory is the direct circuit along which the body is moving at steady velocity. In this case, the vector $\boldsymbol{v} \times \boldsymbol{L}$ is directed perpendicularly to the trajectory. Failing the central force field, we do not have the singular point relative to study the motion. Studying the motion all the points are equivalent and the motion could be examined relatively to any of them. In this case, each point fits its vector $\boldsymbol{A} = \boldsymbol{v} \times \boldsymbol{L}$ according to (7).

Hence, failing the forces the body motion in the space can be considered as the motion in the balanced vector field $\boldsymbol{A} = \boldsymbol{v} \times \boldsymbol{L}$. In the accompanying figure, there is the body motion in the space in accordance with the expression (8) failing the central gravity force:

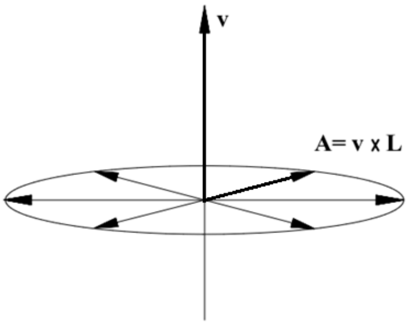

Fig. 2. The body motion in the space in the balanced vector field failing the forces involved.

In case the motion is considered according to the expression (2) when the parameter α is equal to zero, the vector field can be interpreted as the balanced force vector field:

$$\frac{\boldsymbol{A}}{r^2} = \frac{\boldsymbol{v} \times \boldsymbol{L}}{r^2}$$

In the given case, the motion is considered as the motion in the balanced fictitious producing no work forces ($\boldsymbol{F}_2$). As indicated above the mechanical trajectory is the direct circuit. The same result is predicted (when $\alpha = 0$) if the body motion is formally considered in accordance with the Newton's second law ($\boldsymbol{F}_2 = d\boldsymbol{P}/dt$).Thus the use of force $\boldsymbol{F}_2$ to calculate the motion is quite acceptable because it does not violate the Newton's laws.

Conclusion

The representation of the LRL vector character allows to use it to study the motion not only in the force field but also in case of its failing. In that case, the Laplace-Runge-Lenz vector equally with the power and impulse can be widely used to describe the body motion in the space.

The suggested representation of the LRL vector as the parameter which determines the sum of the gravitation forces and fictitious (producing no work) forces should be regarded as conjectural and hypothetical. The employment of the fictitious (producing no work) forces in the form (7) is admitted as it does not contradict the Newton's laws. Failing the gravitation forces or any other real ones, the body motion can be regarded as the uniform rectilinear motion in the balanced force field. Their use is justified as well by the fact that from the Newton's laws standpoint the body motion in the absence of forces and the body motion in the balanced field of forces are equivalent and indistinguishable.

Finally, supporting methods describing motion from the viewpoint of forces acting there can be cited Newton himself: «For all the difficulty of philosophy seems to consist in this – from the phenomena of motions to investigate the forces of nature, and then from these forces to demonstrate the other phenomena» [7]. Even nowadays this statement seems not to be dead.

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