THE RIEMANN HYPOTHESIS

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Abstract. In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. In 1915, Ramanujan proved that under the assumption of the Riemann Hypothesis, the inequality $\sigma(n) < e^{\gamma} \times n \times \log\log n$ holds for all sufficiently large n, where $\sigma(n)$ is the sum-of-divisors function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. In 1984, Guy Robin proved that the inequality is true for all $n > 5040$ if and only if the Riemann Hypothesis is true. In 2002, Lagarias proved that if the inequality $\sigma(n) \leq H_n + exp(H_n) \times \log H_n$ holds for all $n \geq 1$, then the Riemann Hypothesis is true, where H_n is the n^{th} harmonic number. We show certain properties of these both inequalities that leave us to a proof of the Riemann Hypothesis.

1. Introduction

As usual $\sigma(n)$ is the sum-of-divisors function of n [1]:

$$\sum_{d|n} d.$$

Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say $\mathsf{Robins}(n)$ holds provided

$$f(n) < e^{\gamma} \times \log\log n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, and log is the natural logarithm. Let H_n be $\sum_{j=1}^{n} \frac{1}{j}$. Say $\mathsf{Lagarias}(n)$ holds provided

$$\sigma(n) \leq H_n + exp(H_n) \times \log H_n.$$

The importance of these properties is:

Theorem 1.1. *If* $\mathsf{Robins}(n)$ *holds for all* $n > 5040$*, then the Riemann Hypothesis is true* [4]*. If* $\mathsf{Lagarias}(n)$ *holds for all* $n \geq 1$*, then the Riemann Hypothesis is true* [4].

It is known that $\mathsf{Robins}(n)$ and $\mathsf{Lagarias}(n)$ hold for many classes of numbers n. We know this:

Lemma 1.2. *If* $\mathsf{Robins}(n)$ *holds for some* $n > 5040$*, then* $\mathsf{Lagarias}(n)$ *holds* [4].

We prove our main theorems:

Theorem 1.3. $\mathsf{Robins}(n)$ *holds for all* $n > 5040$ *when a prime number* $q_m \nmid n$ *for* $q_m \leq 47$.

Theorem 1.4. *Let* $n > 5040$ *and* $n = r \times q_m$*, where* $q_m \geq 47$ *denotes the largest prime factor of* n*. We prove if* $\mathsf{Lagarias}(r)$ *holds, then* $\mathsf{Lagarias}(n)$ *holds.*

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In this way, we finally conclude that

Theorem 1.5. $\mathsf{Lagarias}(n)$ *holds for all* $n \geq 1$ *and thus, the Riemann Hypothesis is true.*

Proof. Every possible counterexample in $\mathsf{Lagarias}(n)$ for $n > 5040$ must have that its greatest prime factor q_m complies with $q_m \geq 47$ because of lemma 1.2 and theorem 1.3. In addition, $\mathsf{Lagarias}(n)$ has been checked for all $n \leq 5040$ by computer. Moreover, for all $n > 5040$ we have that $\mathsf{Lagarias}(n)$ has been recursively verified when its greatest prime factor q_m complies with $q_m \geq 47$ due to theorems 1.3 and 1.4. In conclusion, we show that $\mathsf{Lagarias}(n)$ holds for all $n \geq 1$ and therefore, the Riemann Hypothesis is true. □

2. Known Results

We use that the following are known:

Lemma 2.1. *From the reference* [1]*:*

$$f(n) < \prod_{p|n} \frac{p}{p-1}. \tag{2.1}$$

Lemma 2.2. *From the reference* [2]*:*

$$\prod_{k=1}^{\infty} \frac{1}{1-\frac{1}{q_k^2}} = \zeta(2) = \frac{\pi^2}{6}. \tag{2.2}$$

Lemma 2.3. *From the reference* [4]*:*

$$\log(e^{\gamma} \times (n+1)) \geq H_n \geq \log(e^{\gamma} \times n). \tag{2.3}$$

3. A Central Lemma

The following is a key lemma. It gives an upper bound on $f(n)$ that holds for all n. The bound is too weak to prove $\mathsf{Robins}(n)$ directly, but is critical because it holds for all n. Further the bound only uses the primes that divide n and not how many times they divide n. This is a key insight.

Lemma 3.1. *Given a natural number*

$$n = q_1^{a_1} \times q_2^{a_2} \times \cdots \times q_m^{a_m}$$

such that $q_1, q_2, \cdots, q_m$ *are prime numbers and* $a_1, a_2, \cdots, a_m$ *are natural numbers, then we obtain the following inequality*

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i+1}{q_i}.$$

Proof. From the lemma 2.1, we know

$$f(n) < \prod_{i=1}^{m} \frac{q_i}{q_i-1}. \tag{3.1}$$

We can easily prove

$$\prod_{i=1}^{m} \frac{q_i}{q_i-1} = \prod_{i=1}^{m} \frac{1}{1-q_i^{-2}} \times \prod_{i=1}^{m} \frac{q_i+1}{q_i}.$$

However, we know

$$\prod_{i=1}^{m} \frac{1}{1 - q_i^{-2}} < \prod_{j=1}^{\infty} \frac{1}{1 - q_j^{-2}}$$

where q_j is the j^{th} prime number and

$$\prod_{j=1}^{\infty} \frac{1}{1 - q_j^{-2}} = \frac{\pi^2}{6}$$

as a consequence of lemma 2.2. Consequently, we obtain

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}$$

and thus,

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}.$$

□

4. A Particular Case

We prove the Robin's inequality for this specific case:

Lemma 4.1. *Given a natural number*

$$n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} \times 7^{a_4} > 5040$$

such that $a_1, a_2, a_3, a_4 \geq 0$ *are integers, then* $\mathsf{Robins}(n)$ *holds for* $n > 5040$.

Proof. Given a natural number $n = q_1^{a_1} \times q_2^{a_2} \times \cdots \times q_m^{a_m} > 5040$ such that $q_1, q_2, \cdots, q_m$ are prime numbers and $a_1, a_2, \cdots, a_m$ are natural numbers, we need to prove

$$f(n) < e^{\gamma} \times \log\log n$$

that is true when

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq e^{\gamma} \times \log\log n$$

according to the lemma 2.1. Given a natural number $n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} > 5040$ such that $a_1, a_2, a_3 \geq 0$ are integers, we have

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5}{1 \times 2 \times 4} = 3.75 < e^{\gamma} \times \log\log(5040) \approx 3.81.$$

However, we know for $n > 5040$

$$e^{\gamma} \times \log\log(5040) < e^{\gamma} \times \log\log n$$

and therefore, the proof is completed for that case. Hence, we only need to prove the Robin's inequality is true for every natural number $n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} \times 7^{a_4} > 5040$ such that $a_1, a_2, a_3 \geq 0$ and $a_4 \geq 1$ are integers. In addition, we know the Robin's inequality is true for every natural number $n > 5040$ such that $7^k \mid n$ and $7^7 \nmid n$ for some integer $1 \leq k \leq 6$ [3]. Therefore, we need to prove this case for those natural numbers $n > 5040$ such that $7^7 \mid n$. In this way, we have

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5 \times 7}{1 \times 2 \times 4 \times 6} = 4.375 < e^{\gamma} \times \log\log(7^7) \approx 4.65.$$

However, we know for $n > 5040$ and $7^7 \mid n$ such that

$$e^{\gamma} \times \log\log(7^7) \leq e^{\gamma} \times \log\log n$$

and as a consequence, the proof is completed. □

5. A Better Upper Bound

Lemma 5.1. *For $x \geq 11$, we have*

$$\sum_{q \leq x} \frac{1}{q} < \log\log x + \gamma - 0.12$$

where $q \leq x$ means all the primes lesser than or equal to x.

Proof. For $x > 1$, we have

$$\sum_{q \leq x} \frac{1}{q} < \log\log x + B + \frac{1}{\log^2 x}$$

where

$$B = 0.2614972128\cdots$$

is the (Meissel-)Mertens constant, since this is a proven result from the article reference [5]. This is the same as

$$\sum_{q \leq x} \frac{1}{q} < \log\log x + \gamma - (C - \frac{1}{\log^2 x})$$

where $\gamma - B = C > 0.31$, because of $\gamma > B$. If we analyze $(C - \frac{1}{\log^2 x})$, then this complies with

$$(C - \frac{1}{\log^2 x}) > (0.31 - \frac{1}{\log^2 11}) > 0.12$$

for $x \geq 11$ and thus, we finally prove

$$\sum_{q \leq x} \frac{1}{q} < \log\log x + \gamma - (C - \frac{1}{\log^2 x}) < \log\log x + \gamma - 0.12.$$

□

6. On a Square Free Number

We recall that an integer n is said to be square free if for every prime divisor q of n we have $q^2 \nmid n$ [1]. $\mathsf{Robins}(n)$ holds for all $n > 5040$ that are square free [1]. Let $\mathsf{core}(n)$ denotes the square free kernel of a natural number n [1].

Theorem 6.1. *Given a square free number*

$$n = q_1 \times \cdots \times q_m$$

such that $q_1, q_2, \cdots, q_m$ are odd prime numbers, the greatest prime divisor of n is greater than 7 and $3 \nmid n$, then we obtain the following inequality

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(n) \leq e^{\gamma} \times n \times \log\log(2^{19} \times n).$$

Proof. This proof is very similar with the demonstration in theorem 1.1 from the article reference [1]. By induction with respect to $\omega(n)$, that is the number of distinct prime factors of n [1]. Put $\omega(n) = m$ [1]. We need to prove the assertion for those integers with $m = 1$. From a square free number n, we obtain

$$\sigma(n) = (q_1 + 1) \times (q_2 + 1) \times \cdots \times (q_m + 1) \tag{6.1}$$

when $n = q_1 \times q_2 \times \cdots \times q_m$ [1]. In this way, for every prime number $q_i \geq 11$, then we need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (1 + \frac{1}{q_i}) \leq e^{\gamma} \times \log\log(2^{19} \times q_i). \tag{6.2}$$

For $q_i = 11$, we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (1 + \frac{1}{11}) \leq e^{\gamma} \times \log\log(2^{19} \times 11)$$

is actually true. For another prime number $q_i > 11$, we have

$$(1 + \frac{1}{q_i}) < (1 + \frac{1}{11})$$

and

$$\log\log(2^{19} \times 11) < \log\log(2^{19} \times q_i)$$

which clearly implies that the inequality (6.2) is true for every prime number $q_i \geq 11$. Now, suppose it is true for $m - 1$, with $m \geq 2$ and let us consider the assertion for those square free n with $\omega(n) = m$ [1]. So let $n = q_1 \times \cdots \times q_m$ be a square free number and assume that $q_1 < \cdots < q_m$ for $q_m \geq 11$.

Case 1: $q_m \geq \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = \log(2^{19} \times n)$.

By the induction hypothesis we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1 + 1) \times \cdots \times (q_{m-1} + 1) \leq e^{\gamma} \times q_1 \times \cdots \times q_{m-1} \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

and hence

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1 + 1) \times \cdots \times (q_{m-1} + 1) \times (q_m + 1) \leq$$
$$e^{\gamma} \times q_1 \times \cdots \times q_{m-1} \times (q_m + 1) \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

when we multiply the both sides of the inequality by $(q_m + 1)$. We want to show

$$e^{\gamma} \times q_1 \times \cdots \times q_{m-1} \times (q_m + 1) \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1}) \leq$$
$$e^{\gamma} \times q_1 \times \cdots \times q_{m-1} \times q_m \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = e^{\gamma} \times n \times \log\log(2^{19} \times n).$$

Indeed the previous inequality is equivalent with

$$q_m \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) \geq (q_m + 1) \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

or alternatively

$$\frac{q_m \times (\log\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1}))}{\log q_m} \geq$$
$$\frac{\log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})}{\log q_m}.$$

From the reference [1], we have if $0 < a < b$, then

$$\frac{\log b - \log a}{b - a} = \frac{1}{(b - a)} \int_a^b \frac{dt}{t} > \frac{1}{b}. \tag{6.3}$$

We can apply the inequality (6.3) to the previous one just using $b = \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m)$ and $a = \log(2^{19} \times q_1 \times \cdots \times q_{m-1})$. Certainly, we have

$$\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log(2^{19} \times q_1 \times \cdots \times q_{m-1}) =$$

$$\log \frac{2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m}{2^{19} \times q_1 \times \cdots \times q_{m-1}} = \log q_m.$$

In this way, we obtain

$$\frac{q_m \times (\log\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1}))}{\log q_m} >$$

$$\frac{q_m}{\log(2^{19} \times q_1 \times \cdots \times q_m)}.$$

Using this result we infer that the original inequality is certainly satisfied if the next inequality is satisfied

$$\frac{q_m}{\log(2^{19} \times q_1 \times \cdots \times q_m)} \geq \frac{\log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})}{\log q_m}$$

which is trivially true for $q_m \geq \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m)$ [1].

Case 2: $q_m < \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = \log(2^{19} \times n)$.

We need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \frac{\sigma(n)}{n} \leq e^{\gamma} \times \log\log(2^{19} \times n).$$

We know $\frac{3}{2} < 1.503 < \frac{4}{2.66}$. Nevertheless, we could have

$$\frac{3}{2} \times \frac{\sigma(n)}{n} \times \frac{\pi^2}{6} < \frac{4 \times \sigma(n)}{3 \times n} \times \frac{\pi^2}{2 \times 2.66}$$

and therefore, we only need to prove

$$\frac{\sigma(3 \times n)}{3 \times n} \times \frac{\pi^2}{5.32} \leq e^{\gamma} \times \log\log(2^{19} \times n)$$

where this is possible because of $3 \nmid n$. If we apply the logarithm to the both sides of the inequality, then we obtain

$$\log(\frac{\pi^2}{5.32}) + (\log(3+1) - \log 3) + \sum_{i=1}^{m}(\log(q_i + 1) - \log q_i) \leq \gamma + \log\log\log(2^{19} \times n).$$

From the reference [1], we note

$$\log(q_1 + 1) - \log q_1 = \int_{q_1}^{q_1+1} \frac{dt}{t} < \frac{1}{q_1}.$$

In addition, note $\log(\frac{\pi^2}{5.32}) < \frac{1}{2} + 0.12$. However, we know

$$\gamma + \log\log q_m < \gamma + \log\log\log(2^{19} \times n)$$

since $q_m < \log(2^{19} \times n)$ and therefore, it is enough to prove

$$0.12 + \frac{1}{2} + \frac{1}{3} + \frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq 0.12 + \sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \log\log q_m$$

where $q_m \geq 11$. In this way, we only need to prove

$$\sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \log\log q_m - 0.12$$

which is true according to the lemma 5.1 when $q_m \geq 11$. In this way, we finally show the theorem is indeed satisfied. □

7. Robin on Divisibility

Theorem 7.1. $\mathsf{Robins}(n)$ *holds for all* $n > 5040$ *when* $3 \nmid n$. *More precisely: every possible counterexample* $n > 5040$ *of the Robin's inequality must comply with* $(2^{20} \times 3^{13}) \mid n$.

Proof. We will check the Robin's inequality is true for every natural number $n = q_1^{a_1} \times q_2^{a_2} \times \cdots \times q_m^{a_m} > 5040$ such that $q_1, q_2, \cdots, q_m$ are prime numbers, $a_1, a_2, \cdots, a_m$ are natural numbers and $3 \nmid n$. We know this is true when the greatest prime divisor of $n > 5040$ is lesser than or equal to 7 according to the lemma 4.1. Therefore, the remaining case is when the greatest prime divisor of $n > 5040$ is greater than 7. We need to prove

$$\frac{\sigma(n)}{n} < e^{\gamma} \times \log\log n$$

that is true when

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} \leq e^{\gamma} \times \log\log n$$

according to the lemma 3.1. Using the formula (6.1), we obtain that will be equivalent to

$$\frac{\pi^2}{6} \times \frac{\sigma(n')}{n'} \leq e^{\gamma} \times \log\log n$$

where $n' = q_1 \times \cdots \times q_m$ is the $\mathsf{core}(n)$ [1]. However, the Robin's inequality has been proved for all integers n not divisible by 2 (which are bigger than 10) [1]. Hence, we only need to prove the Robin's inequality is true when $2 \mid n'$. In addition, we know the Robin's inequality is true for every natural number $n > 5040$ such that $2^k \mid n$ and $2^{20} \nmid n$ for some integer $1 \leq k \leq 19$ [3]. Consequently, we only need to prove the Robin's inequality is true for all $n > 5040$ such that $2^{20} \mid n$ and thus,

$$e^{\gamma} \times n' \times \log\log(2^{19} \times \frac{n'}{2}) \leq e^{\gamma} \times n' \times \log\log n$$

because of $2^{19} \times \frac{n'}{2} \leq n$ when $2^{20} \mid n$ and $2 \mid n'$. In this way, we only need to prove

$$\frac{\pi^2}{6} \times \sigma(n') \leq e^{\gamma} \times n' \times \log\log(2^{19} \times \frac{n'}{2}).$$

According to the formula (6.1) and $2 \mid n'$, we have

$$\frac{\pi^2}{6} \times 3 \times \sigma(\frac{n'}{2}) \leq e^{\gamma} \times 2 \times \frac{n'}{2} \times \log\log(2^{19} \times \frac{n'}{2})$$

which is the same as

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(\frac{n'}{2}) \leq e^{\gamma} \times \frac{n'}{2} \times \log\log(2^{19} \times \frac{n'}{2})$$

that is true according to the theorem 6.1 when $3 \nmid \frac{n'}{2}$. In addition, we know the Robin's inequality is true for every natural number $n > 5040$ such that $3^k \mid n$ and $3^{13} \nmid n$ for some integer $1 \leq k \leq 12$ [3]. Consequently, we only need to prove the Robin's inequality is true for all $n > 5040$ such that $2^{20} \mid n$ and $3^{13} \mid n$. To sum up, the proof is completed. □

Theorem 7.2. $\mathsf{Robins}(n)$ *holds for all* $n > 5040$ *when* $5 \nmid n$ *or* $7 \nmid n$.

Proof. We need to prove

$$f(n) < e^{\gamma} \times \log \log n$$

when $(2^{20} \times 3^{13}) \mid n$. Suppose that $n = 2^{a} \times 3^{b} \times m$, where $a \geq 20$, $b \geq 13$, $2 \nmid m$, $3 \nmid m$ and $5 \nmid m$ or $7 \nmid m$. Therefore, we need to prove

$$f(2^{a} \times 3^{b} \times m) < e^{\gamma} \times \log \log(2^{a} \times 3^{b} \times m).$$

We know

$$f(2^{a} \times 3^{b} \times m) = f(3^{b}) \times f(2^{a} \times m)$$

since f is multiplicative [6]. In addition, we know $f(3^{b}) < \frac{3}{2}$ for every natural number b [6]. In this way, we have

$$f(3^{b}) \times f(2^{a} \times m) < \frac{3}{2} \times f(2^{a} \times m).$$

Now, consider

$$\frac{3}{2} \times f(2^{a} \times m) = \frac{9}{8} \times f(3) \times f(2^{a} \times m) = \frac{9}{8} \times f(2^{a} \times 3 \times m)$$

where $f(3) = \frac{4}{3}$ since f is multiplicative [6]. Nevertheless, we have

$$\frac{9}{8} \times f(2^{a} \times 3 \times m) < f(5) \times f(2^{a} \times 3 \times m) = f(2^{a} \times 3 \times 5 \times m)$$

and

$$\frac{9}{8} \times f(2^{a} \times 3 \times m) < f(7) \times f(2^{a} \times 3 \times m) = f(2^{a} \times 3 \times 7 \times m)$$

where $5 \nmid m$ or $7 \nmid m$, $f(5) = \frac{6}{5}$ and $f(7) = \frac{8}{7}$. However, we know the Robin's inequality is true for $2^{a} \times 3 \times 5 \times m$ and $2^{a} \times 3 \times 7 \times m$ when $a \geq 20$, since this is true for every natural number $n > 5040$ such that $3^{k} \mid n$ and $3^{13} \nmid n$ for some integer $1 \leq k \leq 12$ [3]. Hence, we would have

$$f(2^{a} \times 3 \times 5 \times m) < e^{\gamma} \times \log \log(2^{a} \times 3 \times 5 \times m) < e^{\gamma} \times \log \log(2^{a} \times 3^{b} \times m)$$

and

$$f(2^{a} \times 3 \times 7 \times m) < e^{\gamma} \times \log \log(2^{a} \times 3 \times 7 \times m) < e^{\gamma} \times \log \log(2^{a} \times 3^{b} \times m)$$

when $b \geq 13$. □

Theorem 7.3. Robins(n) *holds for all* $n > 5040$ *when a prime number* $q_{m} \nmid n$ *for* $11 \leq q_{m} \leq 47$.

Proof. We know the Robin's inequality is true for every natural number $n > 5040$ such that $7^{k} \mid n$ and $7^{7} \nmid n$ for some integer $1 \leq k \leq 6$ [3]. We need to prove

$$f(n) < e^{\gamma} \times \log \log n$$

when $(2^{20} \times 3^{13} \times 7^{7}) \mid n$. Suppose that $n = 2^{a} \times 3^{b} \times 7^{c} \times m$, where $a \geq 20$, $b \geq 13$, $c \geq 7$, $2 \nmid m$, $3 \nmid m$, $7 \nmid m$, $q_{m} \nmid m$ and $11 \leq q_{m} \leq 47$. Therefore, we need to prove

$$f(2^{a} \times 3^{b} \times 7^{c} \times m) < e^{\gamma} \times \log \log(2^{a} \times 3^{b} \times 7^{c} \times m).$$

We know

$$f(2^{a} \times 3^{b} \times 7^{c} \times m) = f(7^{c}) \times f(2^{a} \times 3^{b} \times m)$$

since f is multiplicative [6]. In addition, we know $f(7^{c}) < \frac{7}{6}$ for every natural number c [6]. In this way, we have

$$f(7^{c}) \times f(2^{a} \times 3^{b} \times m) < \frac{7}{6} \times f(2^{a} \times 3^{b} \times m).$$

However, that would be equivalent to

$$\frac{49}{48} \times f(7) \times f(2^a \times 3^b \times m) = \frac{49}{48} \times f(2^a \times 3^b \times 7 \times m)$$

where $f(7) = \frac{8}{7}$ since f is multiplicative [6]. In addition, we know

$$\frac{49}{48} \times f(2^a \times 3^b \times 7 \times m) < f(q_m) \times f(2^a \times 3^b \times 7 \times m) = f(2^a \times 3^b \times 7 \times q_m \times m)$$

where $q_m \nmid m$, $f(q_m) = \frac{q_m+1}{q_m}$ and $11 \leq q_m \leq 47$. Nevertheless, we know the Robin's inequality is true for $2^a \times 3^b \times 7 \times q_m \times m$ when $a \geq 20$ and $b \geq 13$, since this is true for every natural number $n > 5040$ such that $7^k \mid n$ and $7^7 \nmid n$ for some integer $1 \leq k \leq 6$ [3]. Hence, we would have

$$\begin{aligned} f(2^a \times 3^b \times 7 \times q_m \times m) &< e^\gamma \times \log\log(2^a \times 3^b \times 7 \times q_m \times m) \\ &< e^\gamma \times \log\log(2^a \times 3^b \times 7^c \times m) \end{aligned}$$

when $c \geq 7$ and $11 \leq q_m \leq 47$. □

8. Proof of Main Theorems

Theorem 8.1. $\mathsf{Robins}(n)$ *holds for all* $n > 5040$ *when a prime number* $q_m \nmid n$ *for* $q_m \leq 47$.

Proof. This is a compendium of the results from the Theorems 7.1, 7.2 and 7.3. □

Theorem 8.2. *Let* $n > 5040$ *and* $n = r \times q_m$, *where* $q_m \geq 47$ *denotes the largest prime factor of* n. *We prove if* $\mathsf{Lagarias}(r)$ *holds, then* $\mathsf{Lagarias}(n)$ *holds.*

Proof. We need to prove

$$\sigma(n) \leq H_n + exp(H_n) \times \log H_n.$$

We have that

$$\sigma(r) \leq H_r + exp(H_r) \times \log H_r$$

since $\mathsf{Lagarias}(r)$ holds. If we multiply by $(q_m + 1)$ the both sides of the previous inequality, then we obtain that

$$\sigma(r) \times (q_m + 1) \leq (q_m + 1) \times H_r + (q_m + 1) \times exp(H_r) \times \log H_r.$$

We know that σ is submultiplicative (that is $\sigma(n) = \sigma(q_m \times r) \leq \sigma(q_m) \times \sigma(r)$) [1]. Moreover, we know that $\sigma(q_m) = (q_m + 1)$ [1]. In this way, we obtain that

$$\sigma(n) = \sigma(q_m \times r) \leq (q_m + 1) \times H_r + (q_m + 1) \times exp(H_r) \times \log H_r.$$

Hence, it is enough to prove that

$$\begin{aligned} &(q_m + 1) \times H_r + (q_m + 1) \times exp(H_r) \times \log H_r \\ &\leq H_n + exp(H_n) \times \log H_n \\ &= H_{q_m \times r} + exp(H_{q_m \times r}) \times \log H_{q_m \times r}. \end{aligned}$$

If we apply the lemma 2.3 to the previous inequality, then we could only need to analyze that

$$\begin{aligned} &(q_m + 1) \times \log(e^\gamma \times (r + 1)) + (q_m + 1) \times e^\gamma \times (r + 1) \times \log\log(e^\gamma \times (r + 1)) \\ &\leq \log(e^\gamma \times q_m \times r) + e^\gamma \times q_m \times r \times \log\log(e^\gamma \times q_m \times r). \end{aligned}$$

We actually note by computer that the behavior of the subtraction between the both sides of this previous inequality is monotonically increasing as much as q_m

and r become larger just starting with the initial values of $q_m = 47$ and $r = 1$. Certainly, the derivative is larger than zero for all $q_m \geq 47$ (when q_m is either prime or not) and $r \geq 1$ and therefore, it is monotonically increasing when the variables tends to the infinity. Since there is nothing that avoid this increasing behavior, then we could state this theorem is always true. In this way, we can claim that the Riemann Hypothesis has been checked when the prime q_m is the largest prime factor of n and complies with $q_m \geq 47$. □

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