Cyclic Reciprocal Sums of Prime Products

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Abstract. We discuss cyclic reciprocal sums of products of primes that appear in distributions of no-prime (NP) sequences. Formulas and asymptotics for cyclic sums of arbitrary order are obtained, and possible connections with certain fields of physics briefly discussed.

Keywords: Number theory; prime number distribution; prime sequences

1 Definitions

A cyclic reciprocal prime sum of order K , and a set of consecutive primes P : $p_1 < p_k < p_N$ of length N is defined as the sum of reciprocal unique products of primes in the set:

$$C_K(P, N) = \sum_{p_1}^{p_1 < p_N, p_i \neq p_j} \frac{1}{p_1 p_2 \dots p_K} \quad (1)$$

Immediately,

$$C_1(P) = \sum_{p_1}^{p_1 \leq p_N} \frac{1}{p_1}, \quad C_N(P) = \frac{1}{p_1 p_2 \dots p_N} \quad (2)$$

For convenience, in the text that follows the notation $q_i = 1 / p_i$ will be used. We will also denote:

$$E_k(P) = \sum_{p_1}^{p_1 \leq p_N} \frac{1}{p_1^k}, k \geq 1$$

From the definition, cyclic sums are invariant with respect to the group of transpositions of the set P .

2 Expressions and Asymptotics

It is well known that E_k , $k > 1$ has constant asymptotics for $N \rightarrow \infty$ directly related to the prime Riemann function $\zeta(k)$ and is unbounded for $k = 1$ with the asymptotics of $\log \log p_N \sim \log \log (N \log N)$ [1,2].

From the definition, for $N = 1$, $C_1(P) = E_1(P)$. Then, for $N = 2$, straightforwardly:

$$E_1 C_1 = \sum_{i=1}^N q_i (C_1 - q_i) + \sum_{i=1}^N q_i^2 = 2 C_2 + E_2$$

$$C_2 = \frac{1}{2} (E_1^2 - E_2) \quad (3)$$

Same strategy allows to express $C_k(P)$, $k = 2 \dots N$ via $E_l \dots E_k$:

$$E_1 C_k = \sum_{i=1}^N q_i (C_k - q_i C_{k-1} \setminus q_i) + \sum_{i=1}^N q_i^2 C_{k-1} \setminus q_i = (k+1) C_{k+1} + O(E_1^{k-1})$$

One can see that $C_k(P)$ has the asymptotics:

$$C_k(P) = \frac{1}{k!} E_1^k + O(E_1^{k-2}) \quad (4)$$

with the second term,

$$O(E_1^{k-2}) = \sum_{i=1}^{k-2} D_{i,j} E_1^i \prod E_j E_l + D_0 E_k \quad (5)$$

where $j, l > 1$.

Hence, the behavior of $C_k(P)$ is polynomial of the order k with respect to $E_l(P)$. As the asymptotics of E_k with $k > l$ are constant, (3) and (4) can be expressed as:

$$C_k(P) = \frac{1}{k!} E_1^k + \sum_{i=1}^{k-2} D_i E_1^i + D_0 E_k \quad (6)$$

with $D_i = \sum_{j=1}^{k-2} D_{i,j} \prod E_j E_l$, the *cyclic coefficients*. It is easy to see that $D_0 = \frac{1}{k}$ and so on.

3 Cyclic Prime Sums and the Naturalness Problem

Let us consider the decomposition of a cyclic sum by dominant terms (4-6) in the form:

$$C_k(P) = \frac{1}{k!} E_1^k \left(1 + \sum_{i=0}^{k-2} D_i E_1^{-(k-i)} \right) \quad (7)$$

Here, $E_l(P) \rightarrow \infty$ with $N \rightarrow \infty$ and for each term T_i in (7) it should hold: $T_{i+1}/T_i \rightarrow 0$ for $N \rightarrow \infty$.

Cyclic prime sums are interesting for their “shaving” effect, whereby the resulting value of the sum can be significantly, many orders of magnitude smaller than the contributing terms.

Consider the examples of cyclic sums of the maximum order N in the Table 1 for the sets of first prime orders with $N = 10; 15; 25$, $p_l = 5$.

Table 1. Cyclic prime sums and term contributions.

N	Cyclic sum, C_N	First term, $\frac{1}{k!} E_1^k$	Last term, $\frac{1}{k} E_k$
10	3.35×10^{-13}	1.11×10^{-8}	1.06×10^{-8}
15	8.70×10^{-22}	4.02×10^{-14}	2.20×10^{-12}
25	3.97×10^{-41}	1.36×10^{-26}	1.34×10^{-19}

With a value of the dominant term for $N = 25$ of the order 10^{-26} , the resulting value of C_N (6) had the order of 10^{-41} . These examples demonstrate that the difference in scale between the resulting value of the sum and the contributing term can reach 10^{-20} even for relatively small sets of primes.

Evidently, with higher N , the cyclic sum C_N is bound to remain positive; and very small, that is, in the order of $1 / p_N \# \sim e^{-N \log N}$ [3], regardless of the arithmetics behind the values of the cyclic coefficients D_{ij} (6). These constraints force the relations between D_{ij} and E_l^k to balance out in a very precise and precarious way to produce extremely small resulting positive values of the higher-order sums, even with the contributing terms of significantly higher magnitude.

Interestingly, this behavior of cyclic sums is reminiscent of some phenomena in High Energy Physics, such as the problem of vacuum contributions of quantum fields and cosmological constant [4-6]. This problem is sometimes commented to have a tension with “the naturalness” assumption, because such a precise balancing of terms can be seen as requiring special assumptions about the contributions. Yet, as seen with the cyclic sums, it can emerge quite naturally in some problems of number theory and prime mathematics.

4 Cyclic Sums and No-Prime Distributions

No-prime (NP) sequences were defined [7] as sequences of pairs of arbitrary length with no primes. While a detailed analysis of NP-sequences and distributions will be a subject of another work, conceptually, an approach to understanding distributions of NP-sequences based on prime modulo cycles is relatively straightforward.

Evidently, the necessary and sufficient condition for a sequence of K consecutive non-prime pairs: $\{k, (x_l, y_l), l_l, (x_2, y_2), \dots\}$ and the key k is a constraint on the prime modulo functions $h_p(k)$ to have at least one value in the range $P = \{(1, 2), (4, 5), (7, 8), \dots\}$ representing the sequence of positions with relative distance to the key that cannot be prime, for prime $p < k$:

$$\forall v \in P \exists p < k: h_p(k) = v \quad (8)$$

Each combination of values satisfying this condition repeats with a period of $3 \times \prod p_k$, leading to an heuristic estimate of the number of NP-sequences of length K in an interval of length d :

$$N_{NP}(K, d) = \frac{d K!}{3} (C_{2K}(p_m) - N_{ov}(p)) \quad (9)$$

where C_{2K} is the cyclic prime sum of the order $2K$, p_m the maximum prime that can be a factor in the range d , $p_m \sim d/3$ and N_{ov} , the count of overlaps between combinations of prime moduli resulting in an NP-condition at certain positions that needs to be subtracted from the sum (9) to prevent counting the positions of overlaps multiple times.

It follows then that distributions of NP-sequences are directly related to cyclic prime sums of even orders.

5 Conclusion

In this work a definition and an analysis of cyclic sums of reciprocal products of primes were given. Cyclic reciprocal sums of prime products emerge in some areas of prime number theory such as distributions of no-prime (NP) sequences and have interesting properties, such as “unnaturally” small resulting value of the sum relative to the order of contributing terms. It is possible for these reasons that cyclic reciprocal sums of prime products will find further applications in the fields and problems of mathematics and theoretical physics.

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