The geometric method for solving a heat equation

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Abstract

In the proposed publication, we present a geometric method for solving a boundary value problem for a nonstationary heat equation. The equivalence of the geometric solution to an explicit difference method is shown with the corresponding initial parameters. There are considered the first and second boundary value problems. The physical interpretation of the method is based on local balance.

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1. Introduction

Let us consider initial-boundary value problem for the heat equation

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2} + f(x), \quad x\in(a,b),\ t>0, \tag{1}$$

$$u(x,t)=U(x), \quad x=a,\ t>0, \tag{2}$$

$$u(x,t)=U(x), \quad x=b,\ t>0, \tag{3}$$

$$u(x,0)=U(x),\ x\in[a,b]. \tag{4}$$

An explicit difference scheme ($f(x)=0$) for equation (1) [1] is written as follows:

$$\frac{u_i^{k+1}-u_i^k}{\Delta t}=D\frac{u_{i-1}^k-2u_i^k+u_{i+1}^k}{h^2}, \tag{5}$$

where $u_i^k=u(x_i,t_k)$ is a grid function,

$$a=x_1<x_2<...<x_i<...<x_N=b,$$

$$h=x_{i+1}-x_i=const,\ i=\overline{1,N-1},\ \Delta t=t_{k+1}-t_k=const,\ k=0,1,2,....$$

Scheme (5) is conditionally stable with respect to perturbation of the initial data. The stability condition is written as $\Delta t \le \dfrac{h^2}{2D}$ [1].

The difference solution of the problem (1) – (4) is calculated as follows:

$$u_1^{k+1} = U(x_1),$$

$$u_i^{k+1} = u_i^k + \frac{D\Delta t}{h^2}(u_{i-1}^k - 2u_i^k + u_{i+1}^k), \quad i = \overline{2, N-1}, \tag{6}$$

$$u_N^{k+1} = U(x_N),$$

где $\quad u_i^0 = U(x_i), \quad i = \overline{1, N}$.

Let us assume that conditions of the first kind are given for equation (1) at the boundary points:

$$\frac{\partial u(a,t)}{\partial x} = 0, \qquad \frac{\partial u(b,t)}{\partial x} = 0. \tag{7}$$

Difference conditions for (7) can be written as follows [2]:

$$\frac{u_1^{k+1} - u_1^k}{\Delta t} = \frac{2}{h}(u_2^k - u_1^k)/h, \tag{8}$$

$$\frac{u_N^{k+1} - u_N^k}{\Delta t} = -\frac{2}{h}(u_N^k - u_{N-1}^k)/h. \tag{9}$$

2. Geometric method for solving the equation (1)

First boundary value problem. Let the initial condition of the equation (1) is given (Figure 1, red line). The node values u_i^0, $i = \overline{1, N}$ are marked with red dots. The calculation of the discrete solution at the points (x_i, t_1), $i = \overline{2, N-1}$ is carried out in two steps:

Step 1 consists in finding the midpoints of the segments (Figure 1, red intervals) which connect the points u_i^0, $i = \overline{1, N}$. The values at these points are denoted $y_{i-1/2}$ where

$$y_{i-1/2} = (u_{i-1}^0 + u_i^0)/2, \quad i = \overline{2, N}.$$

Step 2 consists in connecting the points $y_{i-1/2}$, $i = \overline{2, N}$ (Figure 1, blue dotted line) and calculating values u_i^1 in points x_i, $i = \overline{2, N-1}$

$$u_i^1 = ((u_{i-1}^0 + u_i^0)/2 + (u_i^0 + u_{i+1}^0)/2)/2 = (u_{i-1}^0 + 2u_i^0 + u_{i+1}^0)/4 = u_i^0 + (u_{i-1}^0 - 2u_i^0 + u_{i+1}^0)/4. \tag{10}$$

The values at the boundary points remain unchanged.

Expression (10) can be rewritten as:

$$\frac{u_i^1 - u_i^0}{1/4} = u_{i-1}^0 - 2u_i^0 + u_{i+1}^0, \quad i = \overline{2, N-1}. \tag{11}$$

For $D = h^2$, $\Delta t = 1/4$, $k = 0$ formula (6) coincides with (11). The solution of the boundary value problem (1) – (4) calculated at the moment of time t_k, $k = 1$ by the graphical method coincides with the one calculated according to the scheme (6). In this case, the stability condition is satisfied.
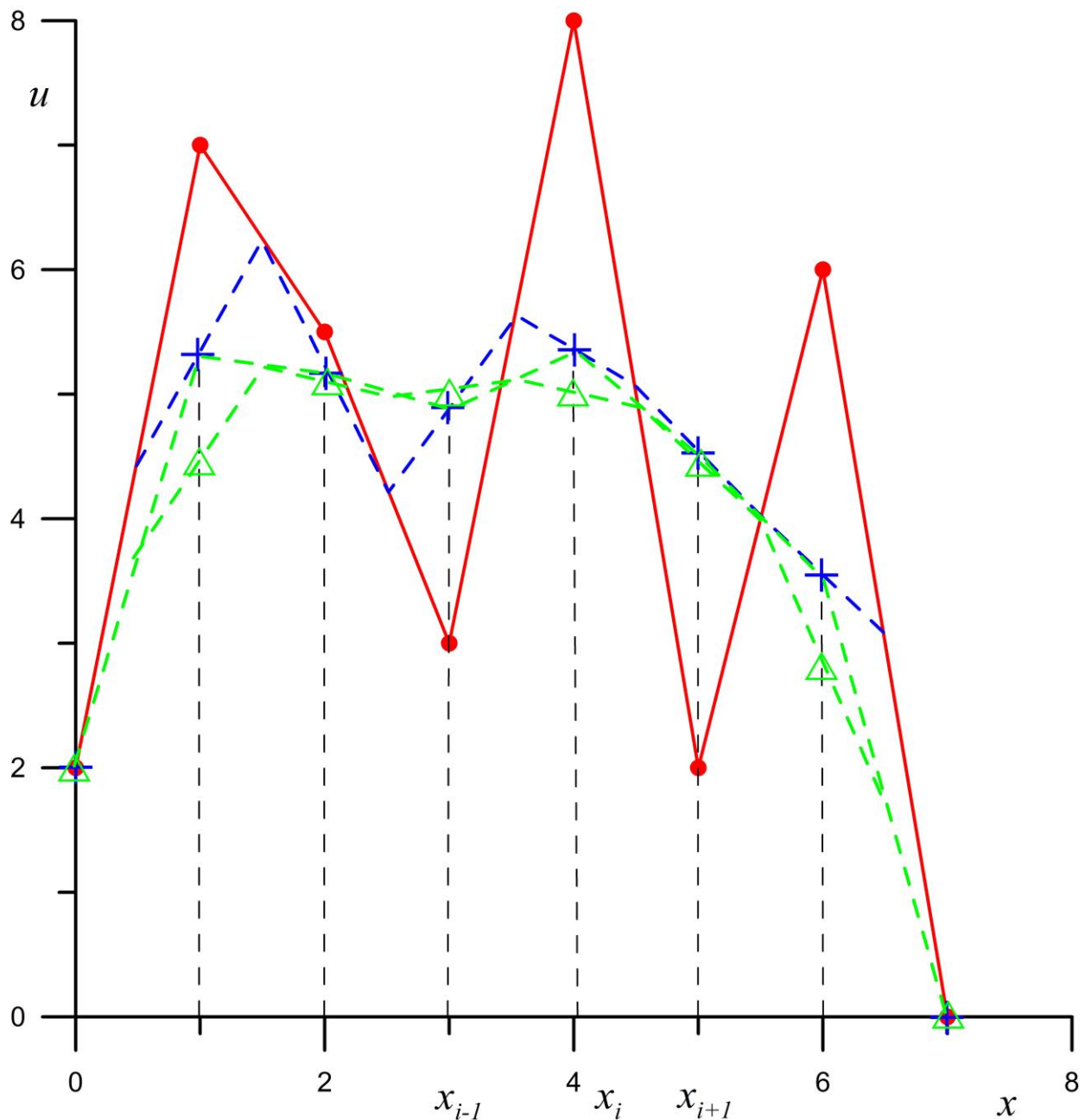

Fig. 1. Graphical method for solving equation (1) ($x \in [0,7]$, $N = 8$, $h = 1$). First boundary value problem. ⚑ – initial condition $U(x)$, $x \in [0,7]$; ✛ – solution computed at time t_k, $k = 1$; △ – solution computed at time t_k, $k = 2$.

The green line in Figure 1 shows a schematic representation of the graphical calculation of the solution at the time t_k, $k = 2$. The solution at the moments of time t_k, $k = 3,4...$ can be calculated analogously.

Second boundary value problem. Physical interpretation of the solution. Let conditions (7) are given at the points $x = a$, $x = b$.
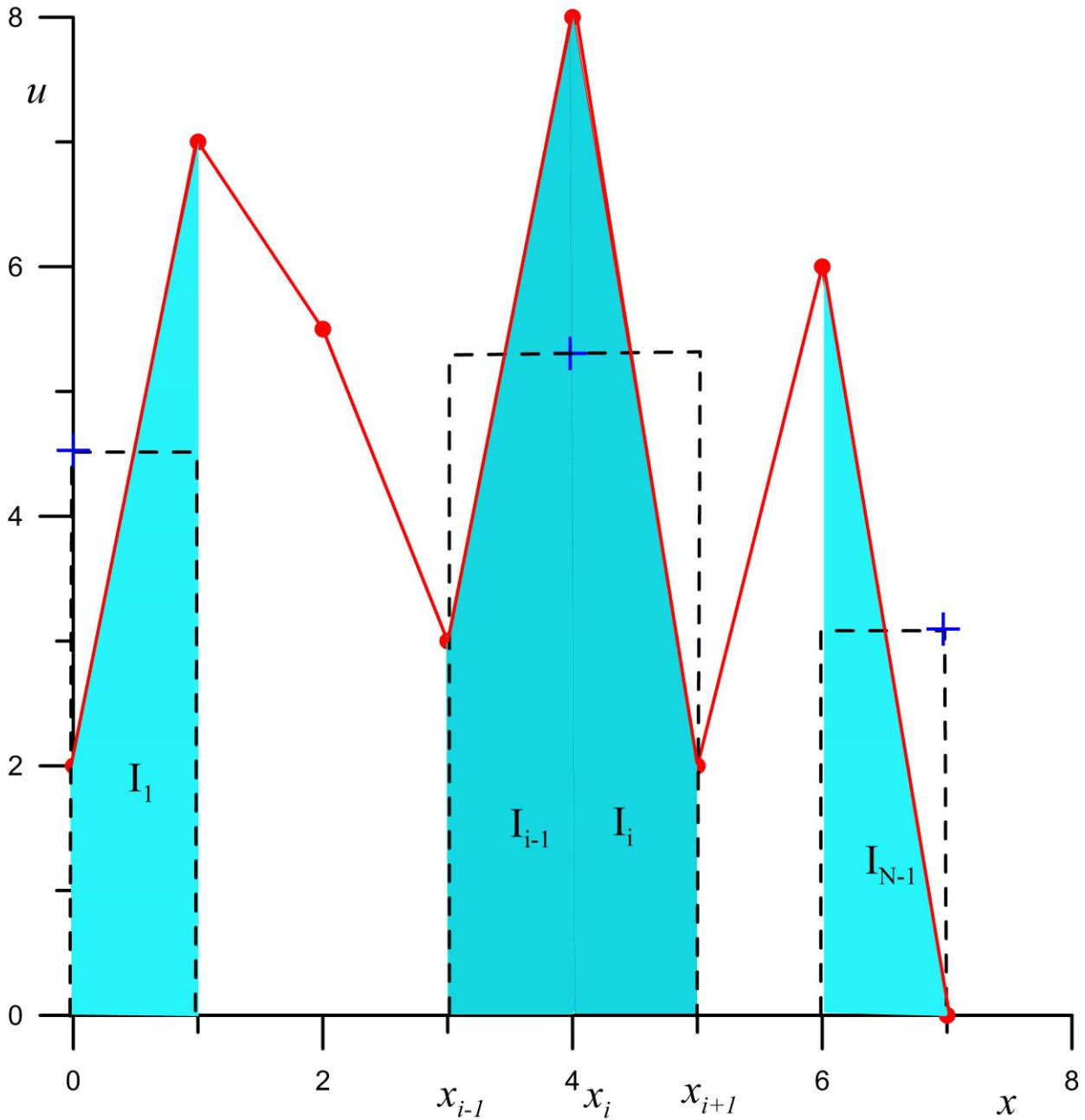

Fig. 2. Physical interpretation of the graphics solution. Second boundary value problem.

In Figure 2, the initial condition $U(x)$ with nodal values u_i^0, $i = \overline{1,N}$ is marked in red. The sum of the areas of the figures I_i, $i = \overline{1,N-1}$ (marked in blue) equals to the

integral $\int_a^b U(x)dx$. The solution at the moment of time t_1 in points x_i, $i = \overline{2, N-1}$ is calculated as the height of a rectangle equal in area to the sum of the areas of the figures I_{i-1} and I_i:

$$u_i^1 = I_{i-1} + I_i = ((u_{i-1}^0 + u_i^0)h/2 + (u_i^0 + u_{i+1}^0)h/2)/2h =$$
$$= (u_{i-1}^0 + 2u_i^0 + u_{i+1}^0)/4 = u_i^0 + (u_{i-1}^0 - 2u_i^0 + u_{i+1}^0)/4. \qquad (12)$$

Expressions (10) and (12) coincide.

Let's calculate the solution at one of the boundary points, for example, at a point x_N.

The area of the figure I_{N-1} (Figure 2) is calculated as follows:

$$I_{N-1} = (u_N^0 + u_{N-1}^0)h/2.$$

The solution at the point (x_N, t_1) is calculated as a height of a rectangle equal in area to the figure I_{N-1}:

$$u_N^1 = (u_N^0 + u_{N-1}^0)h/2h = (\pm 2u_N^0 + u_N^0 + u_{N-1}^0)/2 = u_N^0 + (-2u_N^0 + u_N^0 + u_{N-1}^0)/2 =$$
$$= u_N^0 - (u_N^0 - u_{N-1}^0)/2.$$

Hence, we get the condition:

$$\frac{u_N^1 - u_N^0}{1/4} = -2(u_N^0 - u_{N-1}^0).$$

The obtained condition coincides with the difference condition (9) at $k = 0$.

The solution at a point (x_1, t_1) is calculated analogously. It is easy to obtain a solution for $f(x) \neq 0$.

Physical interpretation. Physical interpretation is based on heat conservation for the figures I_i, $i = \overline{1, N-1}$ (Figure 2). It is assumed that for figures I_1, I_{N-1} and for sums of figures $I_{i-1} + I_i$, $i = \overline{2, N-1}$, there is no heat exchange with neighboring figures. Let the temperature in the selected figure at the initial moment of time t_0 be unevenly distributed. Let us suppose that the temperature is setting and becoming constant. After that, the resulting constant is assigned as a solution at the nodal point x_i at a time t_1. The procedure repeats for the next points in time.

3. Conclusion

The constructed geometric method for solving the first and second boundary value problems for the heat equation gives a good illustration of the use of an explicit finite difference scheme. The physical interpretation corresponds with the process of heat conduction or diffusion. The proposed research has a methodological interest for the

study and illustration of difference methods for solving boundary value problems. The method can be used for computer visualization.

References

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