Robin Criterion on Divisibility

Frank Vega

CopSonic, 1471 Route de Saint-Nauphary 82000 Montauban, France

Abstract

Robin criterion states that the Riemann Hypothesis is true if and only if the inequality $\sigma(n) < e^{\gamma} \times n \times \log\log n$ holds for all $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. This is known as the Robin inequality. In 2007, Choie, Lichiardopol, Moree and Solé have shown that the Robin inequality is true for all $n > 5040$ which are not divisible by 2. We prove that the Robin inequality is true for all $n > 5040$ which are not divisible by any prime number between 3 and 953.

Keywords: Riemann hypothesis, Robin inequality, sum-of-divisors function, prime numbers
2000 MSC: 11M26, 11A41, 11A25

1. Introduction

In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$ [1]. As usual $\sigma(n)$ is the sum-of-divisors function of n [2]:

$$\sum_{d \mid n} d$$

where $d \mid n$ means the integer d divides to n and $d \nmid n$ means the integer d does not divide to n. Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say $\mathsf{Robins}(n)$ holds provided

$$f(n) < e^{\gamma} \times \log\log n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, and log is the natural logarithm. The importance of this property is:

Theorem 1.1. $\mathsf{Robins}(n)$ *holds for all* $n > 5040$ *if and only if the Riemann Hypothesis is true [3].*

It is known that $\mathsf{Robins}(n)$ holds for many classes of numbers n.

Theorem 1.2. $\mathsf{Robins}(n)$ *holds for all* $n > 5040$ *that are not divisible by* 2 *[2].*

In this work, we prove that $\mathsf{Robins}(n)$ holds for all $n > 5040$ that are not divisible by any prime number between 3 and 953.

Email address: `vega.frank@gmail.com` (Frank Vega)

2. A Central Lemma

These are known results:

Lemma 2.1. *[2]. For $n > 1$:*

$$f(n) < \prod_{q|n} \frac{q}{q-1}. \tag{1}$$

Lemma 2.2. *[4].*

$$\prod_{k=1}^{\infty} \frac{1}{1-\frac{1}{q_k^2}} = \zeta(2) = \frac{\pi^2}{6}. \tag{2}$$

The following is a key lemma. It gives an upper bound on $f(n)$ that holds for all n. The bound is too weak to prove $\mathsf{Robins}(n)$ directly, but is critical because it holds for all n. Further the bound only uses the primes that divide n and not how many times they divide n.

Lemma 2.3. *Let $n > 1$ and let all its prime divisors be $q_1 < \cdots < q_m$. Then,*

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i+1}{q_i}.$$

Proof. We use that lemma 2.1:

$$f(n) < \prod_{i=1}^{m} \frac{q_i}{q_i-1}.$$

Now for $q > 1$,

$$\frac{1}{1-\frac{1}{q^2}} = \frac{q^2}{q^2-1}.$$

So

$$\begin{aligned} \frac{1}{1-\frac{1}{q^2}} \times \frac{q+1}{q} &= \frac{q^2}{q^2-1} \times \frac{q+1}{q} \\ &= \frac{q}{q-1}. \end{aligned}$$

Then by lemma 2.2,

$$\prod_{i=1}^{m} \frac{1}{1-\frac{1}{q_i^2}} < \zeta(2) = \frac{\pi^2}{6}.$$

Putting this together yields the proof:

$$\begin{aligned} f(n) &< \prod_{i=1}^{m} \frac{q_i}{q_i-1} \\ &\le \prod_{i=1}^{m} \frac{1}{1-\frac{1}{q_i^2}} \times \frac{q_i+1}{q_i} \\ &< \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i+1}{q_i}. \end{aligned}$$

□

3. About the *p*-adic order

In basic number theory, for a given prime number p, the p-adic order of a natural number n is the highest exponent $\nu_p \geq 1$ such that p^{ν_p} divides n. This is a known result:

Lemma 3.1. *In general, we know that* $\mathsf{Robins}(n)$ *holds for a natural number* $n > 5040$ *that satisfies either* $\nu_2(n) \leq 19$*,* $\nu_3(n) \leq 12$ *or* $\nu_7(n) \leq 6$*, where* $\nu_p(n)$ *is the p-adic order of n [5].*

We know the following lemmas:

Lemma 3.2. *[5]. Let* $\prod_{i=1}^{m} q_i^{a_i}$ *be the representation of n as a product of primes* $q_1 < \cdots < q_m$ *with natural numbers as exponents* $a_1, \ldots, a_m$*. Then,*

$$f(n) = \left(\prod_{i=1}^{m} \frac{q_i}{q_i - 1}\right) \times \prod_{i=1}^{m} \left(1 - \frac{1}{q_i^{a_i+1}}\right).$$

Lemma 3.3. *[5]. Let* $n > e^{e^{23.762143}}$ *and let all its prime divisors be* $q_1 < \cdots < q_m$*, then*

$$\left(\prod_{i=1}^{m} \frac{q_i}{q_i - 1}\right) < \frac{1771561}{1771560} \times e^{\gamma} \times \log\log n.$$

Lemma 3.4. $\mathsf{Robins}(n)$ *holds for all* $10^{10^{10}} \geq n > 5040$ *[5].*

Putting together all these results, then we obtain that

Lemma 3.5. $\mathsf{Robins}(n)$ *holds for* $n > 5040$ *when* $\nu_{31}(n) \leq 3$.

Proof. From lemma 3.2, we note that

$$f(n) = \left(\prod_{i=1}^{m} \frac{q_i}{q_i - 1}\right) \times \prod_{i=1}^{m} \left(1 - \frac{1}{q_i^{a_i+1}}\right) \leq \left(\prod_{i=1}^{m} \frac{q_i}{q_i - 1}\right) \times \left(1 - \frac{1}{31^{\nu_{31}(n)+1}}\right)$$

when $\nu_{31}(n) \leq 3$. We only need to look at the case where $\nu_{31}(n) = 3$ since the weaker cases follow because

$$\left(1 - \frac{1}{31^{1+1}}\right) < \left(1 - \frac{1}{31^{2+1}}\right) < \left(1 - \frac{1}{31^{3+1}}\right).$$

In this way, we obtain that

$$f(n) \leq \left(\prod_{i=1}^{m} \frac{q_i}{q_i - 1}\right) \times \left(1 - \frac{1}{31^{3+1}}\right) = \frac{923520}{923521} \times \left(\prod_{i=1}^{m} \frac{q_i}{q_i - 1}\right)$$

when $\nu_{31}(n) \leq 3$. With lemma 3.3, we have for $n > e^{e^{23.762143}}$

$$\frac{923520}{923521} \times \left(\prod_{i=1}^{m} \frac{q_i}{q_i - 1}\right) < \frac{923520}{923521} \times \frac{1771561}{1771560} \times e^{\gamma} \times \log\log n < e^{\gamma} \times \log\log n$$

since $\frac{923520}{923521} \times \frac{1771561}{1771560} < 1$. In light of lemma 3.4 and the fact that $e^{e^{23.762143}} < 10^{10^{10}}$, we then conclude that $\mathsf{Robins}(n)$ holds for $n > 5040$ when $\nu_{31}(n) \leq 3$. $\square$

4. A Particular Case

We can easily prove that $\mathsf{Robins}(n)$ is true for certain kind of numbers:

Lemma 4.1. $\mathsf{Robins}(n)$ *holds for* $n > 5040$ *when* $q \leq 7$*, where* q *is the largest prime divisor of* n*.*

Proof. Let $n > 5040$ and let all its prime divisors be $q_1 < \cdots < q_m \leq 5$, then we need to prove

$$f(n) < e^{\gamma} \times \log\log n$$

that is true when

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq e^{\gamma} \times \log\log n$$

according to the lemma 2.1. For $q_1 < \cdots < q_m \leq 5$,

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5}{1 \times 2 \times 4} = 3.75 < e^{\gamma} \times \log\log(5040) \approx 3.81.$$

However, we know for $n > 5040$

$$e^{\gamma} \times \log\log(5040) < e^{\gamma} \times \log\log n$$

and therefore, the proof is complete when $q_1 < \cdots < q_m \leq 5$. The remaining case is for $n > 5040$ when all its prime divisors are $q_1 < \cdots < q_m \leq 7$. $\mathsf{Robins}(n)$ holds for $n > 5040$ when $\nu_7(n) \leq 6$ according to the lemma 3.1 [5]. Hence, it is enough to prove this for those natural numbers $n > 5040$ when $7^7 \mid n$. For $q_1 < \cdots < q_m \leq 7$,

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5 \times 7}{1 \times 2 \times 4 \times 6} = 4.375 < e^{\gamma} \times \log\log(7^7) \approx 4.65.$$

However, for $n > 5040$ and $7^7 \mid n$, we know that

$$e^{\gamma} \times \log\log(7^7) \leq e^{\gamma} \times \log\log n$$

and as a consequence, the proof is complete when $q_1 < \cdots < q_m \leq 7$. □

5. A Better Bound

This is a known result:

Lemma 5.1. *[6]. For* $x > 1$*:*

$$\sum_{q \leq x} \frac{1}{q} < \log\log x + B + \frac{1}{\log^2 x} \tag{3}$$

where

$$B = 0.2614972128\cdots$$

denotes the (Meissel-)Mertens constant [6].

We show a better result:

Lemma 5.2. *For $x \geq 11$, we have*

$$\sum_{q \leq x} \frac{1}{q} < \log\log x + \gamma - 0.12.$$

Proof. Let's define $H = \gamma - B$. The lemma 5.1 is the same as

$$\sum_{q \leq x} \frac{1}{q} < \log\log x + \gamma - (H - \frac{1}{\log^2 x}).$$

For $x \geq 11$,

$$(H - \frac{1}{\log^2 x}) > (0.31 - \frac{1}{\log^2 11}) > 0.12$$

and thus,

$$\sum_{q \leq x} \frac{1}{q} < \log\log x + \gamma - (H - \frac{1}{\log^2 x}) < \log\log x + \gamma - 0.12.$$

□

6. On a Square Free Number

We know the following results:

Lemma 6.1. *[2]. For $0 < a < b$:*

$$\frac{\log b - \log a}{b - a} = \frac{1}{(b-a)} \int_a^b \frac{dt}{t} > \frac{1}{b}. \tag{4}$$

Lemma 6.2. *[2]. For $q > 0$:*

$$\log(q+1) - \log q = \int_q^{q+1} \frac{dt}{t} < \frac{1}{q}. \tag{5}$$

We recall that an integer n is said to be square free if for every prime divisor q of n we have $q^2 \nmid n$ [2]. $\mathsf{Robins}(n)$ holds for all $n > 5040$ that are square free [2].

Lemma 6.3. *For a square free number*

$$n = q_1 \times \cdots \times q_m$$

such that $q_1 < q_2 < \cdots < q_m$ are odd prime numbers, $q_m \geq 11$ and $3 \nmid n$, then:

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(n) \leq e^{\gamma} \times n \times \log\log(2^{19} \times n).$$

Proof. By induction with respect to $\omega(n)$, that is the number of distinct prime factors of n [2]. Put $\omega(n) = m$ [2]. We need to prove the assertion for those integers with $m = 1$. From a square free number n, we obtain

$$\sigma(n) = (q_1 + 1) \times (q_2 + 1) \times \cdots \times (q_m + 1) \tag{6}$$

when $n = q_1 \times q_2 \times \cdots \times q_m$ [2]. In this way, for every prime number $q_i \geq 11$, then we need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (1 + \frac{1}{q_i}) \leq e^{\gamma} \times \log\log(2^{19} \times q_i). \tag{7}$$

For $q_i = 11$, we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (1 + \frac{1}{11}) \leq e^{\gamma} \times \log\log(2^{19} \times 11)$$

is actually true. For another prime number $q_i > 11$, we have

$$(1 + \frac{1}{q_i}) < (1 + \frac{1}{11})$$

and

$$\log\log(2^{19} \times 11) < \log\log(2^{19} \times q_i)$$

which clearly implies that the inequality (7) is true for every prime number $q_i \geq 11$. Now, suppose it is true for $m - 1$, with $m \geq 2$ and let us consider the assertion for those square free n with $\omega(n) = m$ [2]. So let $n = q_1 \times \cdots \times q_m$ be a square free number and assume that $q_1 < \cdots < q_m$ for $q_m \geq 11$.

Case 1: $q_m \geq \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = \log(2^{19} \times n)$.

By the induction hypothesis we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1 + 1) \times \cdots \times (q_{m-1} + 1) \leq e^{\gamma} \times q_1 \times \cdots \times q_{m-1} \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

and hence

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1 + 1) \times \cdots \times (q_{m-1} + 1) \times (q_m + 1) \leq$$

$$e^{\gamma} \times q_1 \times \cdots \times q_{m-1} \times (q_m + 1) \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

when we multiply the both sides of the inequality by $(q_m + 1)$. We want to show

$$e^{\gamma} \times q_1 \times \cdots \times q_{m-1} \times (q_m + 1) \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1}) \leq$$

$$e^{\gamma} \times q_1 \times \cdots \times q_{m-1} \times q_m \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = e^{\gamma} \times n \times \log\log(2^{19} \times n).$$

Indeed the previous inequality is equivalent with

$$q_m \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) \geq (q_m + 1) \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

or alternatively

$$\frac{q_m \times (\log\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1}))}{\log q_m} \geq$$

$$\frac{\log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})}{\log q_m}.$$

We can apply the inequality in lemma 6.1 just using $b = \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m)$ and $a = \log(2^{19} \times q_1 \times \cdots \times q_{m-1})$. Certainly, we have

$$\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log(2^{19} \times q_1 \times \cdots \times q_{m-1}) =$$

$$\log \frac{2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m}{2^{19} \times q_1 \times \cdots \times q_{m-1}} = \log q_m.$$

In this way, we obtain

$$\frac{q_m \times (\log\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1}))}{\log q_m} >$$

$$\frac{q_m}{\log(2^{19} \times q_1 \times \cdots \times q_m)}.$$

Using this result we infer that the original inequality is certainly satisfied if the next inequality is satisfied

$$\frac{q_m}{\log(2^{19} \times q_1 \times \cdots \times q_m)} \geq \frac{\log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})}{\log q_m}$$

which is trivially true for $q_m \geq \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m)$ [2].

Case 2: $q_m < \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = \log(2^{19} \times n)$.

We need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \frac{\sigma(n)}{n} \leq e^{\gamma} \times \log\log(2^{19} \times n).$$

We know $\frac{3}{2} < 1.503 < \frac{4}{2.66}$. Nevertheless, we could have

$$\frac{3}{2} \times \frac{\sigma(n)}{n} \times \frac{\pi^2}{6} < \frac{4 \times \sigma(n)}{3 \times n} \times \frac{\pi^2}{2 \times 2.66}$$

and therefore, we only need to prove

$$\frac{\sigma(3 \times n)}{3 \times n} \times \frac{\pi^2}{5.32} \leq e^{\gamma} \times \log\log(2^{19} \times n)$$

where this is possible because of $3 \nmid n$. If we apply the logarithm to the both sides of the inequality, then we obtain

$$\log(\frac{\pi^2}{5.32}) + (\log(3+1) - \log 3) + \sum_{i=1}^{m} (\log(q_i + 1) - \log q_i) \leq \gamma + \log\log\log(2^{19} \times n).$$

In addition, note $\log(\frac{\pi^2}{5.32}) < \frac{1}{2} + 0.12$. However, we know

$$\gamma + \log\log q_m < \gamma + \log\log\log(2^{19} \times n)$$

since $q_m < \log(2^{19} \times n)$. We use that lemma 6.2 for each term $\log(q+1) - \log q$ and thus,

$$0.12 + \frac{1}{2} + \frac{1}{3} + \frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq 0.12 + \sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \log\log q_m$$

where $q_m \geq 11$. Hence, it is enough to prove

$$\sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \log\log q_m - 0.12$$

but this is true according to the lemma 5.2 for $q_m \geq 11$. In this way, we finally show the lemma is indeed satisfied. □

7. Robin on Divisibility

$\mathsf{Robins}(n)$ holds for every $n > 5040$ that is not divisible by 2 [2]. We extend this property to other prime numbers:

Lemma 7.1. $\mathsf{Robins}(n)$ *holds for all* $n > 5040$ *when* $3 \nmid n$. *More precisely: every possible counterexample* $n > 5040$ *of the Robin inequality must comply with* $(2^{20} \times 3^{13}) \mid n$.

Proof. We will check the Robin inequality is true for every natural number $n = q_1^{a_1} \times q_2^{a_2} \times \cdots \times q_m^{a_m} > 5040$ such that $q_1, q_2, \cdots, q_m$ are distinct prime numbers, $a_1, a_2, \cdots, a_m$ are natural numbers and $3 \nmid n$. We know this is true when the greatest prime divisor of $n > 5040$ is lesser than or equal to 7 according to the lemma 4.1. Therefore, the remaining case is when the greatest prime divisor of $n > 5040$ is greater than or equal to 11. We need to prove

$$f(n) < e^{\gamma} \times \log\log n$$

that is true when

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} \leq e^{\gamma} \times \log\log n$$

according to the lemma 2.3. Using the formula (6) for the square free numbers, then we obtain that is equivalent to

$$\frac{\pi^2}{6} \times \frac{\sigma(n')}{n'} \leq e^{\gamma} \times \log\log n$$

where $n' = q_1 \times \cdots \times q_m$ is the square free kernel of a natural number n [2]. The Robin inequality has been proved for all integers n not divisible by 2 (which are bigger than 10) [2]. Hence, we only need to prove the Robin inequality is true when $2 \mid n'$. In addition, we know that $\mathsf{Robins}(n)$ holds for every $n > 5040$ when $\nu_2(n) \leq 19$ according to the lemma 3.1 [5]. Consequently, we only need to prove that $\mathsf{Robins}(n)$ holds for $n > 5040$ when $2^{20} \mid n$ and thus,

$$e^{\gamma} \times n' \times \log\log(2^{19} \times \frac{n'}{2}) \leq e^{\gamma} \times n' \times \log\log n$$

because of $2^{19} \times \frac{n'}{2} \leq n$ where $2^{20} \mid n$ and $2 \mid n'$. So,

$$\frac{\pi^2}{6} \times \sigma(n') \leq e^{\gamma} \times n' \times \log\log(2^{19} \times \frac{n'}{2}).$$

According to the formula (6) for the square free numbers and $2 \mid n'$, then,

$$\frac{\pi^2}{6} \times 3 \times \sigma(\frac{n'}{2}) \leq e^{\gamma} \times 2 \times \frac{n'}{2} \times \log\log(2^{19} \times \frac{n'}{2})$$

which is the same as

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(\frac{n'}{2}) \leq e^{\gamma} \times \frac{n'}{2} \times \log\log(2^{19} \times \frac{n'}{2})$$

where this is true according to the lemma 6.3 when $3 \nmid \frac{n'}{2}$. In addition, we know that $\mathsf{Robins}(n)$ holds for every $n > 5040$ when $\nu_3(n) \leq 12$ according to the lemma 3.1 [5]. Hence, we only need to prove that $\mathsf{Robins}(n)$ holds for every $n > 5040$ when $2^{20} \mid n$ and $3^{13} \mid n$. To sum up, the proof is complete. □

Lemma 7.2. $\mathsf{Robins}(n)$ *holds for all* $n > 5040$ *when* $5 \nmid n$ *or* $7 \nmid n$.

Proof. We need to prove

$$f(n) < e^{\gamma} \times \log\log n$$

when $(2^{20} \times 3^{13}) \mid n$. Suppose that $n = 2^a \times 3^b \times m$, where $a \geq 20$, $b \geq 13$, $2 \nmid m$, $3 \nmid m$ and $5 \nmid m$ or $7 \nmid m$. Therefore, we need to prove

$$f(2^a \times 3^b \times m) < e^{\gamma} \times \log\log(2^a \times 3^b \times m).$$

We know

$$f(2^a \times 3^b \times m) = f(3^b) \times f(2^a \times m)$$

since f is multiplicative [2]. In addition, we know $f(3^b) < \frac{3}{2}$ for every natural number b [2]. In this way, we have

$$f(3^b) \times f(2^a \times m) < \frac{3}{2} \times f(2^a \times m).$$

However, that would be equivalent to

$$\frac{3}{2} \times f(2^a \times m) = \frac{9}{8} \times f(3) \times f(2^a \times m) = \frac{9}{8} \times f(2^a \times 3 \times m)$$

where $f(3) = \frac{4}{3}$ since f is multiplicative [2]. Nevertheless, we have

$$\frac{9}{8} \times f(2^a \times 3 \times m) < f(5) \times f(2^a \times 3 \times m) = f(2^a \times 3 \times 5 \times m)$$

and

$$\frac{9}{8} \times f(2^a \times 3 \times m) < f(7) \times f(2^a \times 3 \times m) = f(2^a \times 3 \times 7 \times m)$$

where $5 \nmid m$ or $7 \nmid m$, $f(5) = \frac{6}{5}$ and $f(7) = \frac{8}{7}$. We know the Robin inequality is true for $2^a \times 3 \times 5 \times m$ and $2^a \times 3 \times 7 \times m$ when $a \geq 20$, since this is true for every natural number $n > 5040$ when $\nu_3(n) \leq 12$ according to the lemma 3.1 [5]. Hence, we would have

$$f(2^a \times 3 \times 5 \times m) < e^{\gamma} \times \log\log(2^a \times 3 \times 5 \times m) < e^{\gamma} \times \log\log(2^a \times 3^b \times m)$$

and

$$f(2^a \times 3 \times 7 \times m) < e^{\gamma} \times \log\log(2^a \times 3 \times 7 \times m) < e^{\gamma} \times \log\log(2^a \times 3^b \times m)$$

when $b \geq 13$. □

Lemma 7.3. $\mathsf{Robins}(n)$ *holds for all* $n > 5040$ *when a prime number* $11 \leq q \leq 47$ *complies with* $q \nmid n$.

Proof. We know that $\mathsf{Robins}(n)$ holds for every $n > 5040$ when $\nu_7(n) \leq 6$ according to the lemma 3.1 [5]. We need to prove

$$f(n) < e^{\gamma} \times \log\log n$$

when $(2^{20} \times 3^{13} \times 7^7) \mid n$. Suppose that $n = 2^a \times 3^b \times 7^c \times m$, where $a \geq 20$, $b \geq 13$, $c \geq 7$, $2 \nmid m$, $3 \nmid m$, $7 \nmid m$, $q \nmid m$ and $11 \leq q \leq 47$. Therefore, we need to prove

$$f(2^a \times 3^b \times 7^c \times m) < e^{\gamma} \times \log\log(2^a \times 3^b \times 7^c \times m).$$

We know

$$f(2^a \times 3^b \times 7^c \times m) = f(7^c) \times f(2^a \times 3^b \times m)$$

since f is multiplicative [2]. In addition, we know $f(7^c) < \frac{7}{6}$ for every natural number c [2]. In this way, we have

$$f(7^c) \times f(2^a \times 3^b \times m) < \frac{7}{6} \times f(2^a \times 3^b \times m).$$

However, that would be equivalent to

$$\frac{7}{6} \times f(2^a \times 3^b \times m) = \frac{49}{48} \times f(7) \times f(2^a \times 3^b \times m) = \frac{49}{48} \times f(2^a \times 3^b \times 7 \times m)$$

where $f(7) = \frac{8}{7}$ since f is multiplicative [2]. In addition, we know

$$\frac{49}{48} \times f(2^a \times 3^b \times 7 \times m) < f(q) \times f(2^a \times 3^b \times 7 \times m) = f(2^a \times 3^b \times 7 \times q \times m)$$

where $q \nmid m$, $f(q) = \frac{q+1}{q}$ and $11 \leq q \leq 47$. Nevertheless, we know the Robin inequality is true for $2^a \times 3^b \times 7 \times q \times m$ when $a \geq 20$ and $b \geq 13$, since this is true for every natural number $n > 5040$ when $\nu_7(n) \leq 6$ according to the lemma 3.1 [5]. Hence, we would have

$$\begin{aligned} f(2^a \times 3^b \times 7 \times q \times m) &< e^{\gamma} \times \log\log(2^a \times 3^b \times 7 \times q \times m) \\ &< e^{\gamma} \times \log\log(2^a \times 3^b \times 7^c \times m) \end{aligned}$$

when $c \geq 7$ and $11 \leq q \leq 47$. □

Lemma 7.4. $\mathsf{Robins}(n)$ *holds for all* $n > 5040$ *when a prime number* $53 \leq q \leq 953$ *complies with* $q \nmid n$.

Proof. We know that $\mathsf{Robins}(n)$ holds for every $n > 5040$ when $\nu_{31}(n) \leq 3$ according to the lemma 3.5. We need to prove that

$$f(n) < e^{\gamma} \times \log\log n$$

when $(2^{20} \times 3^{13} \times 31^4) \mid n$. Suppose that $n = 2^a \times 3^b \times 31^c \times m$, where $a \geq 20$, $b \geq 13$, $c \geq 4$, $2 \nmid m$, $3 \nmid m$, $31 \nmid m$, $q \nmid m$ and $53 \leq q \leq 953$. Therefore, we need to prove that

$$f(2^a \times 3^b \times 31^c \times m) < e^{\gamma} \times \log\log(2^a \times 3^b \times 31^c \times m).$$

We know that

$$f(2^a \times 3^b \times 31^c \times m) = f(31^c) \times f(2^a \times 3^b \times m)$$

since f is multiplicative [2]. In addition, we know that $f(31^c) < \frac{31}{30}$ for every natural number c [2]. In this way, we have that

$$f(31^c) \times f(2^a \times 3^b \times m) < \frac{31}{30} \times f(2^a \times 3^b \times m).$$

However, that would be equivalent to

$$\frac{31}{30} \times f(2^a \times 3^b \times m) = \frac{961}{960} \times f(31) \times f(2^a \times 3^b \times m) = \frac{961}{960} \times f(2^a \times 3^b \times 31 \times m)$$

where $f(31) = \frac{32}{31}$ since f is multiplicative [2]. In addition, we know that

$$\frac{961}{960} \times f(2^a \times 3^b \times 31 \times m) < f(q) \times f(2^a \times 3^b \times 31 \times m) = f(2^a \times 3^b \times 31 \times q \times m)$$

where $q \nmid m$, $f(q) = \frac{q+1}{q}$ and $53 \leq q \leq 953$. Nevertheless, we know the Robin inequality is true for $2^a \times 3^b \times 31 \times q \times m$ when $a \geq 20$ and $b \geq 13$, since this is true for every natural number $n > 5040$ when $\nu_{31}(n) \leq 3$ according to the lemma 3.5. Hence, we would have that

$$\begin{aligned} f(2^a \times 3^b \times 31 \times q \times m) &< e^\gamma \times \log\log(2^a \times 3^b \times 31 \times q \times m) \\ &< e^\gamma \times \log\log(2^a \times 3^b \times 31^c \times m) \end{aligned}$$

when $c \geq 4$ and $53 \leq q \leq 953$. □

8. Proof of Main Theorem

Theorem 8.1. Robins(n) *holds for all* $n > 5040$ *when a prime number* $q \leq 953$ *complies with* $q \nmid n$.

Proof. This is a compendium of the results from the lemmas 7.1, 7.2, 7.3 and 7.4. □

References

[1] P. B. Borwein, S. Choi, B. Rooney, A. Weirathmueller, The Riemann Hypothesis: A Resource for the Afficionado and Virtuoso Alike, Vol. 27, Springer Science & Business Media, 2008.

[2] Y. Choie, N. Lichiardopol, P. Moree, P. Solé, On Robin's criterion for the Riemann hypothesis, Journal de Théorie des Nombres de Bordeaux 19 (2) (2007) 357–372. doi:doi:10.5802/jtnb.591.

[3] G. Robin, Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann, J. Math. pures appl 63 (2) (1984) 187–213.

[4] H. M. Edwards, Riemann's Zeta Function, Dover Publications, 2001.

[5] A. Hertlein, Robin's Inequality for New Families of Integers, Integers 18.

[6] J. B. Rosser, L. Schoenfeld, Approximate Formulas for Some Functions of Prime Numbers, Illinois Journal of Mathematics 6 (1) (1962) 64–94. doi:doi:10.1215/ijm/1255631807.