

The Riemann Hypothesis

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Abstract

Let's define $\delta(x) = (\sum_{q \leq x} \frac{1}{q} - \log \log x - B)$, where $B \approx 0.2614972128$ is the Meissel-Mertens constant. The Robin theorem states that $\delta(x)$ changes sign infinitely often. For $x \geq 2$, Nicolas defined the function $u(x) = \sum_{q > x} \left(\log\left(\frac{q}{q-1}\right) - \frac{1}{q} \right)$ and proved that $0 < u(x) \leq \frac{1}{2 \times (x-1)}$. We define the another function $\varpi(x) = \left(\sum_{q \leq x} \frac{1}{q} - \log \log \theta(x) - B \right)$, where $\theta(x)$ is the Chebyshev function. Using the Nicolas theorem, we demonstrate that the Riemann Hypothesis is true if and only if the inequality $\varpi(x) > u(x)$ is satisfied for all number $x \geq 3$. Consequently, we show that when the inequality $\varpi(x) \leq 0$ is satisfied for some number $x \geq 3$, then the Riemann Hypothesis should be false. In addition, we know that $\lim_{x \rightarrow \infty} \varpi(x) = 0$.

Keywords: Riemann hypothesis, Nicolas theorem, Chebyshev function, prime numbers

2000 MSC: 11M26, 11A41, 11A25

1. Introduction

In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$ [1]. Let $N_n = 2 \times 3 \times 5 \times 7 \times 11 \times \cdots \times p_n$ denotes a primorial number of order n such that p_n is the n^{th} prime number. Say $\text{Nicolas}(p_n)$ holds provided

$$\prod_{q|N_n} \frac{q}{q-1} > e^\gamma \times \log \log N_n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, \log is the natural logarithm, and $q | N_n$ means the prime number q divides to N_n . The importance of this property is:

Theorem 1.1. [2], [3]. $\text{Nicolas}(p_n)$ holds for all prime number $p_n > 2$ if and only if the Riemann Hypothesis is true.

In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

where $p \leq x$ means all the prime numbers p that are less than or equal to x . We know this:

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Theorem 1.2. [4].

$$\lim_{x \rightarrow \infty} \frac{\theta(x)}{x} = 1.$$

Let's define $H = \gamma - B$ such that $B \approx 0.2614972128$ is the Meissel-Mertens constant [5]. We know from the constant H , the following formula:

Theorem 1.3. [6].

$$\sum_q \left(\log\left(\frac{q}{q-1}\right) - \frac{1}{q} \right) = \gamma - B = H.$$

For $x \geq 2$, Nicolas defined the function $u(x)$ as follows

$$u(x) = \sum_{q > x} \left(\log\left(\frac{q}{q-1}\right) - \frac{1}{q} \right).$$

Nicolas showed that

Theorem 1.4. [3]. For $x \geq 2$:

$$0 < u(x) \leq \frac{1}{2 \times (x-1)}.$$

Let's define:

$$\delta(x) = \left(\sum_{q \leq x} \frac{1}{q} - \log \log x - B \right).$$

Robin theorem states the following result:

Theorem 1.5. [7]. $\delta(x)$ changes sign infinitely often.

In addition, the Mertens second theorem states that:

Theorem 1.6. [5].

$$\lim_{x \rightarrow \infty} \delta(x) = 0.$$

We define another function:

$$\varpi(x) = \left(\sum_{q \leq x} \frac{1}{q} - \log \log \theta(x) - B \right).$$

Putting all together yields the proof that the inequality $\varpi(x) > u(x)$ is satisfied for a number $x \geq 3$ if and only if Nicolas(q) holds, where q is the greatest prime number such that $q \leq x$. In this way, we introduce another criterion for the Riemann Hypothesis based on the Nicolas criterion.

2. Results

Theorem 2.1. The inequality $\varpi(x) > u(x)$ is satisfied for a number $x \geq 3$ if and only if Nicolas(q) holds, where q is the greatest prime number such that $q \leq x$.

Proof. We start from the inequality:

$$\varpi(x) > u(x)$$

which is equivalent to

$$\left(\sum_{q \leq x} \frac{1}{q} - \log \log \theta(x) - B \right) > \sum_{q > x} \left(\log \left(\frac{q}{q-1} \right) - \frac{1}{q} \right).$$

Let's add the following formula to the both sides of the inequality,

$$\sum_{q \leq x} \left(\log \left(\frac{q}{q-1} \right) - \frac{1}{q} \right)$$

and due to the theorem 1.3, we obtain that

$$\sum_{q \leq x} \log \left(\frac{q}{q-1} \right) - \log \log \theta(x) - B > H$$

because of

$$H = \sum_{q \leq x} \left(\log \left(\frac{q}{q-1} \right) - \frac{1}{q} \right) + \sum_{q > x} \left(\log \left(\frac{q}{q-1} \right) - \frac{1}{q} \right)$$

and

$$\sum_{q \leq x} \log \left(\frac{q}{q-1} \right) = \sum_{q \leq x} \frac{1}{q} + \sum_{q \leq x} \left(\log \left(\frac{q}{q-1} \right) - \frac{1}{q} \right).$$

Let's distribute it and remove B from the both sides:

$$\sum_{q \leq x} \log \left(\frac{q}{q-1} \right) > \gamma + \log \log \theta(x)$$

since $H = \gamma - B$. If we apply the exponentiation to the both sides of the inequality, then we have that

$$\prod_{q \leq x} \frac{q}{q-1} > e^\gamma \times \log \theta(x)$$

which means that $\text{Nicolas}(q)$ holds, where q is the greatest prime number such that $q \leq x$. The same happens in the reverse implication. \square

Theorem 2.2. *The Riemann Hypothesis is true if and only if the inequality $\varpi(x) > u(x)$ is satisfied for all number $x \geq 3$.*

Proof. This is a direct consequence of theorems 1.1 and 2.1. \square

Lemma 2.3. *If the inequality $\varpi(x) \leq 0$ is satisfied for some number $x \geq 3$, then the Riemann Hypothesis should be false.*

Proof. This is an implication of theorems 1.4, 2.1 and 2.2. \square

Lemma 2.4.

$$\lim_{x \rightarrow \infty} \varpi(x) = 0.$$

Proof. We know that $\lim_{x \rightarrow \infty} \varpi(x) = 0$ for the limits $\lim_{x \rightarrow \infty} \delta(x) = 0$ and $\lim_{x \rightarrow \infty} \frac{\theta(x)}{x} = 1$. In this way, this is a consequence from the theorems 1.6 and 1.2. \square

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