Robin Criterion on Divisibility

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Abstract

Robin criterion states that the Riemann Hypothesis is true if and only if the inequality $\sigma(n) < e^\gamma \times n \times \log \log n$ holds for all $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. This is known as the Robin inequality. We show that the Robin inequality is true for all $n > 5040$ which are not divisible by any prime number between 2 and 953. In addition, from the possible smallest counterexample $n > 5040$ of the Robin inequality, we prove that necessarily $\log(n)^\beta < 1.26 \times \log(N_m)$, where $N_m = \prod_{k=1}^m q_k$ is the primorial number of order m and $\beta = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$ when n is an Hardy-Ramanujan integer of the form $\prod_{i=1}^m q_i^{a_i}$.

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1. Introduction

In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$ [1]. As usual $\sigma(n)$ is the sum-of-divisors function of n [2]:

$$\sum_{d|n} d$$

where $d | n$ means the integer d divides to n and $d \nmid n$ means the integer d does not divide to n . Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say Robins(n) holds provided

$$f(n) < e^\gamma \times \log \log n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, and $\log$ is the natural logarithm. The importance of this property is:

Theorem 1.1. Robins(n) holds for all $n > 5040$ if and only if the Riemann Hypothesis is true [1].

It is known that Robins(n) holds for many classes of numbers n .

Theorem 1.2. Robins(n) holds for all $n > 5040$ that are not divisible by 2 [2].

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On the one hand, we prove that $\text{Robins}(n)$ holds for all $n > 5040$ that are not divisible by any prime number between 3 and 953. Let $q_1 = 2, q_2 = 3, \dots, q_m$ denote the first m consecutive primes, then an integer of the form $\prod_{i=1}^m q_i^{a_i}$ with $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$ is called an Hardy-Ramanujan integer [2]. A natural number n is called superabundant precisely when, for all $m < n$

$$f(m) < f(n).$$

Theorem 1.3. *If n is superabundant, then n is an Hardy-Ramanujan integer [3].*

Theorem 1.4. *The smallest counterexample of the Robin inequality greater than 5040 must be a superabundant number [4].*

On the other hand, suppose that $n > 5040$ might be the possible smallest counterexample of the Robin inequality, then we demonstrate that necessarily $\log(n)^\beta < 1.26 \times \log(N_m)$, where $N_m = \prod_{k=1}^m q_k$ is the primorial number of order m and $\beta = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$ when n is an Hardy-Ramanujan integer of the form $\prod_{i=1}^m q_i^{a_i}$.

2. A Central Lemma

These are known results:

Lemma 2.1. [2]. For $n > 1$:

$$f(n) < \prod_{q|n} \frac{q}{q-1}. \quad (1)$$

Lemma 2.2. [5].

$$\prod_{k=1}^{\infty} \frac{1}{1 - \frac{1}{q_k^2}} = \zeta(2) = \frac{\pi^2}{6}. \quad (2)$$

The following is a key lemma. It gives an upper bound on $f(n)$ that holds for all n . The bound is too weak to prove $\text{Robins}(n)$ directly, but is critical because it holds for all n . Further the bound only uses the primes that divide n and not how many times they divide n .

Lemma 2.3. *Let $n > 1$ and let all its prime divisors be $q_1 < \dots < q_m$. Then,*

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^m \frac{q_i + 1}{q_i}.$$

Proof. We use that lemma 2.1:

$$f(n) < \prod_{i=1}^m \frac{q_i}{q_i - 1}.$$

Now for $q > 1$,

$$\frac{1}{1 - \frac{1}{q^2}} = \frac{q^2}{q^2 - 1}.$$

So

$$\begin{aligned}\frac{1}{1 - \frac{1}{q^2}} \times \frac{q+1}{q} &= \frac{q^2}{q^2 - 1} \times \frac{q+1}{q} \\ &= \frac{q}{q-1}.\end{aligned}$$

Then by lemma 2.2,

$$\prod_{i=1}^m \frac{1}{1 - \frac{1}{q_i^2}} < \zeta(2) = \frac{\pi^2}{6}.$$

Putting this together yields the proof:

$$\begin{aligned}f(n) &< \prod_{i=1}^m \frac{q_i}{q_i - 1} \\ &\leq \prod_{i=1}^m \frac{1}{1 - \frac{1}{q_i^2}} \times \frac{q_i + 1}{q_i} \\ &< \frac{\pi^2}{6} \times \prod_{i=1}^m \frac{q_i + 1}{q_i}.\end{aligned}$$

□

3. About the p -adic order

In basic number theory, for a given prime number p , the p -adic order of a natural number n is the highest exponent $v_p \geq 1$ such that p^{v_p} divides n . This is a known result:

Lemma 3.1. *In general, we know that Robins(n) holds for a natural number $n > 5040$ that satisfies either $v_2(n) \leq 19$, $v_3(n) \leq 12$ or $v_7(n) \leq 6$, where $v_p(n)$ is the p -adic order of n [6].*

We know the following lemmas:

Lemma 3.2. [6]. *Let $\prod_{i=1}^m q_i^{a_i}$ be the representation of n as a product of primes $q_1 < \dots < q_m$ with natural numbers as exponents $a_1, \dots, a_m$. Then,*

$$f(n) = \left(\prod_{i=1}^m \frac{q_i}{q_i - 1} \right) \times \prod_{i=1}^m \left(1 - \frac{1}{q_i^{a_i+1}} \right).$$

Lemma 3.3. [6]. *Let $n > e^{23.762143}$ and let all its prime divisors be $q_1 < \dots < q_m$, then*

$$\left(\prod_{i=1}^m \frac{q_i}{q_i - 1} \right) < \frac{1771561}{1771560} \times e^\gamma \times \log \log n.$$

Lemma 3.4. *Robins(n) holds for all $10^{10^{10}} \geq n > 5040$ [6].*

Putting together all these results, then we obtain that

Lemma 3.5. *Robins(n) holds for $n > 5040$ when $v_{31}(n) \leq 3$.*

Proof. From lemma 3.2, we note that

$$f(n) = \left(\prod_{i=1}^m \frac{q_i}{q_i - 1} \right) \times \prod_{i=1}^m \left(1 - \frac{1}{q_i^{a_i+1}} \right) \leq \left(\prod_{i=1}^m \frac{q_i}{q_i - 1} \right) \times \left(1 - \frac{1}{31^{v_{31}(n)+1}} \right)$$

when $v_{31}(n) \leq 3$. We only need to look at the case where $v_{31}(n) = 3$ since the weaker cases follow because

$$\left(1 - \frac{1}{31^{1+1}} \right) < \left(1 - \frac{1}{31^{2+1}} \right) < \left(1 - \frac{1}{31^{3+1}} \right).$$

In this way, we obtain that

$$f(n) \leq \left(\prod_{i=1}^m \frac{q_i}{q_i - 1} \right) \times \left(1 - \frac{1}{31^{3+1}} \right) = \frac{923520}{923521} \times \left(\prod_{i=1}^m \frac{q_i}{q_i - 1} \right)$$

when $v_{31}(n) \leq 3$. With lemma 3.3, we have for $n > e^{e^{23.762143}}$

$$\frac{923520}{923521} \times \left(\prod_{i=1}^m \frac{q_i}{q_i - 1} \right) < \frac{923520}{923521} \times \frac{1771561}{1771560} \times e^\gamma \times \log \log n < e^\gamma \times \log \log n$$

since $\frac{923520}{923521} \times \frac{1771561}{1771560} < 1$. In light of lemma 3.4 and the fact that $e^{e^{23.762143}} < 10^{10^{10}}$, we then conclude that Robins(n) holds for $n > 5040$ when $v_{31}(n) \leq 3$. $\square$

4. A Particular Case

We can easily prove that Robins(n) is true for certain kind of numbers:

Lemma 4.1. *Robins(n) holds for $n > 5040$ when $q \leq 7$, where q is the largest prime divisor of n .*

Proof. Let $n > 5040$ and let all its prime divisors be $q_1 < \dots < q_m \leq 5$, then we need to prove

$$f(n) < e^\gamma \times \log \log n$$

that is true when

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} \leq e^\gamma \times \log \log n$$

according to the lemma 2.1. For $q_1 < \dots < q_m \leq 5$,

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5}{1 \times 2 \times 4} = 3.75 < e^\gamma \times \log \log(5040) \approx 3.81.$$

However, we know for $n > 5040$

$$e^\gamma \times \log \log(5040) < e^\gamma \times \log \log n$$

and therefore, the proof is complete when $q_1 < \dots < q_m \leq 5$. The remaining case is for $n > 5040$ when all its prime divisors are $q_1 < \dots < q_m \leq 7$. Robins(n) holds for $n > 5040$ when $v_7(n) \leq 6$ according to the lemma 3.1 [6]. Hence, it is enough to prove this for those natural numbers $n > 5040$ when $7^7 \mid n$. For $q_1 < \dots < q_m \leq 7$,

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5 \times 7}{1 \times 2 \times 4 \times 6} = 4.375 < e^\gamma \times \log \log(7^7) \approx 4.65.$$

However, for $n > 5040$ and $7^7 \mid n$, we know that

$$e^\gamma \times \log \log(7^7) \leq e^\gamma \times \log \log n$$

and as a consequence, the proof is complete when $q_1 < \dots < q_m \leq 7$. □

5. A Better Bound

This is a known result:

Lemma 5.1. [7]. For $x > 1$:

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + B + \frac{1}{\log^2 x} \quad (3)$$

where

$$B = 0.2614972128 \dots$$

denotes the (Meissel-)Mertens constant [7].

We show a better result:

Lemma 5.2. For $x \geq 11$, we have

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + \gamma - 0.12.$$

Proof. Let's define $H = \gamma - B$. The lemma 5.1 is the same as

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + \gamma - (H - \frac{1}{\log^2 x}).$$

For $x \geq 11$,

$$(H - \frac{1}{\log^2 x}) > (0.31 - \frac{1}{\log^2 11}) > 0.12$$

and thus,

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + \gamma - (H - \frac{1}{\log^2 x}) < \log \log x + \gamma - 0.12.$$

□

6. On a Square Free Number

We know the following results:

Lemma 6.1. [2]. For $0 < a < b$:

$$\frac{\log b - \log a}{b - a} = \frac{1}{(b - a)} \int_a^b \frac{dt}{t} > \frac{1}{b}. \quad (4)$$

Lemma 6.2. [2]. For $q > 0$:

$$\log(q + 1) - \log q = \int_q^{q+1} \frac{dt}{t} < \frac{1}{q}. \quad (5)$$

We recall that an integer n is said to be square free if for every prime divisor q of n we have $q^2 \nmid n$ [2]. Robins(n) holds for all $n > 5040$ that are square free [2].

Lemma 6.3. For a square free number

$$n = q_1 \times \cdots \times q_m$$

such that $q_1 < q_2 < \cdots < q_m$ are odd prime numbers, $q_m \geq 11$ and $3 \nmid n$, then:

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(n) \leq e^\gamma \times n \times \log \log(2^{19} \times n).$$

Proof. By induction with respect to $\omega(n)$, that is the number of distinct prime factors of n [2]. Put $\omega(n) = m$ [2]. We need to prove the assertion for those integers with $m = 1$. From a square free number n , we obtain

$$\sigma(n) = (q_1 + 1) \times (q_2 + 1) \times \cdots \times (q_m + 1) \quad (6)$$

when $n = q_1 \times q_2 \times \cdots \times q_m$ [2]. In this way, for every prime number $q_i \geq 11$, then we need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \left(1 + \frac{1}{q_i}\right) \leq e^\gamma \times \log \log(2^{19} \times q_i). \quad (7)$$

For $q_i = 11$, we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \left(1 + \frac{1}{11}\right) \leq e^\gamma \times \log \log(2^{19} \times 11)$$

is actually true. For another prime number $q_i > 11$, we have

$$\left(1 + \frac{1}{q_i}\right) < \left(1 + \frac{1}{11}\right)$$

and

$$\log \log(2^{19} \times 11) < \log \log(2^{19} \times q_i)$$

which clearly implies that the inequality (7) is true for every prime number $q_i \geq 11$. Now, suppose it is true for $m - 1$, with $m \geq 2$ and let us consider the assertion for those square free n with $\omega(n) = m$ [2]. So let $n = q_1 \times \cdots \times q_m$ be a square free number and assume that $q_1 < \cdots < q_m$ for $q_m \geq 11$.

Case 1: $q_m \geq \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = \log(2^{19} \times n)$.

By the induction hypothesis we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1 + 1) \times \cdots \times (q_{m-1} + 1) \leq e^\gamma \times q_1 \times \cdots \times q_{m-1} \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

and hence

$$\begin{aligned} & \frac{\pi^2}{6} \times \frac{3}{2} \times (q_1 + 1) \times \cdots \times (q_{m-1} + 1) \times (q_m + 1) \leq \\ & e^\gamma \times q_1 \times \cdots \times q_{m-1} \times (q_m + 1) \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1}) \end{aligned}$$

when we multiply the both sides of the inequality by $(q_m + 1)$. We want to show

$$e^\gamma \times q_1 \times \cdots \times q_{m-1} \times (q_m + 1) \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1}) \leq$$

$$e^\gamma \times q_1 \times \cdots \times q_{m-1} \times q_m \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = e^\gamma \times n \times \log \log(2^{19} \times n).$$

Indeed the previous inequality is equivalent with

$$q_m \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) \geq (q_m + 1) \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

or alternatively

$$\begin{aligned} & \frac{q_m \times (\log \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1}))}{\log q_m} \geq \\ & \frac{\log \log(2^{19} \times q_1 \times \cdots \times q_{m-1})}{\log q_m}. \end{aligned}$$

We can apply the inequality in lemma 6.1 just using $b = \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m)$ and $a = \log(2^{19} \times q_1 \times \cdots \times q_{m-1})$. Certainly, we have

$$\begin{aligned} & \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log(2^{19} \times q_1 \times \cdots \times q_{m-1}) = \\ & \log \frac{2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m}{2^{19} \times q_1 \times \cdots \times q_{m-1}} = \log q_m. \end{aligned}$$

In this way, we obtain

$$\begin{aligned} & \frac{q_m \times (\log \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1}))}{\log q_m} > \\ & \frac{q_m}{\log(2^{19} \times q_1 \times \cdots \times q_m)}. \end{aligned}$$

Using this result we infer that the original inequality is certainly satisfied if the next inequality is satisfied

$$\frac{q_m}{\log(2^{19} \times q_1 \times \cdots \times q_m)} \geq \frac{\log \log(2^{19} \times q_1 \times \cdots \times q_{m-1})}{\log q_m}$$

which is trivially true for $q_m \geq \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m)$ [2].

Case 2: $q_m < \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = \log(2^{19} \times n)$.

We need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \frac{\sigma(n)}{n} \leq e^\gamma \times \log \log(2^{19} \times n).$$

We know $\frac{3}{2} < 1.503 < \frac{4}{2.66}$. Nevertheless, we could have

$$\frac{3}{2} \times \frac{\sigma(n)}{n} \times \frac{\pi^2}{6} < \frac{4 \times \sigma(n)}{3 \times n} \times \frac{\pi^2}{2 \times 2.66}$$

and therefore, we only need to prove

$$\frac{\sigma(3 \times n)}{3 \times n} \times \frac{\pi^2}{5.32} \leq e^\gamma \times \log \log(2^{19} \times n)$$

where this is possible because of $3 \nmid n$. If we apply the logarithm to the both sides of the inequality, then we obtain

$$\log\left(\frac{\pi^2}{5.32}\right) + (\log(3+1) - \log 3) + \sum_{i=1}^m (\log(q_i + 1) - \log q_i) \leq \gamma + \log \log \log(2^{19} \times n).$$

In addition, note that $\log\left(\frac{\pi^2}{5.32}\right) < \frac{1}{2} + 0.12$. However, we know

$$\gamma + \log \log q_m < \gamma + \log \log \log(2^{19} \times n)$$

since $q_m < \log(2^{19} \times n)$. We use that lemma 6.2 for each term $\log(q+1) - \log q$ and thus,

$$0.12 + \frac{1}{2} + \frac{1}{3} + \frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq 0.12 + \sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \log \log q_m$$

where $q_m \geq 11$. Hence, it is enough to prove

$$\sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \log \log q_m - 0.12$$

but this is true according to the lemma 5.2 for $q_m \geq 11$. In this way, we finally show the lemma is indeed satisfied. $\square$

7. Robin on Divisibility

$\text{Robins}(n)$ holds for every $n > 5040$ that is not divisible by 2 [2]. We extend this property to other prime numbers:

Lemma 7.1. *Robins(n) holds for all $n > 5040$ when $3 \nmid n$. More precisely: every possible counterexample $n > 5040$ of the Robin inequality must comply with $(2^{20} \times 3^{13}) \mid n$.*

Proof. We will check the Robin inequality is true for every natural number $n = q_1^{a_1} \times q_2^{a_2} \times \cdots \times q_m^{a_m} > 5040$ such that $q_1, q_2, \dots, q_m$ are distinct prime numbers, $a_1, a_2, \dots, a_m$ are natural numbers and $3 \nmid n$. We know this is true when the greatest prime divisor of $n > 5040$ is lesser

than or equal to 7 according to the lemma 4.1. Therefore, the remaining case is when the greatest prime divisor of $n > 5040$ is greater than or equal to 11. We need to prove

$$f(n) < e^\gamma \times \log \log n$$

that is true when

$$\frac{\pi^2}{6} \times \prod_{i=1}^m \frac{q_i + 1}{q_i} \leq e^\gamma \times \log \log n$$

according to the lemma 2.3. Using the formula (6) for the square free numbers, then we obtain that is equivalent to

$$\frac{\pi^2}{6} \times \frac{\sigma(n')}{n'} \leq e^\gamma \times \log \log n$$

where $n' = q_1 \times \dots \times q_m$ is the square free kernel of a natural number n [2]. The Robin inequality has been proved for all integers n not divisible by 2 (which are bigger than 10) [2]. Hence, we only need to prove the Robin inequality is true when $2 \mid n'$. In addition, we know that $\text{Robins}(n)$ holds for every $n > 5040$ when $v_2(n) \leq 19$ according to the lemma 3.1 [6]. Consequently, we only need to prove that $\text{Robins}(n)$ holds for $n > 5040$ when $2^{20} \mid n$ and thus,

$$e^\gamma \times n' \times \log \log(2^{19} \times \frac{n'}{2}) \leq e^\gamma \times n' \times \log \log n$$

because of $2^{19} \times \frac{n'}{2} \leq n$ where $2^{20} \mid n$ and $2 \mid n'$. So,

$$\frac{\pi^2}{6} \times \sigma(n') \leq e^\gamma \times n' \times \log \log(2^{19} \times \frac{n'}{2}).$$

According to the formula (6) for the square free numbers and $2 \mid n'$, then,

$$\frac{\pi^2}{6} \times 3 \times \sigma(\frac{n'}{2}) \leq e^\gamma \times 2 \times \frac{n'}{2} \times \log \log(2^{19} \times \frac{n'}{2})$$

which is the same as

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(\frac{n'}{2}) \leq e^\gamma \times \frac{n'}{2} \times \log \log(2^{19} \times \frac{n'}{2})$$

where this is true according to the lemma 6.3 when $3 \nmid \frac{n'}{2}$. In addition, we know that $\text{Robins}(n)$ holds for every $n > 5040$ when $v_3(n) \leq 12$ according to the lemma 3.1 [6]. Hence, we only need to prove that $\text{Robins}(n)$ holds for every $n > 5040$ when $2^{20} \mid n$ and $3^{13} \mid n$. To sum up, the proof is complete. $\square$

Lemma 7.2. *Robins(n) holds for all $n > 5040$ when $5 \nmid n$ or $7 \nmid n$.*

Proof. We need to prove

$$f(n) < e^\gamma \times \log \log n$$

when $(2^{20} \times 3^{13}) \mid n$. Suppose that $n = 2^a \times 3^b \times m$, where $a \geq 20$, $b \geq 13$, $2 \nmid m$, $3 \nmid m$ and $5 \nmid m$ or $7 \nmid m$. Therefore, we need to prove

$$f(2^a \times 3^b \times m) < e^\gamma \times \log \log(2^a \times 3^b \times m).$$

We know

$$f(2^a \times 3^b \times m) = f(3^b) \times f(2^a \times m)$$

since f is multiplicative [2]. In addition, we know $f(3^b) < \frac{3}{2}$ for every natural number b [2]. In this way, we have

$$f(3^b) \times f(2^a \times m) < \frac{3}{2} \times f(2^a \times m).$$

However, that would be equivalent to

$$\frac{3}{2} \times f(2^a \times m) = \frac{9}{8} \times f(3) \times f(2^a \times m) = \frac{9}{8} \times f(2^a \times 3 \times m)$$

where $f(3) = \frac{4}{3}$ since f is multiplicative [2]. Nevertheless, we have

$$\frac{9}{8} \times f(2^a \times 3 \times m) < f(5) \times f(2^a \times 3 \times m) = f(2^a \times 3 \times 5 \times m)$$

and

$$\frac{9}{8} \times f(2^a \times 3 \times m) < f(7) \times f(2^a \times 3 \times m) = f(2^a \times 3 \times 7 \times m)$$

where $5 \nmid m$ or $7 \nmid m$, $f(5) = \frac{6}{5}$ and $f(7) = \frac{8}{7}$. We know the Robin inequality is true for $2^a \times 3 \times 5 \times m$ and $2^a \times 3 \times 7 \times m$ when $a \geq 20$, since this is true for every natural number $n > 5040$ when $v_3(n) \leq 12$ according to the lemma 3.1 [6]. Hence, we would have

$$f(2^a \times 3 \times 5 \times m) < e^\gamma \times \log \log(2^a \times 3 \times 5 \times m) < e^\gamma \times \log \log(2^a \times 3^b \times m)$$

and

$$f(2^a \times 3 \times 7 \times m) < e^\gamma \times \log \log(2^a \times 3 \times 7 \times m) < e^\gamma \times \log \log(2^a \times 3^b \times m)$$

when $b \geq 13$. □

Lemma 7.3. *Robins(n) holds for all $n > 5040$ when a prime number $11 \leq q \leq 47$ complies with $q \nmid n$.*

Proof. We know that Robins(n) holds for every $n > 5040$ when $v_7(n) \leq 6$ according to the lemma 3.1 [6]. We need to prove

$$f(n) < e^\gamma \times \log \log n$$

when $(2^{20} \times 3^{13} \times 7^7) \mid n$. Suppose that $n = 2^a \times 3^b \times 7^c \times m$, where $a \geq 20$, $b \geq 13$, $c \geq 7$, $2 \nmid m$, $3 \nmid m$, $7 \nmid m$, $q \nmid m$ and $11 \leq q \leq 47$. Therefore, we need to prove

$$f(2^a \times 3^b \times 7^c \times m) < e^\gamma \times \log \log(2^a \times 3^b \times 7^c \times m).$$

We know

$$f(2^a \times 3^b \times 7^c \times m) = f(7^c) \times f(2^a \times 3^b \times m)$$

since f is multiplicative [2]. In addition, we know $f(7^c) < \frac{7}{6}$ for every natural number c [2]. In this way, we have

$$f(7^c) \times f(2^a \times 3^b \times m) < \frac{7}{6} \times f(2^a \times 3^b \times m).$$

However, that would be equivalent to

$$\frac{7}{6} \times f(2^a \times 3^b \times m) = \frac{49}{48} \times f(7) \times f(2^a \times 3^b \times m) = \frac{49}{48} \times f(2^a \times 3^b \times 7 \times m)$$

where $f(7) = \frac{8}{7}$ since f is multiplicative [2]. In addition, we know

$$\frac{49}{48} \times f(2^a \times 3^b \times 7 \times m) < f(q) \times f(2^a \times 3^b \times 7 \times m) = f(2^a \times 3^b \times 7 \times q \times m)$$

where $q \nmid m$, $f(q) = \frac{q+1}{q}$ and $11 \leq q \leq 47$. Nevertheless, we know the Robin inequality is true for $2^a \times 3^b \times 7 \times q \times m$ when $a \geq 20$ and $b \geq 13$, since this is true for every natural number $n > 5040$ when $v_7(n) \leq 6$ according to the lemma 3.1 [6]. Hence, we would have

$$\begin{aligned} f(2^a \times 3^b \times 7 \times q \times m) &< e^\gamma \times \log \log(2^a \times 3^b \times 7 \times q \times m) \\ &< e^\gamma \times \log \log(2^a \times 3^b \times 7^c \times m) \end{aligned}$$

when $c \geq 7$ and $11 \leq q \leq 47$. □

Lemma 7.4. *Robins(n) holds for all $n > 5040$ when a prime number $53 \leq q \leq 953$ complies with $q \nmid n$.*

Proof. We know that Robins(n) holds for every $n > 5040$ when $v_{31}(n) \leq 3$ according to the lemma 3.5. We need to prove that

$$f(n) < e^\gamma \times \log \log n$$

when $(2^{20} \times 3^{13} \times 31^4) \mid n$. Suppose that $n = 2^a \times 3^b \times 31^c \times m$, where $a \geq 20$, $b \geq 13$, $c \geq 4$, $2 \nmid m$, $3 \nmid m$, $31 \nmid m$, $q \nmid m$ and $53 \leq q \leq 953$. Therefore, we need to prove that

$$f(2^a \times 3^b \times 31^c \times m) < e^\gamma \times \log \log(2^a \times 3^b \times 31^c \times m).$$

We know that

$$f(2^a \times 3^b \times 31^c \times m) = f(31^c) \times f(2^a \times 3^b \times m)$$

since f is multiplicative [2]. In addition, we know that $f(31^c) < \frac{31}{30}$ for every natural number c [2]. In this way, we have that

$$f(31^c) \times f(2^a \times 3^b \times m) < \frac{31}{30} \times f(2^a \times 3^b \times m).$$

However, that would be equivalent to

$$\frac{31}{30} \times f(2^a \times 3^b \times m) = \frac{961}{960} \times f(31) \times f(2^a \times 3^b \times m) = \frac{961}{960} \times f(2^a \times 3^b \times 31 \times m)$$

where $f(31) = \frac{32}{31}$ since f is multiplicative [2]. In addition, we know that

$$\frac{961}{960} \times f(2^a \times 3^b \times 31 \times m) < f(q) \times f(2^a \times 3^b \times 31 \times m) = f(2^a \times 3^b \times 31 \times q \times m)$$

where $q \nmid m$, $f(q) = \frac{q+1}{q}$ and $53 \leq q \leq 953$. Nevertheless, we know the Robin inequality is true for $2^a \times 3^b \times 31 \times q \times m$ when $a \geq 20$ and $b \geq 13$, since this is true for every natural number $n > 5040$ when $v_{31}(n) \leq 3$ according to the lemma 3.5. Hence, we would have that

$$\begin{aligned} f(2^a \times 3^b \times 31 \times q \times m) &< e^\gamma \times \log \log(2^a \times 3^b \times 31 \times q \times m) \\ &< e^\gamma \times \log \log(2^a \times 3^b \times 31^c \times m) \end{aligned}$$

when $c \geq 4$ and $53 \leq q \leq 953$. □

8. A Helpful Lemma

In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

where $p \leq x$ means all the prime numbers p that are less than or equal to x .

Lemma 8.1. [7]. For $x \geq 41$:

$$\theta(x) > \left(1 - \frac{1}{\log(x)}\right) \times x.$$

Besides, we know that

Lemma 8.2. [7]. For $x \geq 286$:

$$\prod_{q \leq x} \frac{q}{q-1} < e^\gamma \times \left(\log x + \frac{1}{2 \times \log(x)}\right).$$

We will prove another important inequality:

Lemma 8.3. Let $q_1 = 2, q_2 = 3, \dots, q_m$ denote the first m consecutive primes such that $q_1 < q_2 < \dots < q_m$ and $q_m \geq 953$. Then

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} < e^\gamma \times \log(1.26 \times \log(q_1 \times q_2 \times \dots \times q_m)).$$

Proof. We have that

$$\theta(q_m) = \log(q_1 \times q_2 \times \dots \times q_m).$$

From the lemma 8.1, we know that

$$\theta(q_m) > \left(1 - \frac{1}{\log(q_m)}\right) \times q_m.$$

In this way, we can show that

$$\begin{aligned} \log(1.26 \times \log(q_1 \times q_2 \times \dots \times q_m)) &= \log(1.26 \times \theta(q_m)) \\ &> \log\left(1.26 \times \left(1 - \frac{1}{\log(q_m)}\right) \times q_m\right) \\ &= \log q_m + \log\left(1.26 \times \left(1 - \frac{1}{\log(q_m)}\right)\right). \end{aligned}$$

We know that

$$\begin{aligned} \log\left(1.26 \times \left(1 - \frac{1}{\log(q_m)}\right)\right) &\geq \log\left(1.26 \times \left(1 - \frac{1}{\log(953)}\right)\right) \\ &> \frac{1}{2 \times \log(953)} \\ &\geq \frac{1}{2 \times \log(q_m)}. \end{aligned}$$

Certainly, we know that

$$\log\left(1.26 \times \left(1 - \frac{1}{\log(953)}\right)\right) - \frac{1}{2 \times \log(953)} \approx 0.000653913437463.$$

Consequently, we obtain that

$$\log q_m + \log\left(1.26 \times \left(1 - \frac{1}{\log(q_m)}\right)\right) > \left(\log q_m + \frac{1}{2 \times \log(q_m)}\right).$$

Due to the lemma 8.2, we finally prove that

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} < e^\gamma \times \left(\log q_m + \frac{1}{2 \times \log(q_m)}\right) < e^\gamma \times \log(1.26 \times \log(q_1 \times q_2 \times \cdots \times q_m))$$

when $q_m \geq 953$. □

9. Proof of Main Theorems

Theorem 9.1. *Robins(n) holds for all $n > 5040$ when a prime number $q \leq 953$ complies with $q \nmid n$.*

Proof. This is a compendium of the results from the lemmas 7.1, 7.2, 7.3 and 7.4. □

Theorem 9.2. *Let $\prod_{i=1}^m q_i^{a_i}$ be the representation of n as a product of the first m consecutive primes $q_1 < \cdots < q_m$ with natural numbers as exponents $a_1, \dots, a_m$. If $n > 5040$ is the smallest integer such that Robins(n) does not hold, then $\log(n)^\beta < 1.26 \times \log(N_m)$, where $N_m = \prod_{k=1}^m q_k$ is the primorial number of order m and $\beta = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1} - 1}$.*

Proof. According to the theorems 1.3 and 1.4, the primes $q_1 < \cdots < q_m$ must be the first m consecutive primes from $q_1 = 2, q_2 = 3, \dots, q_m$ since $n > 5040$ should be an Hardy-Ramanujan integer. From the theorem 9.1, we know that necessarily $q_m \geq 953$. From the lemma 3.2, we note that

$$f(n) = \left(\prod_{i=1}^m \frac{q_i}{q_i - 1}\right) \times \prod_{i=1}^m \left(1 - \frac{1}{q_i^{a_i+1}}\right).$$

However, we know that

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} < e^\gamma \times \log(1.26 \times \log(N_m))$$

because of the lemma 8.3. If we multiply by $\prod_{i=1}^m \left(1 - \frac{1}{q_i^{a_i+1}}\right)$ the both sides of the previous inequality, then we obtain that

$$f(n) < e^\gamma \times \log(1.26 \times \log(N_m)) \times \prod_{i=1}^m \left(1 - \frac{1}{q_i^{a_i+1}}\right).$$

If n is the smallest integer exceeding 5040 that does not satisfy the Robin inequality, then

$$e^\gamma \times \log \log n < e^\gamma \times \log(1.26 \times \log(N_m)) \times \prod_{i=1}^m \left(1 - \frac{1}{q_i^{a_i+1}}\right)$$

because of

$$e^\gamma \times \log \log n \leq f(n).$$

That is the same as

$$\prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1} - 1} \times \log \log n < \log(1.26 \times \log(N_m))$$

which is equivalent to

$$\log(n)^\beta < 1.26 \times \log(N_m)$$

where $\beta = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1} - 1}$. Therefore, the main proof is done. $\square$

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