Another Criterion For The Riemann Hypothesis

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Abstract

Let's define $\delta(x) = (\sum_{q \leq x} \frac{1}{q} - \log \log x - B)$, where $B \approx 0.2614972128$ is the Meissel-Mertens constant. The Robin theorem states that $\delta(x)$ changes sign infinitely often. Let's also define $S(x) = \theta(x) - x$, where $\theta(x)$ is the Chebyshev function. It is known that $S(x)$ changes sign infinitely often. We define the another function $\varpi(x) = \left(\sum_{q \leq x} \frac{1}{q} - \log \log \theta(x) - B\right)$. We prove that when the inequality $\varpi(x) \leq 0$ is satisfied for some number $x \geq 3$, then the Riemann Hypothesis should be false. The Riemann Hypothesis is also false when the inequalities $\delta(x) \leq 0$ and $S(x) \geq 0$ are satisfied for some number $x \geq 3$ or when $\int_{x}^{\infty} \frac{S(y) \times (1 + \log y)}{y^2 \times \log^2 y} dy \geq \frac{S(x)^2}{x^2 \times \log x}$ is satisfied for some number $x \geq 121$.

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1. Introduction

In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$ [1]. Let $N_n = 2 \times 3 \times 5 \times 7 \times 11 \times \cdots \times p_n$ denotes a primorial number of order n such that p_n is the n^{th} prime number. Say $\mathsf{Nicolas}(p_n)$ holds provided

$$\prod_{q \mid N_n} \frac{q}{q-1} > e^{\gamma} \times \log \log N_n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, log is the natural logarithm, and $q \mid N_n$ means the prime number q divides to N_n. The importance of this property is:

Theorem 1.1. *[2], [3].* $\mathsf{Nicolas}(p_n)$ *holds for all prime numbers* $p_n > 2$ *if and only if the Riemann Hypothesis is true.*

In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

where $p \leq x$ means all the prime numbers p that are less than or equal to x. We know this:

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Theorem 1.2. *[4].*
$$\lim_{x\to\infty} \frac{\theta(x)}{x} = 1.$$

Let's define $S(x) = \theta(x) - x$. It is a known result that:

Theorem 1.3. *[5]. $S(x)$ changes sign infinitely often.*

We also know that

Theorem 1.4. *[3]. For $x \geq 121$:*

$$\log\log\theta(x) \geq \log\log x + \frac{S(x)}{x \times \log x} - \frac{S(x)^2}{x^2 \times \log x}$$

Let's define $H = \gamma - B$ such that $B \approx 0.2614972128$ is the Meissel-Mertens constant [6]. We know from the constant H, the following formula:

Theorem 1.5. *[7].*
$$\sum_{q}\left(\log(\frac{q}{q-1}) - \frac{1}{q}\right) = \gamma - B = H.$$

For $x \geq 2$, the function $u(x)$ is defined as follows

$$u(x) = \sum_{q>x}\left(\log(\frac{q}{q-1}) - \frac{1}{q}\right).$$

Nicolas showed that

Theorem 1.6. *[3]. For $x \geq 2$:*
$$0 < u(x) \leq \frac{1}{2 \times (x-1)}.$$

Let's define:
$$\delta(x) = \left(\sum_{q\leq x}\frac{1}{q} - \log\log x - B\right).$$

Robin theorem states the following result:

Theorem 1.7. *[8]. $\delta(x)$ changes sign infinitely often.*

In addition, the Mertens second theorem states that:

Theorem 1.8. *[6].*
$$\lim_{x\to\infty}\delta(x) = 0.$$

Besides, Rosser and Schoenfeld derived a remarkable identity:

Theorem 1.9. *[9].*

$$\sum_{q\leq x}\frac{1}{q} = \log\log x + B + \frac{S(x)}{x \times \log x} - \int_{x}^{\infty}\frac{S(y) \times (1 + \log y)}{y^2 \times \log^2 y}dy.$$

We define another function:

$$\varpi(x) = \left(\sum_{q \leq x} \frac{1}{q} - \log \log \theta(x) - B \right).$$

Putting all together yields the proof that the inequality $\varpi(x) > u(x)$ is satisfied for a number $x \geq 3$ if and only if $\mathsf{Nicolas}(p)$ holds, where p is the greatest prime number such that $p \leq x$. In this way, we introduce another criterion for the Riemann Hypothesis based on the Nicolas criterion and deduce some of its consequences.

2. Results

Theorem 2.1. *The inequality $\varpi(x) > u(x)$ is satisfied for a number $x \geq 3$ if and only if $\mathsf{Nicolas}(p)$ holds, where p is the greatest prime number such that $p \leq x$.*

Proof. We start from the inequality:

$$\varpi(x) > u(x)$$

which is equivalent to

$$\left(\sum_{q \leq x} \frac{1}{q} - \log \log \theta(x) - B \right) > \sum_{q > x} \left(\log(\frac{q}{q-1}) - \frac{1}{q} \right).$$

Let's add the following formula to the both sides of the inequality,

$$\sum_{q \leq x} \left(\log(\frac{q}{q-1}) - \frac{1}{q} \right)$$

and due to the theorem 1.5, we obtain that

$$\sum_{q \leq x} \log(\frac{q}{q-1}) - \log \log \theta(x) - B > H$$

because of

$$H = \sum_{q \leq x} \left(\log(\frac{q}{q-1}) - \frac{1}{q} \right) + \sum_{q > x} \left(\log(\frac{q}{q-1}) - \frac{1}{q} \right)$$

and

$$\sum_{q \leq x} \log(\frac{q}{q-1}) = \sum_{q \leq x} \frac{1}{q} + \sum_{q \leq x} \left(\log(\frac{q}{q-1}) - \frac{1}{q} \right).$$

Let's distribute it and remove B from the both sides:

$$\sum_{q \leq x} \log(\frac{q}{q-1}) > \gamma + \log \log \theta(x)$$

since $H = \gamma - B$. If we apply the exponentiation to the both sides of the inequality, then we have that

$$\prod_{q \leq x} \frac{q}{q-1} > e^{\gamma} \times \log \theta(x)$$

which means that $\mathsf{Nicolas}(p)$ holds, where p is the greatest prime number such that $p \leq x$. The same happens in the reverse implication. □

Theorem 2.2. *The Riemann Hypothesis is true if and only if the inequality $\varpi(x) > u(x)$ is satisfied for all numbers $x \geq 3$.*

Proof. This is a direct consequence of theorems 1.1 and 2.1. □

Theorem 2.3. *If the inequality $\varpi(x) \leq 0$ is satisfied for some number $x \geq 3$, then the Riemann Hypothesis should be false.*

Proof. This is an implication of theorems 1.6, 2.1 and 2.2. □

Theorem 2.4. *If the inequalities $\delta(x) \leq 0$ and $S(x) \geq 0$ are satisfied for some number $x \geq 3$, then the Riemann Hypothesis should be false.*

Proof. If the inequalities $\delta(x) \leq 0$ and $S(x) \geq 0$ are satisfied for some number $x \geq 3$, then we obtain that $\varpi(x) \leq 0$ is also satisfied, which means that the Riemann Hypothesis should be false according to the theorem 2.3. □

Theorem 2.5.

$$\lim_{x\to\infty} \varpi(x) = 0.$$

Proof. We know that $\lim_{x\to\infty} \varpi(x) = 0$ for the limits $\lim_{x\to\infty} \delta(x) = 0$ and $\lim_{x\to\infty} \frac{\theta(x)}{x} = 1$. In this way, this is a consequence from the theorems 1.8 and 1.2. □

Theorem 2.6. *Under the assumption that*

$$\int_x^\infty \frac{S(y) \times (1 + \log y)}{y^2 \times \log^2 y} dy \geq \frac{S(x)^2}{x^2 \times \log x}$$

is satisfied for some number $x \geq 121$, then the Riemann Hypothesis should be false.

Proof. Under the assumption that

$$\int_x^\infty \frac{S(y) \times (1 + \log y)}{y^2 \times \log^2 y} dy \geq \frac{S(x)^2}{x^2 \times \log x}$$

for some number $x \geq 121$, then we can deduce that

$$\sum_{q \leq x} \frac{1}{q} \leq \log\log x + B + \frac{S(x)}{x \times \log x} - \frac{S(x)^2}{x^2 \times \log x}$$

according to the theorem 1.9. Using the theorem 1.4, then we obtain that

$$\sum_{q \leq x} \frac{1}{q} \leq \log\log \theta(x) + B$$

due to $x \geq 121$. However, that would mean

$$\varpi(x) = \sum_{q \leq x} \frac{1}{q} - \log\log \theta(x) - B \leq 0$$

and therefore, the Riemann Hypothesis should be false because of the theorem 2.3. □

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