

# Robin Criterion on Divisibility

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**Abstract** Robin criterion states that the Riemann Hypothesis is true if and only if the inequality  $\sigma(n) < e^\gamma \times n \times \log \log n$  holds for all  $n > 5040$ , where  $\sigma(n)$  is the sum-of-divisors function and  $\gamma \approx 0.57721$  is the Euler-Mascheroni constant. We show that the Robin inequality is true for all  $n > 5040$  which are not divisible by any prime number between 2 and 953. We prove that the Robin inequality holds when  $\frac{\pi^2}{6} \times \log \log n' \leq \log \log n$  for some  $n > 5040$  where  $n'$  is the square free kernel of the natural number  $n$ . The possible smallest counterexample  $n > 5040$  of the Robin inequality complies that necessarily  $(\log n)^\beta < 1.2592 \times \log(N_m)$ , where  $N_m = \prod_{i=1}^m q_i$  is the primorial number of order  $m$  and  $\beta = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$  when  $n$  is an Hardy-Ramanujan integer of the form  $\prod_{i=1}^m q_i^{a_i}$ .

**Keywords** Riemann hypothesis · Robin inequality · sum-of-divisors function · prime numbers

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## 1 Introduction

In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part  $\frac{1}{2}$  [6]. As usual  $\sigma(n)$  is the sum-of-divisors function of  $n$  [3]:

$$\sum_{d|n} d$$

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where  $d \mid n$  means the integer  $d$  divides to  $n$  and  $d \nmid n$  means the integer  $d$  does not divide to  $n$ . Define  $f(n)$  to be  $\frac{\sigma(n)}{n}$ . Say Robins( $n$ ) holds provided

$$f(n) < e^\gamma \times \log \log n.$$

The constant  $\gamma \approx 0.57721$  is the Euler-Mascheroni constant, and  $\log$  is the natural logarithm. The importance of this property is:

**Theorem 1.1** Robins( $n$ ) holds for all  $n > 5040$  if and only if the Riemann Hypothesis is true [6].

It is known that Robins( $n$ ) holds for many classes of numbers  $n$ .

**Theorem 1.2** Robins( $n$ ) holds for all  $n > 5040$  that are not divisible by 2 [3].

On the one hand, we prove that Robins( $n$ ) holds for all  $n > 5040$  that are not divisible by any prime number between 3 and 953. We recall that an integer  $n$  is said to be square free if for every prime divisor  $q$  of  $n$  we have  $q^2 \nmid n$  [3]. Robins( $n$ ) holds for all  $n > 5040$  that are square free [3]. In addition, we show that Robins( $n$ ) holds for some  $n > 5040$  when  $\frac{\pi^2}{6} \times \log \log n' \leq \log \log n$  such that  $n'$  is the square free kernel of the natural number  $n$ . Let  $q_1 = 2, q_2 = 3, \dots, q_m$  denote the first  $m$  consecutive primes, then an integer of the form  $\prod_{i=1}^m q_i^{a_i}$  with  $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$  is called an Hardy-Ramanujan integer [3]. A natural number  $n$  is called superabundant precisely when, for all  $m < n$

$$f(m) < f(n).$$

**Theorem 1.3** If  $n$  is superabundant, then  $n$  is an Hardy-Ramanujan integer [2].

**Theorem 1.4** The smallest counterexample of the Robin inequality greater than 5040 must be a superabundant number [1].

On the other hand, suppose that  $n > 5040$  might be the possible smallest counterexample of the Robin inequality, then we demonstrate that necessarily  $(\log n)^\beta < 1.2592 \times \log(N_m)$ , where  $N_m = \prod_{i=1}^m q_i$  is the primorial number of order  $m$  and  $\beta = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$  when  $n$  is an Hardy-Ramanujan integer of the form  $\prod_{i=1}^m q_i^{a_i}$ .

## 2 A Central Lemma

These are known results:

**Lemma 2.1** [3]. For  $n > 1$ :

$$f(n) < \prod_{q \mid n} \frac{q}{q-1}. \quad (2.1)$$

**Lemma 2.2** [4].

$$\prod_{k=1}^{\infty} \frac{1}{1 - \frac{1}{q_k^2}} = \zeta(2) = \frac{\pi^2}{6}. \quad (2.2)$$

The following is a key lemma. It gives an upper bound on  $f(n)$  that holds for all  $n$ . The bound is too weak to prove  $\text{Robins}(n)$  directly, but is critical because it holds for all  $n$ . Further the bound only uses the primes that divide  $n$  and not how many times they divide  $n$ .

**Lemma 2.3** *Let  $n > 1$  and let all its prime divisors be  $q_1 < \dots < q_m$ . Then,*

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^m \frac{q_i + 1}{q_i}.$$

*Proof* We use that lemma 2.1:

$$f(n) < \prod_{i=1}^m \frac{q_i}{q_i - 1}.$$

Now for  $q > 1$ ,

$$\frac{1}{1 - \frac{1}{q^2}} = \frac{q^2}{q^2 - 1}.$$

So

$$\begin{aligned} \frac{1}{1 - \frac{1}{q^2}} \times \frac{q+1}{q} &= \frac{q^2}{q^2 - 1} \times \frac{q+1}{q} \\ &= \frac{q}{q-1}. \end{aligned}$$

Then by lemma 2.2,

$$\prod_{i=1}^m \frac{1}{1 - \frac{1}{q_i^2}} < \zeta(2) = \frac{\pi^2}{6}.$$

Putting this together yields the proof:

$$\begin{aligned} f(n) &< \prod_{i=1}^m \frac{q_i}{q_i - 1} \\ &\leq \prod_{i=1}^m \frac{1}{1 - \frac{1}{q_i^2}} \times \frac{q_i + 1}{q_i} \\ &< \frac{\pi^2}{6} \times \prod_{i=1}^m \frac{q_i + 1}{q_i}. \end{aligned}$$

### 3 About the $p$ -adic order

In basic number theory, for a given prime number  $p$ , the  $p$ -adic order of a natural number  $n$  is the highest exponent  $v_p \geq 1$  such that  $p^{v_p}$  divides  $n$ . This is a known result:

**Lemma 3.1** *In general, we know that  $\text{Robins}(n)$  holds for a natural number  $n > 5040$  that satisfies either  $v_2(n) \leq 19$ ,  $v_3(n) \leq 12$  or  $v_7(n) \leq 6$ , where  $v_p(n)$  is the  $p$ -adic order of  $n$  [5].*

We know the following lemmas:

**Lemma 3.2** [5]. Let  $\prod_{i=1}^m q_i^{a_i}$  be the representation of  $n$  as a product of primes  $q_1 < \dots < q_m$  with natural numbers as exponents  $a_1, \dots, a_m$ . Then,

$$f(n) = \left( \prod_{i=1}^m \frac{q_i}{q_i - 1} \right) \times \prod_{i=1}^m \left( 1 - \frac{1}{q_i^{a_i+1}} \right).$$

**Lemma 3.3** [5]. Let  $n > e^{23.762143}$  and let all its prime divisors be  $q_1 < \dots < q_m$ , then

$$\left( \prod_{i=1}^m \frac{q_i}{q_i - 1} \right) < \frac{1771561}{1771560} \times e^\gamma \times \log \log n.$$

**Lemma 3.4** Robins( $n$ ) holds for all  $10^{10^{10}} \geq n > 5040$  [5].

Putting together all these results, then we obtain that

**Lemma 3.5** Robins( $n$ ) holds for  $n > 5040$  when  $v_{31}(n) \leq 3$ .

*Proof* From lemma 3.2, we note that

$$f(n) = \left( \prod_{i=1}^m \frac{q_i}{q_i - 1} \right) \times \prod_{i=1}^m \left( 1 - \frac{1}{q_i^{a_i+1}} \right) \leq \left( \prod_{i=1}^m \frac{q_i}{q_i - 1} \right) \times \left( 1 - \frac{1}{31^{v_{31}(n)+1}} \right)$$

when  $v_{31}(n) \leq 3$ . We only need to look at the case where  $v_{31}(n) = 3$  since the weaker cases follow because

$$\left( 1 - \frac{1}{31^{1+1}} \right) < \left( 1 - \frac{1}{31^{2+1}} \right) < \left( 1 - \frac{1}{31^{3+1}} \right).$$

In this way, we obtain that

$$f(n) \leq \left( \prod_{i=1}^m \frac{q_i}{q_i - 1} \right) \times \left( 1 - \frac{1}{31^{3+1}} \right) = \frac{923520}{923521} \times \left( \prod_{i=1}^m \frac{q_i}{q_i - 1} \right)$$

when  $v_{31}(n) \leq 3$ . With lemma 3.3, we have for  $n > e^{23.762143}$

$$\frac{923520}{923521} \times \left( \prod_{i=1}^m \frac{q_i}{q_i - 1} \right) < \frac{923520}{923521} \times \frac{1771561}{1771560} \times e^\gamma \times \log \log n < e^\gamma \times \log \log n$$

since  $\frac{923520}{923521} \times \frac{1771561}{1771560} < 1$ . In light of lemma 3.4 and the fact that  $e^{23.762143} < 10^{10^{10}}$ , we then conclude that Robins( $n$ ) holds for  $n > 5040$  when  $v_{31}(n) \leq 3$ .

#### 4 A Particular Case

We can easily prove that  $\text{Robins}(n)$  is true for certain kind of numbers:

**Lemma 4.1**  *$\text{Robins}(n)$  holds for  $n > 5040$  when  $q \leq 7$ , where  $q$  is the largest prime divisor of  $n$ .*

*Proof* Let  $n > 5040$  and let all its prime divisors be  $q_1 < \dots < q_m \leq 5$ , then we need to prove

$$f(n) < e^\gamma \times \log \log n$$

that is true when

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} \leq e^\gamma \times \log \log n$$

according to the lemma 2.1. For  $q_1 < \dots < q_m \leq 5$ ,

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5}{1 \times 2 \times 4} = 3.75 < e^\gamma \times \log \log(5040) \approx 3.81.$$

However, we know for  $n > 5040$

$$e^\gamma \times \log \log(5040) < e^\gamma \times \log \log n$$

and therefore, the proof is complete when  $q_1 < \dots < q_m \leq 5$ . The remaining case is for  $n > 5040$  when all its prime divisors are  $q_1 < \dots < q_m \leq 7$ .  $\text{Robins}(n)$  holds for  $n > 5040$  when  $v_7(n) \leq 6$  according to the lemma 3.1 [5]. Hence, it is enough to prove this for those natural numbers  $n > 5040$  when  $7^7 \mid n$ . For  $q_1 < \dots < q_m \leq 7$ ,

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5 \times 7}{1 \times 2 \times 4 \times 6} = 4.375 < e^\gamma \times \log \log(7^7) \approx 4.65.$$

However, for  $n > 5040$  and  $7^7 \mid n$ , we know that

$$e^\gamma \times \log \log(7^7) \leq e^\gamma \times \log \log n$$

and as a consequence, the proof is complete when  $q_1 < \dots < q_m \leq 7$ .

#### 5 A Better Bound

This is a known result:

**Lemma 5.1** [7]. For  $x > 1$ :

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + B + \frac{1}{\log^2 x} \quad (5.1)$$

where

$$B = 0.2614972128 \dots$$

denotes the (Meissel-)Mertens constant [7].

We show a better result:

**Lemma 5.2** *For  $x \geq 11$ , we have*

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + \gamma - 0.12.$$

*Proof* Let's define  $H = \gamma - B$ . The lemma 5.1 is the same as

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + \gamma - \left(H - \frac{1}{\log^2 x}\right).$$

For  $x \geq 11$ ,

$$\left(H - \frac{1}{\log^2 x}\right) > \left(0.31 - \frac{1}{\log^2 11}\right) > 0.12$$

and thus,

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + \gamma - \left(H - \frac{1}{\log^2 x}\right) < \log \log x + \gamma - 0.12.$$

## 6 On a Square Free Number

We know the following results:

**Lemma 6.1** [3]. *For  $0 < a < b$ :*

$$\frac{\log b - \log a}{b - a} = \frac{1}{(b - a)} \int_a^b \frac{dt}{t} > \frac{1}{b}. \quad (6.1)$$

**Lemma 6.2** [3]. *For  $q > 0$ :*

$$\log(q + 1) - \log q = \int_q^{q+1} \frac{dt}{t} < \frac{1}{q}. \quad (6.2)$$

We recall that an integer  $n$  is said to be square free if for every prime divisor  $q$  of  $n$  we have  $q^2 \nmid n$  [3]. Robins( $n$ ) holds for all  $n > 5040$  that are square free [3].

**Lemma 6.3** *For a square free number*

$$n = q_1 \times \cdots \times q_m$$

*such that  $q_1 < q_2 < \cdots < q_m$  are odd prime numbers,  $q_m \geq 11$  and  $3 \nmid n$ , then:*

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(n) \leq e^\gamma \times n \times \log \log(2^{19} \times n).$$

*Proof* By induction with respect to  $\omega(n)$ , that is the number of distinct prime factors of  $n$  [3]. Put  $\omega(n) = m$  [3]. We need to prove the assertion for those integers with  $m = 1$ . From a square free number  $n$ , we obtain

$$\sigma(n) = (q_1 + 1) \times (q_2 + 1) \times \cdots \times (q_m + 1) \quad (6.3)$$

when  $n = q_1 \times q_2 \times \cdots \times q_m$  [3]. In this way, for every prime number  $q_i \geq 11$ , then we need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \left(1 + \frac{1}{q_i}\right) \leq e^\gamma \times \log \log(2^{19} \times q_i). \quad (6.4)$$

For  $q_i = 11$ , we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \left(1 + \frac{1}{11}\right) \leq e^\gamma \times \log \log(2^{19} \times 11)$$

is actually true. For another prime number  $q_i > 11$ , we have

$$\left(1 + \frac{1}{q_i}\right) < \left(1 + \frac{1}{11}\right)$$

and

$$\log \log(2^{19} \times 11) < \log \log(2^{19} \times q_i)$$

which clearly implies that the inequality (6.4) is true for every prime number  $q_i \geq 11$ . Now, suppose it is true for  $m - 1$ , with  $m \geq 2$  and let us consider the assertion for those square free  $n$  with  $\omega(n) = m$  [3]. So let  $n = q_1 \times \cdots \times q_m$  be a square free number and assume that  $q_1 < \cdots < q_m$  for  $q_m \geq 11$ .

*Case 1:*  $q_m \geq \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = \log(2^{19} \times n)$ .

By the induction hypothesis we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1 + 1) \times \cdots \times (q_{m-1} + 1) \leq e^\gamma \times q_1 \times \cdots \times q_{m-1} \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

and hence

$$\begin{aligned} & \frac{\pi^2}{6} \times \frac{3}{2} \times (q_1 + 1) \times \cdots \times (q_{m-1} + 1) \times (q_m + 1) \leq \\ & e^\gamma \times q_1 \times \cdots \times q_{m-1} \times (q_m + 1) \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1}) \end{aligned}$$

when we multiply the both sides of the inequality by  $(q_m + 1)$ . We want to show

$$e^\gamma \times q_1 \times \cdots \times q_{m-1} \times (q_m + 1) \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1}) \leq$$

$$e^\gamma \times q_1 \times \cdots \times q_{m-1} \times q_m \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = e^\gamma \times n \times \log \log(2^{19} \times n).$$

Indeed the previous inequality is equivalent with

$$q_m \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) \geq (q_m + 1) \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

or alternatively

$$\frac{q_m \times (\log \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1}))}{\log q_m} \geq$$

$$\frac{\log \log(2^{19} \times q_1 \times \cdots \times q_{m-1})}{\log q_m}.$$

We can apply the inequality in lemma 6.1 just using  $b = \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m)$  and  $a = \log(2^{19} \times q_1 \times \cdots \times q_{m-1})$ . Certainly, we have

$$\begin{aligned} \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log(2^{19} \times q_1 \times \cdots \times q_{m-1}) = \\ \log \frac{2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m}{2^{19} \times q_1 \times \cdots \times q_{m-1}} = \log q_m. \end{aligned}$$

In this way, we obtain

$$\begin{aligned} \frac{q_m \times (\log \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1}))}{\log q_m} > \\ \frac{q_m}{\log(2^{19} \times q_1 \times \cdots \times q_m)}. \end{aligned}$$

Using this result we infer that the original inequality is certainly satisfied if the next inequality is satisfied

$$\frac{q_m}{\log(2^{19} \times q_1 \times \cdots \times q_m)} \geq \frac{\log \log(2^{19} \times q_1 \times \cdots \times q_{m-1})}{\log q_m}$$

which is trivially true for  $q_m \geq \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m)$  [3].

Case 2:  $q_m < \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = \log(2^{19} \times n)$ .

We need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \frac{\sigma(n)}{n} \leq e^\gamma \times \log \log(2^{19} \times n).$$

We know  $\frac{3}{2} < 1.503 < \frac{4}{2.66}$ . Nevertheless, we could have

$$\frac{3}{2} \times \frac{\sigma(n)}{n} \times \frac{\pi^2}{6} < \frac{4 \times \sigma(n)}{3 \times n} \times \frac{\pi^2}{2 \times 2.66}$$

and therefore, we only need to prove

$$\frac{\sigma(3 \times n)}{3 \times n} \times \frac{\pi^2}{5.32} \leq e^\gamma \times \log \log(2^{19} \times n)$$

where this is possible because of  $3 \nmid n$ . If we apply the logarithm to the both sides of the inequality, then we obtain

$$\log\left(\frac{\pi^2}{5.32}\right) + (\log(3+1) - \log 3) + \sum_{i=1}^m (\log(q_i + 1) - \log q_i) \leq \gamma + \log \log \log(2^{19} \times n).$$

In addition, note that  $\log\left(\frac{\pi^2}{5.32}\right) < \frac{1}{2} + 0.12$ . However, we know

$$\gamma + \log \log q_m < \gamma + \log \log \log(2^{19} \times n)$$



since  $q_m < \log(2^{19} \times n)$ . We use that lemma 6.2 for each term  $\log(q+1) - \log q$  and thus,

$$0.12 + \frac{1}{2} + \frac{1}{3} + \frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq 0.12 + \sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \log \log q_m$$

where  $q_m \geq 11$ . Hence, it is enough to prove

$$\sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \log \log q_m - 0.12$$

but this is true according to the lemma 5.2 for  $q_m \geq 11$ . In this way, we finally show the lemma is indeed satisfied.

## 7 Robin on Divisibility

$\text{Robins}(n)$  holds for every  $n > 5040$  that is not divisible by 2 [3]. We extend this property to other prime numbers:

**Lemma 7.1**  *$\text{Robins}(n)$  holds for all  $n > 5040$  when  $3 \nmid n$ . More precisely: every possible counterexample  $n > 5040$  of the Robin inequality must comply with  $(2^{20} \times 3^{13}) \mid n$ .*

*Proof* We will check the Robin inequality is true for every natural number  $n = q_1^{a_1} \times q_2^{a_2} \times \cdots \times q_m^{a_m} > 5040$  such that  $q_1, q_2, \dots, q_m$  are distinct prime numbers,  $a_1, a_2, \dots, a_m$  are natural numbers and  $3 \nmid n$ . We know this is true when the greatest prime divisor of  $n > 5040$  is lesser than or equal to 7 according to the lemma 4.1. Therefore, the remaining case is when the greatest prime divisor of  $n > 5040$  is greater than or equal to 11. We need to prove

$$f(n) < e^\gamma \times \log \log n$$

that is true when

$$\frac{\pi^2}{6} \times \prod_{i=1}^m \frac{q_i + 1}{q_i} \leq e^\gamma \times \log \log n$$

according to the lemma 2.3. Using the formula (6.3) for the square free numbers, then we obtain that is equivalent to

$$\frac{\pi^2}{6} \times \frac{\sigma(n')}{n'} \leq e^\gamma \times \log \log n$$

where  $n' = q_1 \times \cdots \times q_m$  is the square free kernel of the natural number  $n$  [3]. The Robin inequality has been proved for all integers  $n$  not divisible by 2 (which are bigger than 10) [3]. Hence, we only need to prove the Robin inequality is true when  $2 \mid n'$ . In addition, we know that  $\text{Robins}(n)$  holds for every  $n > 5040$  when  $v_2(n) \leq 19$  according to the lemma 3.1 [5]. Consequently, we only need to prove that  $\text{Robins}(n)$  holds for  $n > 5040$  when  $2^{20} \mid n$  and thus,

$$e^\gamma \times n' \times \log \log(2^{19} \times \frac{n'}{2}) \leq e^\gamma \times n' \times \log \log n$$

because of  $2^{19} \times \frac{n'}{2} \leq n$  where  $2^{20} \mid n$  and  $2 \mid n'$ . So,

$$\frac{\pi^2}{6} \times \sigma(n') \leq e^\gamma \times n' \times \log \log(2^{19} \times \frac{n'}{2}).$$

According to the formula (6.3) for the square free numbers and  $2 \mid n'$ , then,

$$\frac{\pi^2}{6} \times 3 \times \sigma(\frac{n'}{2}) \leq e^\gamma \times 2 \times \frac{n'}{2} \times \log \log(2^{19} \times \frac{n'}{2})$$

which is the same as

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(\frac{n'}{2}) \leq e^\gamma \times \frac{n'}{2} \times \log \log(2^{19} \times \frac{n'}{2})$$

where this is true according to the lemma 6.3 when  $3 \nmid \frac{n'}{2}$ . In addition, we know that Robins( $n$ ) holds for every  $n > 5040$  when  $v_3(n) \leq 12$  according to the lemma 3.1 [5]. Hence, we only need to prove that Robins( $n$ ) holds for every  $n > 5040$  when  $2^{20} \mid n$  and  $3^{13} \mid n$ . To sum up, the proof is complete.

**Lemma 7.2** Robins( $n$ ) holds for all  $n > 5040$  when  $5 \nmid n$  or  $7 \nmid n$ .

*Proof* We need to prove

$$f(n) < e^\gamma \times \log \log n$$

when  $(2^{20} \times 3^{13}) \mid n$ . Suppose that  $n = 2^a \times 3^b \times m$ , where  $a \geq 20$ ,  $b \geq 13$ ,  $2 \nmid m$ ,  $3 \nmid m$  and  $5 \nmid m$  or  $7 \nmid m$ . Therefore, we need to prove

$$f(2^a \times 3^b \times m) < e^\gamma \times \log \log(2^a \times 3^b \times m).$$

We know

$$f(2^a \times 3^b \times m) = f(3^b) \times f(2^a \times m)$$

since  $f$  is multiplicative [3]. In addition, we know  $f(3^b) < \frac{3}{2}$  for every natural number  $b$  [3]. In this way, we have

$$f(3^b) \times f(2^a \times m) < \frac{3}{2} \times f(2^a \times m).$$

However, that would be equivalent to

$$\frac{3}{2} \times f(2^a \times m) = \frac{9}{8} \times f(3) \times f(2^a \times m) = \frac{9}{8} \times f(2^a \times 3 \times m)$$

where  $f(3) = \frac{4}{3}$  since  $f$  is multiplicative [3]. Nevertheless, we have

$$\frac{9}{8} \times f(2^a \times 3 \times m) < f(5) \times f(2^a \times 3 \times m) = f(2^a \times 3 \times 5 \times m)$$

and

$$\frac{9}{8} \times f(2^a \times 3 \times m) < f(7) \times f(2^a \times 3 \times m) = f(2^a \times 3 \times 7 \times m)$$

where  $5 \nmid m$  or  $7 \nmid m$ ,  $f(5) = \frac{6}{5}$  and  $f(7) = \frac{8}{7}$ . We know the Robin inequality is true for  $2^a \times 3 \times 5 \times m$  and  $2^a \times 3 \times 7 \times m$  when  $a \geq 20$ , since this is true for every natural number  $n > 5040$  when  $v_3(n) \leq 12$  according to the lemma 3.1 [5]. Hence, we would have

$$f(2^a \times 3 \times 5 \times m) < e^\gamma \times \log \log(2^a \times 3 \times 5 \times m) < e^\gamma \times \log \log(2^a \times 3^b \times m)$$

and

$$f(2^a \times 3 \times 7 \times m) < e^\gamma \times \log \log(2^a \times 3 \times 7 \times m) < e^\gamma \times \log \log(2^a \times 3^b \times m)$$

when  $b \geq 13$ .

**Lemma 7.3** *Robins( $n$ ) holds for all  $n > 5040$  when a prime number  $11 \leq q \leq 47$  complies with  $q \nmid n$ .*

*Proof* We know that Robins( $n$ ) holds for every  $n > 5040$  when  $v_7(n) \leq 6$  according to the lemma 3.1 [5]. We need to prove

$$f(n) < e^\gamma \times \log \log n$$

when  $(2^{20} \times 3^{13} \times 7^7) \mid n$ . Suppose that  $n = 2^a \times 3^b \times 7^c \times m$ , where  $a \geq 20$ ,  $b \geq 13$ ,  $c \geq 7$ ,  $2 \nmid m$ ,  $3 \nmid m$ ,  $7 \nmid m$ ,  $q \nmid m$  and  $11 \leq q \leq 47$ . Therefore, we need to prove

$$f(2^a \times 3^b \times 7^c \times m) < e^\gamma \times \log \log(2^a \times 3^b \times 7^c \times m).$$

We know

$$f(2^a \times 3^b \times 7^c \times m) = f(7^c) \times f(2^a \times 3^b \times m)$$

since  $f$  is multiplicative [3]. In addition, we know  $f(7^c) < \frac{7}{6}$  for every natural number  $c$  [3]. In this way, we have

$$f(7^c) \times f(2^a \times 3^b \times m) < \frac{7}{6} \times f(2^a \times 3^b \times m).$$

However, that would be equivalent to

$$\frac{7}{6} \times f(2^a \times 3^b \times m) = \frac{49}{48} \times f(7) \times f(2^a \times 3^b \times m) = \frac{49}{48} \times f(2^a \times 3^b \times 7 \times m)$$

where  $f(7) = \frac{8}{7}$  since  $f$  is multiplicative [3]. In addition, we know

$$\frac{49}{48} \times f(2^a \times 3^b \times 7 \times m) < f(q) \times f(2^a \times 3^b \times 7 \times m) = f(2^a \times 3^b \times 7 \times q \times m)$$

where  $q \nmid m$ ,  $f(q) = \frac{q+1}{q}$  and  $11 \leq q \leq 47$ . Nevertheless, we know the Robin inequality is true for  $2^a \times 3^b \times 7 \times q \times m$  when  $a \geq 20$  and  $b \geq 13$ , since this is true for every natural number  $n > 5040$  when  $v_7(n) \leq 6$  according to the lemma 3.1 [5]. Hence, we would have

$$\begin{aligned} f(2^a \times 3^b \times 7 \times q \times m) &< e^\gamma \times \log \log(2^a \times 3^b \times 7 \times q \times m) \\ &< e^\gamma \times \log \log(2^a \times 3^b \times 7^c \times m) \end{aligned}$$

when  $c \geq 7$  and  $11 \leq q \leq 47$ .

**Lemma 7.4** *Robins( $n$ ) holds for all  $n > 5040$  when a prime number  $53 \leq q \leq 953$  complies with  $q \nmid n$ .*

*Proof* We know that Robins( $n$ ) holds for every  $n > 5040$  when  $v_{31}(n) \leq 3$  according to the lemma 3.5. We need to prove that

$$f(n) < e^\gamma \times \log \log n$$

when  $(2^{20} \times 3^{13} \times 31^4) \mid n$ . Suppose that  $n = 2^a \times 3^b \times 31^c \times m$ , where  $a \geq 20, b \geq 13, c \geq 4, 2 \nmid m, 3 \nmid m, 31 \nmid m, q \nmid m$  and  $53 \leq q \leq 953$ . Therefore, we need to prove that

$$f(2^a \times 3^b \times 31^c \times m) < e^\gamma \times \log \log (2^a \times 3^b \times 31^c \times m).$$

We know that

$$f(2^a \times 3^b \times 31^c \times m) = f(31^c) \times f(2^a \times 3^b \times m)$$

since  $f$  is multiplicative [3]. In addition, we know that  $f(31^c) < \frac{31}{30}$  for every natural number  $c$  [3]. In this way, we have that

$$f(31^c) \times f(2^a \times 3^b \times m) < \frac{31}{30} \times f(2^a \times 3^b \times m).$$

However, that would be equivalent to

$$\frac{31}{30} \times f(2^a \times 3^b \times m) = \frac{961}{960} \times f(31) \times f(2^a \times 3^b \times m) = \frac{961}{960} \times f(2^a \times 3^b \times 31 \times m)$$

where  $f(31) = \frac{32}{31}$  since  $f$  is multiplicative [3]. In addition, we know that

$$\frac{961}{960} \times f(2^a \times 3^b \times 31 \times m) < f(q) \times f(2^a \times 3^b \times 31 \times m) = f(2^a \times 3^b \times 31 \times q \times m)$$

where  $q \nmid m, f(q) = \frac{q+1}{q}$  and  $53 \leq q \leq 953$ . Nevertheless, we know the Robin inequality is true for  $2^a \times 3^b \times 31 \times q \times m$  when  $a \geq 20$  and  $b \geq 13$ , since this is true for every natural number  $n > 5040$  when  $v_{31}(n) \leq 3$  according to the lemma 3.5. Hence, we would have that

$$\begin{aligned} f(2^a \times 3^b \times 31 \times q \times m) &< e^\gamma \times \log \log (2^a \times 3^b \times 31 \times q \times m) \\ &< e^\gamma \times \log \log (2^a \times 3^b \times 31^c \times m) \end{aligned}$$

when  $c \geq 4$  and  $53 \leq q \leq 953$ .

## 8 Helpful Lemmas

For every prime number  $p_n > 2$ , we define sequence  $Y_n = \frac{e^{\frac{1}{2 \times \log(p_n)}}}{(1 - \frac{1}{\log(p_n)})}$ .

**Lemma 8.1** *For every prime number  $p_n > 2$ , the sequence  $Y_n$  is strictly decreasing.*

*Proof* For every real value  $x \geq 3$ , we state the function

$$f(x) = \frac{e^{\frac{1}{2 \times \log(x)}}}{(1 - \frac{1}{\log(x)})}$$

which is equivalent to

$$f(x) = g(x) \times h(u)$$

where  $g(x) = e^{\frac{1}{2 \times \log(x)}}$  and  $h(u) = \frac{u}{u-1}$  for  $u = \log(x)$ . We know that  $g(x)$  decreases as  $x \geq 3$  increases, Moreover, we note that  $h(u)$  decreases as  $u > 1$  increases where  $u = \log(x) > 1$  for  $x \geq 3$ . In conclusion, we can see that the function  $f(x)$  is monotonically decreasing for every real value  $x \geq 3$  and therefore, the sequence  $Y_n$  is monotonically decreasing as well. In addition,  $Y_n$  is essentially a strictly decreasing sequence, since there is not any natural number  $n > 1$  such that  $Y_n = Y_{n+1}$ .

In mathematics, the Chebyshev function  $\theta(x)$  is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

where  $p \leq x$  means all the prime numbers  $p$  that are less than or equal to  $x$ .

**Lemma 8.2** [7]. *For  $x \geq 41$ :*

$$\theta(x) > (1 - \frac{1}{\log(x)}) \times x.$$

Besides, we know that

**Lemma 8.3** [7]. *For  $x \geq 286$ :*

$$\prod_{q \leq x} \frac{q}{q-1} < e^\gamma \times (\log x + \frac{1}{2 \times \log(x)}).$$

We will prove another important inequality:

**Theorem 8.4** *Let  $q_1, q_2, \dots, q_m$  denote the first  $m$  consecutive primes such that  $q_1 < q_2 < \dots < q_m$  and  $q_m > 286$ . Then*

$$\prod_{i=1}^m \frac{q_i}{q_i-1} < e^\gamma \times \log(Y_m \times \theta(q_m)).$$

*Proof* From the theorem 8.2, we know that

$$\theta(q_m) > \left(1 - \frac{1}{\log(q_m)}\right) \times q_m.$$

In this way, we can show that

$$\begin{aligned} \log(Y_m \times \theta(q_m)) &> \log\left(Y_m \times \left(1 - \frac{1}{\log(q_m)}\right) \times q_m\right) \\ &= \log q_m + \log\left(Y_m \times \left(1 - \frac{1}{\log(q_m)}\right)\right). \end{aligned}$$

We know that

$$\begin{aligned} \log\left(Y_m \times \left(1 - \frac{1}{\log(q_m)}\right)\right) &= \log\left(\frac{e^{\frac{1}{2 \times \log(q_m)}}}{\left(1 - \frac{1}{\log(q_m)}\right)} \times \left(1 - \frac{1}{\log(q_m)}\right)\right) \\ &= \log\left(e^{\frac{1}{2 \times \log(q_m)}}\right) \\ &= \frac{1}{2 \times \log(q_m)}. \end{aligned}$$

Consequently, we obtain that

$$\log q_m + \log\left(Y_m \times \left(1 - \frac{1}{\log(q_m)}\right)\right) \geq \left(\log q_m + \frac{1}{2 \times \log(q_m)}\right).$$

Due to the theorem 8.3, we prove that

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} < e^\gamma \times \left(\log q_m + \frac{1}{2 \times \log(q_m)}\right) < e^\gamma \times \log(Y_m \times \theta(q_m))$$

when  $q_m > 286$ .

## 9 Proof of Main Theorems

**Theorem 9.1** *Robins( $n$ ) holds for all  $n > 5040$  when a prime number  $q \leq 953$  complies with  $q \nmid n$ .*

*Proof* This is a compendium of the results from the theorem 1.2 and the lemmas 7.1, 7.2, 7.3 and 7.4.

**Theorem 9.2** *Let  $\frac{\pi^2}{6} \times \log \log n' \leq \log \log n$  for some  $n > 5040$  such that  $n'$  is the square free kernel of the natural number  $n$ . Then Robins( $n$ ) holds.*

*Proof* Let  $n'$  be the square free kernel of the natural number  $n$ . Let  $n'$  be the product of the distinct primes  $q_1, \dots, q_m$ . By assumption we have that

$$\frac{\pi^2}{6} \times \log \log n' \leq \log \log n.$$

For all square free  $n' \leq 5040$ , Robins( $n'$ ) holds if and only if  $n' \notin \{2, 3, 5, 6, 10, 30\}$  [3]. However, Robins( $n$ ) holds for all  $n > 5040$  when  $n' \in \{2, 3, 5, 6, 10, 15, 30\}$  due to the lemma 4.1. When  $n' > 5040$ , we know that Robins( $n'$ ) holds and so

$$f(n') < e^\gamma \times \log \log n'.$$

By the previous lemma 2.3:

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^m \frac{q_i + 1}{q_i}.$$

Suppose by way of contradiction that Robins( $n$ ) fails. Then

$$f(n) \geq e^\gamma \times \log \log n.$$

We claim that

$$\frac{\pi^2}{6} \times \prod_{i=1}^m \frac{q_i + 1}{q_i} > e^\gamma \times \log \log n.$$

Since otherwise we would have a contradiction. This shows that

$$\frac{\pi^2}{6} \times \prod_{i=1}^m \frac{q_i + 1}{q_i} > \frac{\pi^2}{6} \times e^\gamma \times \log \log n'.$$

Thus

$$\prod_{i=1}^m \frac{q_i + 1}{q_i} > e^\gamma \times \log \log n',$$

and

$$\prod_{i=1}^m \frac{q_i + 1}{q_i} > f(n'),$$

This is a contradiction since  $f(n')$  is equal to

$$\frac{(q_1 + 1) \times \dots \times (q_m + 1)}{q_1 \times \dots \times q_m}$$

according to the formula (6.3) for the square free numbers.

**Theorem 9.3** Let  $\prod_{i=1}^m q_i^{a_i}$  be the representation of  $n$  as a product of the first  $m$  consecutive primes  $q_1 < \dots < q_m$  with natural numbers as exponents  $a_1, \dots, a_m$ . If  $n > 5040$  is the smallest integer such that Robins( $n$ ) does not hold, then  $(\log n)^\beta < Y_m \times \log(N_m)$ , where  $N_m = \prod_{i=1}^m q_i$  is the primorial number of order  $m$  and  $\beta = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$ .

*Proof* According to the theorems 1.3 and 1.4, the primes  $q_1 < \dots < q_m$  must be the first  $m$  consecutive primes since  $n > 5040$  should be an Hardy-Ramanujan integer. From the theorem 9.1, we know that necessarily  $q_m \geq 953$ . From the lemma 3.2, we note that

$$f(n) = \left( \prod_{i=1}^m \frac{q_i}{q_i - 1} \right) \times \prod_{i=1}^m \left( 1 - \frac{1}{q_i^{a_i+1}} \right).$$

However, we know that

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} < e^\gamma \times \log(Y_m \times \log(N_m))$$

because of the lemma 8.4 when  $q_m \geq 953$ . If we multiply by  $\prod_{i=1}^m \left( 1 - \frac{1}{q_i^{a_i+1}} \right)$  the both sides of the previous inequality, then we obtain that

$$f(n) < e^\gamma \times \log(Y_m \times \log(N_m)) \times \prod_{i=1}^m \left( 1 - \frac{1}{q_i^{a_i+1}} \right).$$

If  $n$  is the smallest integer exceeding 5040 that does not satisfy the Robin inequality, then

$$e^\gamma \times \log \log n < e^\gamma \times \log(Y_m \times \log(N_m)) \times \prod_{i=1}^m \left( 1 - \frac{1}{q_i^{a_i+1}} \right)$$

because of

$$e^\gamma \times \log \log n \leq f(n).$$

That is the same as

$$\prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1} - 1} \times \log \log n < \log(Y_m \times \log(N_m))$$

which is equivalent to

$$(\log n)^\beta < Y_m \times \log(N_m)$$

where  $\beta = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1} - 1}$ . Therefore, the proof is done.

**Theorem 9.4** Let  $\prod_{i=1}^m q_i^{a_i}$  be the representation of  $n$  as a product of the first  $m$  consecutive primes  $q_1 < \dots < q_m$  with natural numbers as exponents  $a_1, \dots, a_m$ . If  $n > 5040$  is the smallest integer such that  $\text{Robins}(n)$  does not hold, then  $(\log n)^\beta < 1.2592 \times \log(N_m)$ , where  $N_m = \prod_{i=1}^m q_i$  is the primorial number of order  $m$  and  $\beta = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1} - 1}$ .

*Proof* From the theorem 9.1, we know that necessarily  $q_m \geq 953$ . Using the theorem 9.3, we obtain that

$$(\log n)^\beta < 1.2592 \times \log(N_m)$$

because of the lemma 8.1 since we have that  $Y_m < 1.2592$  when  $q_m \geq 953$ .



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