Robin Criterion on Divisibility

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the date of receipt and acceptance should be inserted later

Abstract Robin criterion states that the Riemann Hypothesis is true if and only if the inequality $\sigma(n) < e^\gamma \times n \times \log \log n$ holds for all $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. This is known as the Robin inequality. We prove that the Robin inequality is true for all $n > 5040$ which are not divisible by any prime number between 2 and 953. In addition, we demonstrate that the Robin inequality holds when $\frac{\pi^2}{6} \times \log \log n' \leq \log \log n$ for some number $n > 5040$ where n' is the square free kernel of the natural number n . Using these results, we show some properties of the possible smallest counterexample $n > 5040$ of the Robin inequality.

Keywords Riemann hypothesis · Robin inequality · sum-of-divisors function · prime numbers

Mathematics Subject Classification (2010) MSC 11M26 · MSC 11A41 · MSC 11A25

1 Introduction

In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$ [6]. As usual $\sigma(n)$ is the sum-of-divisors function of n [3]:

$$\sum_{d|n} d$$

where $d \mid n$ means the integer d divides to n and $d \nmid n$ means the integer d does not divide to n . Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say Robins(n) holds provided

$$f(n) < e^\gamma \times \log \log n.$$

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The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, and $\log$ is the natural logarithm. The importance of this property is:

Theorem 1.1 *Robins(n) holds for all $n > 5040$ if and only if the Riemann Hypothesis is true [6].*

It is known that Robins(n) holds for many classes of numbers n .

Theorem 1.2 *Robins(n) holds for all $n > 5040$ that are not divisible by 2 [3].*

On the one hand, we prove that Robins(n) holds for all $n > 5040$ that are not divisible by any prime number between 3 and 953. We recall that an integer n is said to be square free if for every prime divisor q of n we have $q^2 \nmid n$ [3]. Robins(n) holds for all $n > 5040$ that are square free [3]. In addition, we show that Robins(n) holds for some number $n > 5040$ when $\frac{\pi^2}{6} \times \log \log n' \leq \log \log n$ such that n' is the square free kernel of the natural number n . Let $q_1 = 2, q_2 = 3, \dots, q_m$ denote the first m consecutive primes, then an integer of the form $\prod_{i=1}^m q_i^{a_i}$ with $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$ is called an Hardy-Ramanujan integer [3]. A natural number n is called superabundant precisely when, for all $m < n$

$$f(m) < f(n).$$

Theorem 1.3 *If n is superabundant, then n is an Hardy-Ramanujan integer [2].*

Theorem 1.4 *The smallest counterexample of the Robin inequality greater than 5040 must be a superabundant number [1].*

On the other hand, suppose that $n > 5040$ might be the possible smallest counterexample of the Robin inequality, then we demonstrate that necessarily $(\log n)^\beta < 1.2592 \times \log(N_m)$, where $N_m = \prod_{i=1}^m q_i$ is the primorial number of order m and $\beta = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$ when n is an Hardy-Ramanujan integer of the form $\prod_{i=1}^m q_i^{a_i}$. Moreover, we show that $(\beta - 1) \times \log \log n < 0.379121$ for this number $n > 5040$ which might be the possible smallest counterexample of the Robin inequality.

2 A Central Lemma

These are known results:

Lemma 2.1 [3]. *For $n > 1$:*

$$f(n) < \prod_{q|n} \frac{q}{q-1}. \quad (2.1)$$

Lemma 2.2 [4].

$$\prod_{k=1}^{\infty} \frac{1}{1 - \frac{1}{q_k^2}} = \zeta(2) = \frac{\pi^2}{6}. \quad (2.2)$$

The following is a key lemma. It gives an upper bound on $f(n)$ that holds for all n . The bound is too weak to prove $\text{Robins}(n)$ directly, but is critical because it holds for all n . Further the bound only uses the primes that divide n and not how many times they divide n .

Lemma 2.3 *Let $n > 1$ and let all its prime divisors be $q_1 < \dots < q_m$. Then,*

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^m \frac{q_i + 1}{q_i}.$$

Proof We use that lemma 2.1:

$$f(n) < \prod_{i=1}^m \frac{q_i}{q_i - 1}.$$

Now for $q > 1$,

$$\frac{1}{1 - \frac{1}{q^2}} = \frac{q^2}{q^2 - 1}.$$

So

$$\begin{aligned} \frac{1}{1 - \frac{1}{q^2}} \times \frac{q+1}{q} &= \frac{q^2}{q^2 - 1} \times \frac{q+1}{q} \\ &= \frac{q}{q-1}. \end{aligned}$$

Then by lemma 2.2,

$$\prod_{i=1}^m \frac{1}{1 - \frac{1}{q_i^2}} < \zeta(2) = \frac{\pi^2}{6}.$$

Putting this together yields the proof:

$$\begin{aligned} f(n) &< \prod_{i=1}^m \frac{q_i}{q_i - 1} \\ &\leq \prod_{i=1}^m \frac{1}{1 - \frac{1}{q_i^2}} \times \frac{q_i + 1}{q_i} \\ &< \frac{\pi^2}{6} \times \prod_{i=1}^m \frac{q_i + 1}{q_i}. \end{aligned}$$

3 About the p -adic order

In basic number theory, for a given prime number p , the p -adic order of a natural number n is the highest exponent $v_p \geq 1$ such that p^{v_p} divides n . This is a known result:

Lemma 3.1 *In general, we know that $\text{Robins}(n)$ holds for a natural number $n > 5040$ that satisfies either $v_2(n) \leq 19$, $v_3(n) \leq 12$ or $v_7(n) \leq 6$, where $v_p(n)$ is the p -adic order of n [5].*

We know the following lemmas:

Lemma 3.2 [5]. Let $\prod_{i=1}^m q_i^{a_i}$ be the representation of n as a product of primes $q_1 < \dots < q_m$ with natural numbers as exponents $a_1, \dots, a_m$. Then,

$$f(n) = \left(\prod_{i=1}^m \frac{q_i}{q_i - 1} \right) \times \prod_{i=1}^m \left(1 - \frac{1}{q_i^{a_i+1}} \right).$$

Lemma 3.3 [5]. Let $n > e^{23.762143}$ and let all its prime divisors be $q_1 < \dots < q_m$, then

$$\left(\prod_{i=1}^m \frac{q_i}{q_i - 1} \right) < \frac{1771561}{1771560} \times e^\gamma \times \log \log n.$$

Lemma 3.4 Robins(n) holds for all $10^{10^{10}} \geq n > 5040$ [5].

Putting together all these results, then we obtain that

Lemma 3.5 Robins(n) holds for $n > 5040$ when $v_{31}(n) \leq 3$.

Proof From lemma 3.2, we note that

$$f(n) = \left(\prod_{i=1}^m \frac{q_i}{q_i - 1} \right) \times \prod_{i=1}^m \left(1 - \frac{1}{q_i^{a_i+1}} \right) \leq \left(\prod_{i=1}^m \frac{q_i}{q_i - 1} \right) \times \left(1 - \frac{1}{31^{v_{31}(n)+1}} \right)$$

when $v_{31}(n) \leq 3$. We only need to look at the case where $v_{31}(n) = 3$ since the weaker cases follow because

$$\left(1 - \frac{1}{31^{1+1}} \right) < \left(1 - \frac{1}{31^{2+1}} \right) < \left(1 - \frac{1}{31^{3+1}} \right).$$

In this way, we obtain that

$$f(n) \leq \left(\prod_{i=1}^m \frac{q_i}{q_i - 1} \right) \times \left(1 - \frac{1}{31^{3+1}} \right) = \frac{923520}{923521} \times \left(\prod_{i=1}^m \frac{q_i}{q_i - 1} \right)$$

when $v_{31}(n) \leq 3$. With lemma 3.3, we have for $n > e^{23.762143}$

$$\frac{923520}{923521} \times \left(\prod_{i=1}^m \frac{q_i}{q_i - 1} \right) < \frac{923520}{923521} \times \frac{1771561}{1771560} \times e^\gamma \times \log \log n < e^\gamma \times \log \log n$$

since $\frac{923520}{923521} \times \frac{1771561}{1771560} < 1$. In light of lemma 3.4 and the fact that $e^{23.762143} < 10^{10^{10}}$, we then conclude that Robins(n) holds for $n > 5040$ when $v_{31}(n) \leq 3$.

4 A Particular Case

We can easily prove that $\text{Robins}(n)$ is true for certain kind of numbers:

Lemma 4.1 *$\text{Robins}(n)$ holds for $n > 5040$ when $q \leq 7$, where q is the largest prime divisor of n .*

Proof Let $n > 5040$ and let all its prime divisors be $q_1 < \dots < q_m \leq 5$, then we need to prove

$$f(n) < e^\gamma \times \log \log n$$

that is true when

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} \leq e^\gamma \times \log \log n$$

according to the lemma 2.1. For $q_1 < \dots < q_m \leq 5$,

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5}{1 \times 2 \times 4} = 3.75 < e^\gamma \times \log \log(5040) \approx 3.81.$$

However, we know for $n > 5040$

$$e^\gamma \times \log \log(5040) < e^\gamma \times \log \log n$$

and therefore, the proof is complete when $q_1 < \dots < q_m \leq 5$. The remaining case is for $n > 5040$ when all its prime divisors are $q_1 < \dots < q_m \leq 7$. $\text{Robins}(n)$ holds for $n > 5040$ when $v_7(n) \leq 6$ according to the lemma 3.1 [5]. Hence, it is enough to prove this for those natural numbers $n > 5040$ when $7^7 \mid n$. For $q_1 < \dots < q_m \leq 7$,

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5 \times 7}{1 \times 2 \times 4 \times 6} = 4.375 < e^\gamma \times \log \log(7^7) \approx 4.65.$$

However, for $n > 5040$ and $7^7 \mid n$, we know that

$$e^\gamma \times \log \log(7^7) \leq e^\gamma \times \log \log n$$

and as a consequence, the proof is complete when $q_1 < \dots < q_m \leq 7$.

5 A Better Bound

This is a known result:

Lemma 5.1 [7]. For $x > 1$:

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + B + \frac{1}{\log^2 x} \quad (5.1)$$

where

$$B = 0.2614972128 \dots$$

denotes the (Meissel-)Mertens constant [7].

We show a better result:

Lemma 5.2 *For $x \geq 11$, we have*

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + \gamma - 0.12.$$

Proof Let's define $H = \gamma - B$. The lemma 5.1 is the same as

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + \gamma - \left(H - \frac{1}{\log^2 x}\right).$$

For $x \geq 11$,

$$\left(H - \frac{1}{\log^2 x}\right) > \left(0.31 - \frac{1}{\log^2 11}\right) > 0.12$$

and thus,

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + \gamma - \left(H - \frac{1}{\log^2 x}\right) < \log \log x + \gamma - 0.12.$$

6 On a Square Free Number

We know the following results:

Lemma 6.1 [3]. *For $0 < a < b$:*

$$\frac{\log b - \log a}{b - a} = \frac{1}{(b - a)} \int_a^b \frac{dt}{t} > \frac{1}{b}. \quad (6.1)$$

Lemma 6.2 [3]. *For $q > 0$:*

$$\log(q + 1) - \log q = \int_q^{q+1} \frac{dt}{t} < \frac{1}{q}. \quad (6.2)$$

We recall that an integer n is said to be square free if for every prime divisor q of n we have $q^2 \nmid n$ [3]. Robins(n) holds for all $n > 5040$ that are square free [3].

Lemma 6.3 *For a square free number*

$$n = q_1 \times \cdots \times q_m$$

such that $q_1 < q_2 < \cdots < q_m$ are odd prime numbers, $q_m \geq 11$ and $3 \nmid n$, then:

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(n) \leq e^\gamma \times n \times \log \log(2^{19} \times n).$$

Proof By induction with respect to $\omega(n)$, that is the number of distinct prime factors of n [3]. Put $\omega(n) = m$ [3]. We need to prove the assertion for those integers with $m = 1$. From a square free number n , we obtain

$$\sigma(n) = (q_1 + 1) \times (q_2 + 1) \times \cdots \times (q_m + 1) \quad (6.3)$$

when $n = q_1 \times q_2 \times \cdots \times q_m$ [3]. In this way, for every prime number $q_i \geq 11$, then we need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \left(1 + \frac{1}{q_i}\right) \leq e^\gamma \times \log \log(2^{19} \times q_i). \quad (6.4)$$

For $q_i = 11$, we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \left(1 + \frac{1}{11}\right) \leq e^\gamma \times \log \log(2^{19} \times 11)$$

is actually true. For another prime number $q_i > 11$, we have

$$\left(1 + \frac{1}{q_i}\right) < \left(1 + \frac{1}{11}\right)$$

and

$$\log \log(2^{19} \times 11) < \log \log(2^{19} \times q_i)$$

which clearly implies that the inequality (6.4) is true for every prime number $q_i \geq 11$. Now, suppose it is true for $m - 1$, with $m \geq 2$ and let us consider the assertion for those square free n with $\omega(n) = m$ [3]. So let $n = q_1 \times \cdots \times q_m$ be a square free number and assume that $q_1 < \cdots < q_m$ for $q_m \geq 11$.

Case 1: $q_m \geq \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = \log(2^{19} \times n)$.

By the induction hypothesis we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1 + 1) \times \cdots \times (q_{m-1} + 1) \leq e^\gamma \times q_1 \times \cdots \times q_{m-1} \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

and hence

$$\begin{aligned} & \frac{\pi^2}{6} \times \frac{3}{2} \times (q_1 + 1) \times \cdots \times (q_{m-1} + 1) \times (q_m + 1) \leq \\ & e^\gamma \times q_1 \times \cdots \times q_{m-1} \times (q_m + 1) \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1}) \end{aligned}$$

when we multiply the both sides of the inequality by $(q_m + 1)$. We want to show

$$e^\gamma \times q_1 \times \cdots \times q_{m-1} \times (q_m + 1) \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1}) \leq$$

$$e^\gamma \times q_1 \times \cdots \times q_{m-1} \times q_m \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = e^\gamma \times n \times \log \log(2^{19} \times n).$$

Indeed the previous inequality is equivalent with

$$q_m \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) \geq (q_m + 1) \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

or alternatively

$$\frac{q_m \times (\log \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1}))}{\log q_m} \geq$$

$$\frac{\log \log(2^{19} \times q_1 \times \cdots \times q_{m-1})}{\log q_m}.$$

We can apply the inequality in lemma 6.1 just using $b = \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m)$ and $a = \log(2^{19} \times q_1 \times \cdots \times q_{m-1})$. Certainly, we have

$$\begin{aligned} \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log(2^{19} \times q_1 \times \cdots \times q_{m-1}) = \\ \log \frac{2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m}{2^{19} \times q_1 \times \cdots \times q_{m-1}} = \log q_m. \end{aligned}$$

In this way, we obtain

$$\begin{aligned} \frac{q_m \times (\log \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1}))}{\log q_m} > \\ \frac{q_m}{\log(2^{19} \times q_1 \times \cdots \times q_m)}. \end{aligned}$$

Using this result we infer that the original inequality is certainly satisfied if the next inequality is satisfied

$$\frac{q_m}{\log(2^{19} \times q_1 \times \cdots \times q_m)} \geq \frac{\log \log(2^{19} \times q_1 \times \cdots \times q_{m-1})}{\log q_m}$$

which is trivially true for $q_m \geq \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m)$ [3].

Case 2: $q_m < \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = \log(2^{19} \times n)$.

We need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \frac{\sigma(n)}{n} \leq e^\gamma \times \log \log(2^{19} \times n).$$

We know $\frac{3}{2} < 1.503 < \frac{4}{2.66}$. Nevertheless, we could have

$$\frac{3}{2} \times \frac{\sigma(n)}{n} \times \frac{\pi^2}{6} < \frac{4 \times \sigma(n)}{3 \times n} \times \frac{\pi^2}{2 \times 2.66}$$

and therefore, we only need to prove

$$\frac{\sigma(3 \times n)}{3 \times n} \times \frac{\pi^2}{5.32} \leq e^\gamma \times \log \log(2^{19} \times n)$$

where this is possible because of $3 \nmid n$. If we apply the logarithm to the both sides of the inequality, then we obtain

$$\log\left(\frac{\pi^2}{5.32}\right) + (\log(3+1) - \log 3) + \sum_{i=1}^m (\log(q_i + 1) - \log q_i) \leq \gamma + \log \log \log(2^{19} \times n).$$

In addition, note that $\log\left(\frac{\pi^2}{5.32}\right) < \frac{1}{2} + 0.12$. However, we know

$$\gamma + \log \log q_m < \gamma + \log \log \log(2^{19} \times n)$$

since $q_m < \log(2^{19} \times n)$. We use that lemma 6.2 for each term $\log(q+1) - \log q$ and thus,

$$0.12 + \frac{1}{2} + \frac{1}{3} + \frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq 0.12 + \sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \log \log q_m$$

where $q_m \geq 11$. Hence, it is enough to prove

$$\sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \log \log q_m - 0.12$$

but this is true according to the lemma 5.2 for $q_m \geq 11$. In this way, we finally show the lemma is indeed satisfied.

7 Robin on Divisibility

$\text{Robins}(n)$ holds for every $n > 5040$ that is not divisible by 2 [3]. We extend this property to other prime numbers:

Lemma 7.1 *$\text{Robins}(n)$ holds for all $n > 5040$ when $3 \nmid n$. More precisely: every possible counterexample $n > 5040$ of the Robin inequality must comply with $(2^{20} \times 3^{13}) \mid n$.*

Proof We will check the Robin inequality is true for every natural number $n = q_1^{a_1} \times q_2^{a_2} \times \cdots \times q_m^{a_m} > 5040$ such that $q_1, q_2, \dots, q_m$ are distinct prime numbers, $a_1, a_2, \dots, a_m$ are natural numbers and $3 \nmid n$. We know this is true when the greatest prime divisor of $n > 5040$ is lesser than or equal to 7 according to the lemma 4.1. Therefore, the remaining case is when the greatest prime divisor of $n > 5040$ is greater than or equal to 11. We need to prove

$$f(n) < e^\gamma \times \log \log n$$

that is true when

$$\frac{\pi^2}{6} \times \prod_{i=1}^m \frac{q_i + 1}{q_i} \leq e^\gamma \times \log \log n$$

according to the lemma 2.3. Using the formula (6.3) for the square free numbers, then we obtain that is equivalent to

$$\frac{\pi^2}{6} \times \frac{\sigma(n')}{n'} \leq e^\gamma \times \log \log n$$

where $n' = q_1 \times \cdots \times q_m$ is the square free kernel of the natural number n [3]. The Robin inequality has been proved for all integers n not divisible by 2 (which are bigger than 10) [3]. Hence, we only need to prove the Robin inequality is true when $2 \mid n'$. In addition, we know that $\text{Robins}(n)$ holds for every $n > 5040$ when $v_2(n) \leq 19$ according to the lemma 3.1 [5]. Consequently, we only need to prove that $\text{Robins}(n)$ holds for $n > 5040$ when $2^{20} \mid n$ and thus,

$$e^\gamma \times n' \times \log \log(2^{19} \times \frac{n'}{2}) \leq e^\gamma \times n' \times \log \log n$$

because of $2^{19} \times \frac{n'}{2} \leq n$ where $2^{20} \mid n$ and $2 \mid n'$. So,

$$\frac{\pi^2}{6} \times \sigma(n') \leq e^\gamma \times n' \times \log \log(2^{19} \times \frac{n'}{2}).$$

According to the formula (6.3) for the square free numbers and $2 \mid n'$, then,

$$\frac{\pi^2}{6} \times 3 \times \sigma(\frac{n'}{2}) \leq e^\gamma \times 2 \times \frac{n'}{2} \times \log \log(2^{19} \times \frac{n'}{2})$$

which is the same as

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(\frac{n'}{2}) \leq e^\gamma \times \frac{n'}{2} \times \log \log(2^{19} \times \frac{n'}{2})$$

where this is true according to the lemma 6.3 when $3 \nmid \frac{n'}{2}$. In addition, we know that Robins(n) holds for every $n > 5040$ when $v_3(n) \leq 12$ according to the lemma 3.1 [5]. Hence, we only need to prove that Robins(n) holds for every $n > 5040$ when $2^{20} \mid n$ and $3^{13} \mid n$. To sum up, the proof is complete.

Lemma 7.2 Robins(n) holds for all $n > 5040$ when $5 \nmid n$ or $7 \nmid n$.

Proof We need to prove

$$f(n) < e^\gamma \times \log \log n$$

when $(2^{20} \times 3^{13}) \mid n$. Suppose that $n = 2^a \times 3^b \times m$, where $a \geq 20$, $b \geq 13$, $2 \nmid m$, $3 \nmid m$ and $5 \nmid m$ or $7 \nmid m$. Therefore, we need to prove

$$f(2^a \times 3^b \times m) < e^\gamma \times \log \log(2^a \times 3^b \times m).$$

We know

$$f(2^a \times 3^b \times m) = f(3^b) \times f(2^a \times m)$$

since f is multiplicative [3]. In addition, we know $f(3^b) < \frac{3}{2}$ for every natural number b [3]. In this way, we have

$$f(3^b) \times f(2^a \times m) < \frac{3}{2} \times f(2^a \times m).$$

However, that would be equivalent to

$$\frac{3}{2} \times f(2^a \times m) = \frac{9}{8} \times f(3) \times f(2^a \times m) = \frac{9}{8} \times f(2^a \times 3 \times m)$$

where $f(3) = \frac{4}{3}$ since f is multiplicative [3]. Nevertheless, we have

$$\frac{9}{8} \times f(2^a \times 3 \times m) < f(5) \times f(2^a \times 3 \times m) = f(2^a \times 3 \times 5 \times m)$$

and

$$\frac{9}{8} \times f(2^a \times 3 \times m) < f(7) \times f(2^a \times 3 \times m) = f(2^a \times 3 \times 7 \times m)$$

where $5 \nmid m$ or $7 \nmid m$, $f(5) = \frac{6}{5}$ and $f(7) = \frac{8}{7}$. We know the Robin inequality is true for $2^a \times 3 \times 5 \times m$ and $2^a \times 3 \times 7 \times m$ when $a \geq 20$, since this is true for every natural number $n > 5040$ when $v_3(n) \leq 12$ according to the lemma 3.1 [5]. Hence, we would have

$$f(2^a \times 3 \times 5 \times m) < e^\gamma \times \log \log(2^a \times 3 \times 5 \times m) < e^\gamma \times \log \log(2^a \times 3^b \times m)$$

and

$$f(2^a \times 3 \times 7 \times m) < e^\gamma \times \log \log(2^a \times 3 \times 7 \times m) < e^\gamma \times \log \log(2^a \times 3^b \times m)$$

when $b \geq 13$.

Lemma 7.3 *Robins(n) holds for all $n > 5040$ when a prime number $11 \leq q \leq 47$ complies with $q \nmid n$.*

Proof We know that Robins(n) holds for every $n > 5040$ when $v_7(n) \leq 6$ according to the lemma 3.1 [5]. We need to prove

$$f(n) < e^\gamma \times \log \log n$$

when $(2^{20} \times 3^{13} \times 7^7) \mid n$. Suppose that $n = 2^a \times 3^b \times 7^c \times m$, where $a \geq 20$, $b \geq 13$, $c \geq 7$, $2 \nmid m$, $3 \nmid m$, $7 \nmid m$, $q \nmid m$ and $11 \leq q \leq 47$. Therefore, we need to prove

$$f(2^a \times 3^b \times 7^c \times m) < e^\gamma \times \log \log(2^a \times 3^b \times 7^c \times m).$$

We know

$$f(2^a \times 3^b \times 7^c \times m) = f(7^c) \times f(2^a \times 3^b \times m)$$

since f is multiplicative [3]. In addition, we know $f(7^c) < \frac{7}{6}$ for every natural number c [3]. In this way, we have

$$f(7^c) \times f(2^a \times 3^b \times m) < \frac{7}{6} \times f(2^a \times 3^b \times m).$$

However, that would be equivalent to

$$\frac{7}{6} \times f(2^a \times 3^b \times m) = \frac{49}{48} \times f(7) \times f(2^a \times 3^b \times m) = \frac{49}{48} \times f(2^a \times 3^b \times 7 \times m)$$

where $f(7) = \frac{8}{7}$ since f is multiplicative [3]. In addition, we know

$$\frac{49}{48} \times f(2^a \times 3^b \times 7 \times m) < f(q) \times f(2^a \times 3^b \times 7 \times m) = f(2^a \times 3^b \times 7 \times q \times m)$$

where $q \nmid m$, $f(q) = \frac{q+1}{q}$ and $11 \leq q \leq 47$. Nevertheless, we know the Robin inequality is true for $2^a \times 3^b \times 7 \times q \times m$ when $a \geq 20$ and $b \geq 13$, since this is true for every natural number $n > 5040$ when $v_7(n) \leq 6$ according to the lemma 3.1 [5]. Hence, we would have

$$\begin{aligned} f(2^a \times 3^b \times 7 \times q \times m) &< e^\gamma \times \log \log(2^a \times 3^b \times 7 \times q \times m) \\ &< e^\gamma \times \log \log(2^a \times 3^b \times 7^c \times m) \end{aligned}$$

when $c \geq 7$ and $11 \leq q \leq 47$.

Lemma 7.4 *Robins(n) holds for all $n > 5040$ when a prime number $53 \leq q \leq 953$ complies with $q \nmid n$.*

Proof We know that Robins(n) holds for every $n > 5040$ when $v_{31}(n) \leq 3$ according to the lemma 3.5. We need to prove that

$$f(n) < e^\gamma \times \log \log n$$

when $(2^{20} \times 3^{13} \times 31^4) \mid n$. Suppose that $n = 2^a \times 3^b \times 31^c \times m$, where $a \geq 20$, $b \geq 13$, $c \geq 4$, $2 \nmid m$, $3 \nmid m$, $31 \nmid m$, $q \nmid m$ and $53 \leq q \leq 953$. Therefore, we need to prove that

$$f(2^a \times 3^b \times 31^c \times m) < e^\gamma \times \log \log (2^a \times 3^b \times 31^c \times m).$$

We know that

$$f(2^a \times 3^b \times 31^c \times m) = f(31^c) \times f(2^a \times 3^b \times m)$$

since f is multiplicative [3]. In addition, we know that $f(31^c) < \frac{31}{30}$ for every natural number c [3]. In this way, we have that

$$f(31^c) \times f(2^a \times 3^b \times m) < \frac{31}{30} \times f(2^a \times 3^b \times m).$$

However, that would be equivalent to

$$\frac{31}{30} \times f(2^a \times 3^b \times m) = \frac{961}{960} \times f(31) \times f(2^a \times 3^b \times m) = \frac{961}{960} \times f(2^a \times 3^b \times 31 \times m)$$

where $f(31) = \frac{32}{31}$ since f is multiplicative [3]. In addition, we know that

$$\frac{961}{960} \times f(2^a \times 3^b \times 31 \times m) < f(q) \times f(2^a \times 3^b \times 31 \times m) = f(2^a \times 3^b \times 31 \times q \times m)$$

where $q \nmid m$, $f(q) = \frac{q+1}{q}$ and $53 \leq q \leq 953$. Nevertheless, we know the Robin inequality is true for $2^a \times 3^b \times 31 \times q \times m$ when $a \geq 20$ and $b \geq 13$, since this is true for every natural number $n > 5040$ when $v_{31}(n) \leq 3$ according to the lemma 3.5. Hence, we would have that

$$\begin{aligned} f(2^a \times 3^b \times 31 \times q \times m) &< e^\gamma \times \log \log (2^a \times 3^b \times 31 \times q \times m) \\ &< e^\gamma \times \log \log (2^a \times 3^b \times 31^c \times m) \end{aligned}$$

when $c \geq 4$ and $53 \leq q \leq 953$.

8 Helpful Lemmas

For every prime number $p_n > 2$, we define sequence $Y_n = \frac{e^{\frac{1}{2 \times \log(p_n)}}}{(1 - \frac{1}{\log(p_n)})}$.

Lemma 8.1 *For every prime number $p_n > 2$, the sequence Y_n is strictly decreasing.*

Proof For every real value $x \geq 3$, we state the function

$$f(x) = \frac{e^{\frac{1}{2 \times \log(x)}}}{(1 - \frac{1}{\log(x)})}$$

which is equivalent to

$$f(x) = g(x) \times h(u)$$

where $g(x) = e^{\frac{1}{2 \times \log(x)}}$ and $h(u) = \frac{u}{u-1}$ for $u = \log(x)$. We know that $g(x)$ decreases as $x \geq 3$ increases, Moreover, we note that $h(u)$ decreases as $u > 1$ increases where $u = \log(x) > 1$ for $x \geq 3$. In conclusion, we can see that the function $f(x)$ is monotonically decreasing for every real value $x \geq 3$ and therefore, the sequence Y_n is monotonically decreasing as well. In addition, Y_n is essentially a strictly decreasing sequence, since there is not any natural number $n > 1$ such that $Y_n = Y_{n+1}$.

In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

where $p \leq x$ means all the prime numbers p that are less than or equal to x .

Lemma 8.2 [7]. *For $x \geq 41$:*

$$\theta(x) > (1 - \frac{1}{\log(x)}) \times x.$$

Besides, we know that

Lemma 8.3 [7]. *For $x \geq 286$:*

$$\prod_{q \leq x} \frac{q}{q-1} < e^\gamma \times (\log x + \frac{1}{2 \times \log(x)}).$$

We will prove another important inequality:

Theorem 8.4 *Let $q_1, q_2, \dots, q_m$ denote the first m consecutive primes such that $q_1 < q_2 < \dots < q_m$ and $q_m > 286$. Then*

$$\prod_{i=1}^m \frac{q_i}{q_i-1} < e^\gamma \times \log(Y_m \times \theta(q_m)).$$

Proof From the theorem 8.2, we know that

$$\theta(q_m) > \left(1 - \frac{1}{\log(q_m)}\right) \times q_m.$$

In this way, we can show that

$$\begin{aligned} \log(Y_m \times \theta(q_m)) &> \log\left(Y_m \times \left(1 - \frac{1}{\log(q_m)}\right) \times q_m\right) \\ &= \log q_m + \log\left(Y_m \times \left(1 - \frac{1}{\log(q_m)}\right)\right). \end{aligned}$$

We know that

$$\begin{aligned} \log\left(Y_m \times \left(1 - \frac{1}{\log(q_m)}\right)\right) &= \log\left(\frac{e^{\frac{1}{2 \times \log(q_m)}}}{\left(1 - \frac{1}{\log(q_m)}\right)} \times \left(1 - \frac{1}{\log(q_m)}\right)\right) \\ &= \log\left(e^{\frac{1}{2 \times \log(q_m)}}\right) \\ &= \frac{1}{2 \times \log(q_m)}. \end{aligned}$$

Consequently, we obtain that

$$\log q_m + \log\left(Y_m \times \left(1 - \frac{1}{\log(q_m)}\right)\right) \geq \left(\log q_m + \frac{1}{2 \times \log(q_m)}\right).$$

Due to the theorem 8.3, we prove that

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} < e^\gamma \times \left(\log q_m + \frac{1}{2 \times \log(q_m)}\right) < e^\gamma \times \log(Y_m \times \theta(q_m))$$

when $q_m > 286$.

9 Proof of Main Theorems

Theorem 9.1 *Robins(n) holds for all $n > 5040$ when a prime number $q \leq 953$ complies with $q \nmid n$.*

Proof This is a compendium of the results from the theorem 1.2 and the lemmas 7.1, 7.2, 7.3 and 7.4.

Theorem 9.2 *Let $\frac{\pi^2}{6} \times \log \log n' \leq \log \log n$ for some number $n > 5040$ such that n' is the square free kernel of the natural number n . Then Robins(n) holds.*

Proof Let n' be the square free kernel of the natural number n . Let n' be the product of the distinct primes $q_1, \dots, q_m$. By assumption we have that

$$\frac{\pi^2}{6} \times \log \log n' \leq \log \log n.$$

For all square free $n' \leq 5040$, Robins(n') holds if and only if $n' \notin \{2, 3, 5, 6, 10, 30\}$ [3]. However, Robins(n) holds for all $n > 5040$ when $n' \in \{2, 3, 5, 6, 10, 15, 30\}$ due to the lemma 4.1. When $n' > 5040$, we know that Robins(n') holds and so

$$f(n') < e^\gamma \times \log \log n'.$$

By the previous lemma 2.3:

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^m \frac{q_i + 1}{q_i}.$$

Suppose by way of contradiction that Robins(n) fails. Then

$$f(n) \geq e^\gamma \times \log \log n.$$

We claim that

$$\frac{\pi^2}{6} \times \prod_{i=1}^m \frac{q_i + 1}{q_i} > e^\gamma \times \log \log n.$$

Since otherwise we would have a contradiction. This shows that

$$\frac{\pi^2}{6} \times \prod_{i=1}^m \frac{q_i + 1}{q_i} > \frac{\pi^2}{6} \times e^\gamma \times \log \log n'.$$

Thus

$$\prod_{i=1}^m \frac{q_i + 1}{q_i} > e^\gamma \times \log \log n',$$

and

$$\prod_{i=1}^m \frac{q_i + 1}{q_i} > f(n'),$$

This is a contradiction since $f(n')$ is equal to

$$\frac{(q_1 + 1) \times \dots \times (q_m + 1)}{q_1 \times \dots \times q_m}$$

according to the formula (6.3) for the square free numbers.

10 Consequence of Main Theorems

Theorem 10.1 *Let $\prod_{i=1}^m q_i^{a_i}$ be the representation of n as a product of the first m consecutive primes $q_1 < \dots < q_m$ with natural numbers as exponents $a_1, \dots, a_m$. If $n > 5040$ is the smallest integer such that Robins(n) does not hold, then $(\log n)^\beta < Y_m \times \log(N_m)$, where $N_m = \prod_{i=1}^m q_i$ is the primorial number of order m and $\beta = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$.*

Proof According to the theorems 1.3 and 1.4, the primes $q_1 < \dots < q_m$ must be the first m consecutive primes since $n > 5040$ should be an Hardy-Ramanujan integer. From the theorem 9.1, we know that necessarily $q_m \geq 953$. From the lemma 3.2, we note that

$$f(n) = \left(\prod_{i=1}^m \frac{q_i}{q_i - 1} \right) \times \prod_{i=1}^m \left(1 - \frac{1}{q_i^{a_i+1}} \right).$$

However, we know that

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} < e^\gamma \times \log(Y_m \times \log(N_m))$$

because of the lemma 8.4 when $q_m \geq 953$. If we multiply by $\prod_{i=1}^m \left(1 - \frac{1}{q_i^{a_i+1}} \right)$ the both sides of the previous inequality, then we obtain that

$$f(n) < e^\gamma \times \log(Y_m \times \log(N_m)) \times \prod_{i=1}^m \left(1 - \frac{1}{q_i^{a_i+1}} \right).$$

If n is the smallest integer exceeding 5040 that does not satisfy the Robin inequality, then

$$e^\gamma \times \log \log n < e^\gamma \times \log(Y_m \times \log(N_m)) \times \prod_{i=1}^m \left(1 - \frac{1}{q_i^{a_i+1}} \right)$$

because of

$$e^\gamma \times \log \log n \leq f(n).$$

That is the same as

$$\prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1} \times \log \log n < \log(Y_m \times \log(N_m))$$

which is equivalent to

$$(\log n)^\beta < Y_m \times \log(N_m)$$

where $\beta = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$. Therefore, the proof is done.

Theorem 10.2 Let $\prod_{i=1}^m q_i^{a_i}$ be the representation of n as a product of the first m consecutive primes $q_1 < \dots < q_m$ with natural numbers as exponents $a_1, \dots, a_m$. If $n > 5040$ is the smallest integer such that Robins(n) does not hold, then $(\log n)^\beta < 1.2592 \times \log(N_m)$, where $N_m = \prod_{i=1}^m q_i$ is the primorial number of order m and $\beta = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$.

Proof From the theorem 9.1, we know that necessarily $q_m \geq 953$. Using the theorem 10.1, we obtain that

$$(\log n)^\beta < 1.2592 \times \log(N_m)$$

because of the lemma 8.1 since we have that $Y_m < 1.2592$ when $q_m \geq 953$.

Theorem 10.3 Let $\prod_{i=1}^m q_i^{a_i}$ be the representation of n as a product of the first m consecutive primes $q_1 < \dots < q_m$ with natural numbers as exponents $a_1, \dots, a_m$. If $n > 5040$ is the smallest integer such that Robins(n) does not hold, then $(\beta - 1) \times \log \log n < 0.379121$, where $\beta = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$.

Proof Using the theorem 10.2, we obtain that

$$\beta \times \log \log n - \log \log(N_m) < \log 1.2592.$$

That is equivalent to

$$(\beta - 1) \times \log \log(N_m) < \log 1.2592$$

since we know that

$$\beta \times \log \log n - \log \log(N_m) > \beta \times \log \log(N_m) - \log \log(N_m) = (\beta - 1) \times \log \log(N_m).$$

If we multiply by $\frac{\pi^2}{6}$ the both sides of the previous inequality, then we have that

$$(\beta - 1) \times \frac{\pi^2}{6} \times \log \log(N_m) < \frac{\pi^2}{6} \times \log 1.2592 < 0.379121.$$

According to the theorem 9.2, we obtain that

$$(\beta - 1) \times \log \log n < 0.379121$$

since that necessarily

$$\frac{\pi^2}{6} \times \log \log(N_m) > \log \log n$$

when $n > 5040$ is a possible counterexample of the Robin inequality. Therefore, the proof is complete.

Acknowledgments

I thank Richard J. Lipton for helpful comments.

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