

Short Note on the Riemann Hypothesis

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Abstract

Robin criterion states that the Riemann Hypothesis is true if and only if the inequality $\sigma(n) < e^\gamma \times n \times \log \log n$ holds for all $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. This is known as the Robin inequality. We know that the Robin inequality is true for all $n > 5040$ which are not divisible by 2. In addition, we prove the Robin inequality is true for all $n > 5040$ which are divisible by 2. In this way, we show the Robin inequality is true for all $n > 5040$ and thus, the Riemann Hypothesis is true.

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2000 MSC: 11M26, 11A41, 11A25

1. Results

In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$ [1]. As usual $\sigma(n)$ is the sum-of-divisors function of n [2]:

$$\sum_{d|n} d$$

where $d \mid n$ means the integer d divides to n . Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say Robins(n) holds provided

$$f(n) < e^\gamma \times \log \log n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and \log is the natural logarithm. The importance of this property is:

Theorem 1.1. Robins(n) holds for all $n > 5040$ if and only if the Riemann Hypothesis is true [1].

It is known that Robins(n) holds for many classes of numbers n .

Theorem 1.2. Robins(n) holds for all $n > 5040$ that are not divisible by 2 [2].

In addition, we know that:

Theorem 1.3. Robins(n) holds for all $10^{10} \geq n > 5040$ [3].

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Let $h(n)$ be defined as

$$h(n) = \prod_{q|n} \frac{q}{q-1}.$$

These are known results:

Theorem 1.4. [2]. For $n > 1$:

$$f(n) < h(n).$$

Theorem 1.5. [4]. For $n \geq 3$:

$$h(n) < e^\gamma \times \log \log n + \frac{2.50637}{\log \log n}.$$

Let's prove our main result:

Theorem 1.6. Robins(n) holds for all $n > 5040$ that are divisible by 2.

Proof. Let's assume that $n > 5040$ is divisible by 2. We have that

$$f(n) \leq f(2) \times f\left(\frac{n}{2}\right)$$

since the function $f(n)$ is submultiplicative (that is $f(q \times r) \leq f(q) \times f(r)$) [2]. We use that theorem 1.4 to show that

$$f(2) \times f\left(\frac{n}{2}\right) \leq f(2) \times h\left(\frac{n}{2}\right) = \frac{f(2)}{h(2)} \times h(n) = \frac{3}{4} \times h(n)$$

since $f(2) = \frac{3}{2}$ and $h(2) = 2$. According to the theorem 1.5, we obtain that

$$f(n) \leq \frac{3}{4} \times h(n) < \frac{3}{4} \times \left(e^\gamma \times \log \log n + \frac{2.50637}{\log \log n} \right).$$

Hence, it is enough to prove that

$$\frac{3}{4} \times \left(e^\gamma \times \log \log n + \frac{2.50637}{\log \log n} \right) \leq e^\gamma \times \log \log n$$

which is equivalent to

$$\frac{3}{4} \times \left(1 + \frac{2.50637}{e^\gamma \times (\log \log n)^2} \right) \leq 1$$

after of dividing the both sides of the inequality by $e^\gamma \times \log \log n$. We know that Robins(n) holds for all $10^{10^{10}} \geq n > 5040$ due to the theorem 1.3. Consequently, we would have that

$$\left(\frac{3}{4} + \frac{3}{4} \times \frac{2.50637}{e^\gamma \times (\log \log n)^2} \right) < \left(\frac{3}{4} + \frac{3}{4} \times \frac{2.50637}{e^\gamma \times (\log \log 10^{10^{10}})^2} \right)$$

for $n > 10^{10^{10}}$. In this way, it is enough to show that

$$\left(\frac{3}{4} + \frac{3}{4} \times \frac{2.50637}{e^\gamma \times (\log \log 10^{10^{10}})^2} \right) \leq 1$$

which is the same as

$$\frac{3}{4} \times \frac{2.50637}{e^\gamma \times (\log \log 10^{10^{10}})^2} \leq \frac{1}{4}$$

that is equal to

$$\frac{3 \times 2.50637}{e^\gamma \times (\log \log 10^{10^{10}})^2} \leq 1$$

after of multiplying by 4. Finally, we need to prove that

$$3 \times 2.50637 \leq e^\gamma \times (\log \log 10^{10^{10}})^2$$

which is trivially true and therefore, the proof is complete. \square

This result implies the following consequences:

Theorem 1.7. *Robins(n) holds for all $n > 5040$.*

Proof. This is a direct consequence of theorems 1.2 and 1.6 \square

Theorem 1.8. *The Riemann Hypothesis is true*

Proof. This is true because of the theorems 1.1 and 1.7. \square

References

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