WHEN THE ROBIN INEQUALITY DOES NOT HOLD

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Abstract. In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US 1,000,000 prize for the first correct solution. In 1915, Ramanujan proved that under the assumption of the Riemann Hypothesis, the inequality $\sigma(n) < e^{\gamma} \times n \times \log\log n$ holds for all sufficiently large n, where $\sigma(n)$ is the sum-of-divisors function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. In 1984, Guy Robin proved that the inequality is true for all $n > 5040$ if and only if the Riemann Hypothesis is true. Let $n > 5040$ be $n = r \times q$, where q denotes the largest prime factor of n. If $n > 5040$ is the smallest number such that Robin inequality does not hold, then we show the following inequality is also satisfied: $\sqrt[q]{e} + \frac{\log\log r}{\log\log n} > 2$.

1. Introduction

As usual $\sigma(n)$ is the sum-of-divisors function of n [Cho+07]:

$$\sum_{d|n} d.$$

Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say $\mathsf{Robins}(n)$ holds provided

$$f(n) < e^{\gamma} \times \log\log n.$$

The constant γ is the Euler-Mascheroni constant, and log is the natural logarithm. The importance of this property is:

Theorem 1.1. [RH] *If* $\mathsf{Robins}(n)$ *holds for all* $n > 5040$*, then the Riemann Hypothesis is true [Rob84].*

There are several known results about the possible counterexamples of $\mathsf{Robins}(n)$ when $n > 5040$ [Cho+07]. In addition, we show that

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Theorem 1.2. [counterexample] *Let $n > 5040$ be $n = r \times q$, where q denotes the largest prime factor of n. If $n > 5040$ is the smallest number such that* $\mathsf{Robins}(n)$ *does not hold, then*

$$\sqrt[q]{e} + \frac{\log \log r}{\log \log n} > 2.$$

2. Some Useful Lemmas

The following lemma is a very helpful inequality:

Lemma 2.1. [ineq] *We have*

$$\frac{x}{1-x} \leq \frac{1}{y + y^2 + \frac{y^3}{2}}$$

where $y = 1 - x$.

Proof. We know $1 + x \leq e^x$ [Koz21]. Therefore,

$$\frac{x}{1-x} \leq \frac{e^{x-1}}{1-x} = \frac{1}{(1-x) \times e^{1-x}} = \frac{1}{y \times e^y}.$$

However, for every real number $y \in \mathbb{R}$ [Koz21]:

$$y \times e^y \geq y + y^2 + \frac{y^3}{2}$$

and this can be transformed into

$$\frac{1}{y \times e^y} \leq \frac{1}{y + y^2 + \frac{y^3}{2}}.$$

Consequently, we show

$$\frac{x}{1-x} \leq \frac{1}{y + y^2 + \frac{y^3}{2}}.$$

□

Here, it is another practical result:

Lemma 2.2. [prop] *Suppose that $n > 5040$ and let $n = r \times q$, where q denotes the largest prime factor of n. We have*

$$f(n) \leq (1 + \frac{1}{q}) \times f(r).$$

Proof. Suppose that n is the form of $m \times q^k$ where $q \nmid m$ and m and k are natural numbers. We have that

$$f(n) = f(m \times q^k) = f(m) \times f(q^k)$$

since f is multiplicative and m and q are coprimes [Voj20]. However, we know that

$$f(q^k) \leq f(q^{k-1}) \times f(q)$$

because of we notice that $f(a \times b) \leq f(a) \times f(b)$ when $a, b \geq 2$ [Voj20]. In this way, we obtain that

$$f(q^{k-1}) \times f(q) = f(q^{k-1}) \times (1 + \frac{1}{q})$$

according to the value of $f(q)$ [Voj20]. In addition, we analyze that

$$f(m) \times f(q^{k-1}) = f(m \times q^{k-1}) = f(r)$$

because f is multiplicative and m and q are coprimes [Voj20]. Finally, we obtain that

$$f(n) = f(m) \times f(q^k) \leq f(m) \times f(q^{k-1}) \times f(q) = f(r) \times (1 + \frac{1}{q})$$

and as a consequence, the proof is completed. □

3. Proof of Main Theorem

Theorem 3.1. *Let $n > 5040$ be $n = r \times q$, where q denotes the largest prime factor of n. If $n > 5040$ is the smallest number such that* $\mathsf{Robins}(n)$ *does not hold, then*

$$\sqrt[q]{e} + \frac{\log \log r}{\log \log n} > 2.$$

Proof. Suppose that n is the smallest integer exceeding 5040 that does not satisfy the Robin's inequality. Let $n = r \times q$, where q denotes the largest prime factor of n. In this way, the following inequality

$$f(n) \geq e^{\gamma} \times \log \log n$$

should be true. We know that

$$(1 + \frac{1}{q}) \times f(r) \geq f(q \times r) \geq f(n) \geq e^{\gamma} \times \log \log n$$

due to lemma 2.2 [prop]. Besides, this shows that

$$(1 + \frac{1}{q}) \times e^{\gamma} \times \log \log r > e^{\gamma} \times \log \log n$$

should be true as well. Certainly, if n is the smallest counterexample exceeding 5040 of the Robin's inequality, then $\mathsf{Robins}(r)$ holds [Cho+07]. That is the same as

$$(1 + \frac{1}{q}) \times \log \log r > \log \log n.$$

We have that

$$(1+\frac{1}{q}) \times \log\log r > \log(\log r + \log q)$$

where we notice that $\log(a+c) = \log a + \log(1+\frac{c}{a})$. This follows

$$(1+\frac{1}{q}) \times \log\log r > \log\log r + \log(1+\frac{\log q}{\log r})$$

which is equal to

$$(1+q) \times \log\log r > q \times \log\log r + q \times \log(1+\frac{\log q}{\log r})$$

and thus,

$$\log\log r > q \times \log(1+\frac{\log q}{\log r}).$$

This implies that

$$\begin{aligned} \frac{\log\log r}{\log(1+\frac{\log q}{\log r})} &= \\ \frac{\log\log r}{\log\frac{\log r + \log q}{\log r}} &= \\ \frac{\log\log r}{\log\frac{\log n}{\log r}} &= \\ \frac{\log\log r}{\log\log n - \log\log r} &= \\ \frac{\log\log r}{\log\log n \times (1-\frac{\log\log r}{\log\log n})} &= \\ \frac{\frac{\log\log r}{\log\log n}}{(1-\frac{\log\log r}{\log\log n})} &> q \end{aligned}$$

should be true. If we assume that $y = 1 - \frac{\log\log r}{\log\log n}$, then we analyze that

$$\frac{1}{y+y^2+\frac{y^3}{2}} \geq \frac{\frac{\log\log r}{\log\log n}}{(1-\frac{\log\log r}{\log\log n})}$$

because of lemma 2.1 [ineq]. As result, we have that

$$\frac{1}{y+y^2+\frac{y^3}{2}} > q$$

and therefore,

$$\frac{1}{1+y+\frac{y^2}{2}} > q \times y.$$

Since we have

$$1 + y + \frac{y^2}{2} > 1$$

then

$$\frac{1}{1 + y + \frac{y^2}{2}} < 1.$$

Consequently, we obtain that

$$1 > q \times y$$

which is the same as

$$e > e^{q \times y}.$$

Because of we have that $1 + y \leq e^y$ [Koz21], then

$$e > e^{q \times y} \geq (1 + y)^q = (2 - \frac{\log \log r}{\log \log n})^q$$

that is

$$\sqrt[q]{e} > (2 - \frac{\log \log r}{\log \log n})$$

and finally,

$$\sqrt[q]{e} + \frac{\log \log r}{\log \log n} > 2.$$

□

References

[Cho+07] YoungJu Choie et al. "On Robin's criterion for the Riemann hypothesis". In: *Journal de Théorie des Nombres de Bordeaux* 19.2 (2007), pp. 357–372. DOI: 10.5802/jtnb.591.

[Koz21] László Kozma. *Useful Inequalities.* http://www.lkozma.net/inequalities_cheat_sheet/ineq.pdf. Accessed on 2021-09-18. 2021.

[Rob84] Guy Robin. "Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann". In: *J. Math. pures appl* 63.2 (1984), pp. 187–213.

[Voj20] Robert Vojak. "On numbers satisfying Robin's inequality, properties of the next counterexample and improved specific bounds". In: *arXiv preprint arXiv:2005.09307* (2020).

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