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The Riemann Hypothesis

Frank Vega

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Abstract Robin criterion states that the Riemann Hypothesis is true if and only if the inequality $\sigma(n) < e^{\gamma} \times n \times \log\log n$ holds for all $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. We prove in another paper that the Robin inequality is true for all $n > 5040$ which are not divisible by any prime number between 2 and 953. Using this result, we show there is a contradiction just assuming the possible smallest counterexample $n > 5040$ of the Robin inequality. In this way, we prove that the Robin inequality is true for all $n > 5040$ and thus, the Riemann Hypothesis is true.

Keywords Riemann hypothesis · Robin inequality · sum-of-divisors function · prime numbers

Mathematics Subject Classification (2010) MSC 11M26 · MSC 11A41 · MSC 11A25

1 Introduction

In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$ [6]. As usual $\sigma(n)$ is the sum-of-divisors function of n [3]:

$$\sum_{d|n} d$$

where $d \mid n$ means the integer d divides to n. Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say $\mathsf{Robins}(n)$ holds provided

$$f(n) < e^{\gamma} \times \log\log n.$$

F. Vega
CopSonic, 1471 Route de Saint-Nauphary 82000 Montauban, France
ORCiD: 0000-0001-8210-4126
E-mail: vega.frank@gmail.com

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, and log is the natural logarithm. The importance of this property is:

Theorem 1.1 $\mathsf{Robins}(n)$ *holds for all* $n > 5040$ *if and only if the Riemann Hypothesis is true [6].*

It is known that $\mathsf{Robins}(n)$ holds for many classes of numbers n [8]. Let $q_1 = 2, q_2 = 3, \ldots, q_m$ denote the first m consecutive primes, then an integer of the form $\prod_{i=1}^{m} q_i^{a_i}$ with $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ is called an Hardy-Ramanujan integer [3]. A natural number n is called superabundant precisely when, for all $m < n$

$$f(m) < f(n).$$

Theorem 1.2 *If n is superabundant, then n is an Hardy-Ramanujan integer [2].*

Theorem 1.3 *The smallest counterexample of the Robin inequality greater than* 5040 *must be a superabundant number [1].*

We prove the nonexistence of such counterexample and therefore, the Riemann Hypothesis is true.

2 Known Results

These are known results:

Lemma 2.1 *[3]. For* $n > 1$*:*

$$f(n) < \prod_{q \mid n} \frac{q}{q-1}. \tag{2.1}$$

In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{q \leq x} \log q$$

where $q \leq x$ means all the prime numbers q that are less than or equal to x.

Lemma 2.2 *[7]. For* $x \geq 41$*:*

$$\theta(x) > (1 - \frac{1}{\log(x)}) \times x.$$

Besides, we know that

Lemma 2.3 *[7]. For* $x \geq 286$*:*

$$\prod_{q \leq x} \frac{q}{q-1} < e^{\gamma} \times (\log x + \frac{1}{2 \times \log(x)}).$$

For the counting prime function $\pi(x)$, we know that

Lemma 2.4 *[7]. For $x \geq 17$:*

$$\frac{x}{\log x} < \pi(x) < 1.25506 \times \frac{x}{\log x}.$$

The following lemma is crucial

Lemma 2.5 *[5]. For $x > -1$:*

$$\frac{x}{x+1} \leq \log(1+x) \leq x.$$

The smallest counterexample of the Robin inequality greater than 5040 complies with

Lemma 2.6 *If $n > 5040$ is the smallest counterexample of the Robin inequality, then $q < \log n$ where q denotes the largest prime factor of n [3].*

In addition, we know that

Lemma 2.7 *$\sigma(n)$ and $f(n)$ are multiplicatives [3]. Besides, for a prime number q and a positive integer $a \geq 0$, we have that $\sigma(q^a) = \frac{q^{a+1}-1}{q-1}$ [3]. We know that $f(q^{a+1}) > f(q^a)$ for all primes q and all $a \geq 0$.*

In basic number theory, for a given prime number q, the q-adic order of a natural number n is the highest exponent $\nu_q \geq 1$ such that q^{ν_q} divides n. This is a known result:

Lemma 2.8 *In general, we know that $\mathsf{Robins}(n)$ holds for a natural number $n > 5040$ that satisfies $\nu_2(n) \leq 19$, where $\nu_q(n)$ is the q-adic order of n [4].*

Moreover, we have that

Lemma 2.9 *$\mathsf{Robins}(n)$ holds for all $10^{10^{10}} \geq n > 5040$ [4].*

3 Useful Lemmas

We show some tools that could help us in the final proof.

Lemma 3.1 *Let $q \geq 2$ be a prime and let $b \geq 0$ be a positive integer. If $q^a \| n$, then*

$$f(q^b \times n) = f(n) \times \frac{q^{a+b+1}-1}{q^{a+b+1}-q^b}$$

where $q^a \| n$ signifies that q^a divides n, but q^{a+1} does not divide n.

Proof We assume that $q^a \| n$. Since $\sigma(n)$ and $f(n)$ are multiplicatives according to the lemma 2.7, then we would only need to study $f(q^{a+b})$ where we know from the lemma 2.7 that $\sigma(q^a) = \frac{q^{a+1}-1}{q-1}$. Then,

$$\begin{aligned} f(q^{a+b}) &= \frac{q^{a+b+1}-1}{q^{a+b}\times(q-1)} \times \frac{q^{a+1}-1}{q^a\times(q-1)} \times \frac{q^a\times(q-1)}{q^{a+1}-1} \\ &= f(q^a) \times \frac{q^{a+b+1}-1}{q^{a+b}\times(q-1)} \times \frac{q^a\times(q-1)}{q^{a+1}-1} \\ &= f(q^a) \times \frac{q^{a+b+1}-1}{q^b} \times \frac{1}{q^{a+1}-1} \\ &= f(q^a) \times \frac{q^{a+b+1}-1}{q^{a+b+1}-q^b}. \end{aligned}$$

Let's see another inequalities:

Lemma 3.2 *If* $n > 5040$ *is the smallest counterexample of the Robin inequality, then*

$$\frac{\log\log n}{\log q} < \left(1 + \frac{1}{2\times\log^2 q}\right)$$

and

$$\frac{\log\log\log n}{\log q} < \frac{\log\log q}{\log q} + \frac{1}{2\times\log^3 q}$$

when we assume that $q \geq 953$ *is the largest prime factor of* n.

Proof Let $\prod_{i=1}^{m} q_i^{a_i}$ be the representation of n as a product of the first m consecutive primes $q_1 < \cdots < q_m$ with natural numbers as exponents $a_1, \ldots, a_m$. According to the theorems 1.2 and 1.3, the primes $q_1 < \cdots < q_m$ must be the first m consecutive primes since $n > 5040$ should be an Hardy-Ramanujan integer. We assume that $q_m \geq 953$. For $q_m \geq 953$, we have that

$$\prod_{q\leq q_m} \frac{q}{q-1} < e^{\gamma} \times (\log q_m + \frac{1}{2\times\log(q_m)})$$

because of the lemma 2.3. We use that lemma 2.1 to show that

$$e^{\gamma} \times \log\log n \leq f(n) < \prod_{q\leq q_m} \frac{q}{q-1} < e^{\gamma} \times (\log q_m + \frac{1}{2\times\log(q_m)})$$

since we assume that n is a counterexample of the Robin inequality. In this way, we obtain that

$$\log\log n < (\log q_m + \frac{1}{2\times\log(q_m)})$$

which is the same as

$$\frac{\log\log n}{\log q_m} < (1 + \frac{1}{2\times\log^2(q_m)}).$$

Besides, if we apply the logarithm to the both sides of the inequality, then

$$\log\log\log n < \log\left(\log q_m \times (1 + \frac{1}{2 \times \log^2(q_m)})\right)$$

that is equivalent to

$$\log\log\log n < \log\log q_m + \log(1 + \frac{1}{2 \times \log^2(q_m)}).$$

We use that lemma 2.5 to show that

$$\log(1 + \frac{1}{2 \times \log^2(q_m)}) \leq \frac{1}{2 \times \log^2(q_m)}.$$

Therefore, we finally have that

$$\frac{\log\log\log n}{\log q_m} < \frac{\log\log q_m}{\log q_m} + \frac{1}{2 \times \log^3 q_m}.$$

Let's show another inequality

Lemma 3.3 *For all primes $q_m \geq 953$, we have that*

$$\sum_{q \leq q_m} \frac{\log\log q}{q_m} > \frac{1}{\log q_m}.$$

Proof This is the same as

$$\sum_{q \leq q_m} \log\log q > \frac{q_m}{\log q_m}.$$

According to the lemma 2.4, it is enough to show that

$$\sum_{q \leq q_m} \log\log q \geq \pi(q_m) > \frac{q_m}{\log q_m}$$

when $q_m \geq 953$. We know that for all primes $p > q_m \geq 953$, then

$$\log\log p > 1.$$

Hence, it is enough to prove that

$$\sum_{q \leq q_m} \log\log q \geq \sum_{q \leq 953} \log\log q \geq \pi(953).$$

We compute that

$$\sum_{q \leq 953} \log\log q > 274.$$

However, we know that $q_{274} = 1759 > 953$ and thus,

$$274 \geq \pi(953).$$

Therefore, the proof is done.

4 Proof of Main Theorems

Theorem 4.1 *Let $\prod_{i=1}^{m} q_i^{a_i}$ be the representation of n as a product of the first m consecutive primes $q_1 < \cdots < q_m$ with natural numbers as exponents $a_1, \ldots, a_m$. We obtain a contradiction just assuming that $n > 5040$ is the smallest integer such that $\mathsf{Robins}(n)$ does not hold.*

Proof According to the theorems 1.2 and 1.3, the primes $q_1 < \cdots < q_m$ must be the first m consecutive primes since $n > 5040$ should be an Hardy-Ramanujan integer. From the recent article [8], we know that necessarily $q_m \geq 953$. Under our assumption, we know that

$$f(n) \geq e^{\gamma} \times \log\log n.$$

For $b = 1$ and the lemma 3.1, we know that

$$f(n) = f(q_i \times m) = f(m) \times \frac{q_i^{a_i+2} - 1}{q_i^{a_i+2} - q_i}$$

for every prime q_i that divides n where $m = \frac{n}{q_i}$. If we subtract $f(m)$ to both sides of the inequality, then we obtain that

$$f(n) - f(m) \geq e^{\gamma} \times \log\log n - f(m).$$

Then,

$$\begin{aligned}
f(n) - f(m) &= f(m) \times \frac{q_i^{a_i+2} - 1}{q_i^{a_i+2} - q_i} - f(m) \\
&= f(m) \times \left(\frac{q_i^{a_i+2} - 1}{q_i^{a_i+2} - q_i} - 1 \right) \\
&= f(m) \times \left(\frac{q_i - 1}{q_i^{a_i+2} - q_i} \right) \\
&= f(m) \times \left(\frac{q_i - 1}{q_i \times (q_i^{a_i+1} - 1)} \right) \\
&= f(m) \times \left(\frac{1}{q_i \times \sigma(q_i^{a_i})} \right) \\
&= f(m') \times f(q_i^{a_i-1}) \times \left(\frac{1}{q_i \times \sigma(q_i^{a_i})} \right) \\
&= f(m') \times \frac{\sigma(q_i^{a_i-1})}{q_i^{a_i-1}} \times \left(\frac{1}{q_i \times \sigma(q_i^{a_i})} \right) \\
&< f(m') \times \frac{\sigma(q_i^{a_i})}{q_i^{a_i}} \times \left(\frac{1}{q_i \times \sigma(q_i^{a_i})} \right) \\
&= f(m') \times \frac{1}{q_i^{a_i+1}}
\end{aligned}$$

where $m' = \frac{n}{q_i^{a_i}}$ and we know that $q_i^{a_i} \| n$ and $\frac{\sigma(q_i^{a_i})}{q_i^{a_i}} > \frac{\sigma(q_i^{a_i-1})}{q_i^{a_i-1}}$ because of the lemma 2.7. In this way, we have that

$$f(m') \times \frac{1}{q_i^{a_i+1}} \geq e^{\gamma} \times \log\log n - f(m).$$

We know that $\mathsf{Robins}(m')$ and $\mathsf{Robins}(m)$ hold, since $n > 5040$ is the smallest integer such that $\mathsf{Robins}(n)$ does not hold. Consequently, we only need to prove that

$$\begin{aligned} e^{\gamma} \times \log\log m' \times \frac{1}{q_i^{a_i+1}} &> f(m') \times \frac{1}{q_i^{a_i+1}} \\ &\geq e^{\gamma} \times \log\log n - f(m) \\ &> e^{\gamma} \times \log\log n - e^{\gamma} \times \log\log m. \end{aligned}$$

As result, we have that

$$\log\log m' \times \frac{1}{q_i^{a_i+1}} > \log\log(q_i \times m) - \log\log m$$

since $m = \frac{n}{q_i}$. We know that

$$\begin{aligned} \log\log(q_i \times m) - \log\log m &= \log\left(\log q_i + \log m\right) - \log\log m \\ &= \log\left(\log m \times (1 + \frac{\log q_i}{\log m})\right) - \log\log m \\ &= \log\log m + \log(1 + \frac{\log q_i}{\log m}) - \log\log m \\ &= \log(1 + \frac{\log q_i}{\log m}). \end{aligned}$$

In addition, we know that

$$\log(1 + \frac{\log q_i}{\log m}) \geq \frac{\log q_i}{\log n}$$

using the lemma 2.5. Certainly, we will have that

$$\log(1 + \frac{\log q_i}{\log m}) \geq \frac{\frac{\log q_i}{\log m}}{\frac{\log q_i}{\log m} + 1} = \frac{\log q_i}{\log q_i + \log m} = \frac{\log q_i}{\log n}.$$

As a consequence, we would have

$$\log\log m' \times \frac{1}{q_i^{a_i+1}} > \frac{\log q_i}{\log n}$$

which is equivalent to

$$\log n \times \log\log m' > q_i^{a_i+1} \times \log q_i.$$

However, we know that

$$\log n \times \log\log n > \log n \times \log\log m'$$

and thus

$$\log n \times \log\log n > q_i^{a_i+1} \times \log q_i.$$

For $n > 10^{10^{10}}$, we have that $\log n \times \log\log n > 1$ according to the lemma 2.9. Moreover, for $q_i \geq 3$, then $q_i^{a_i+1} \times \log q_i > 1$. In addition, for $q_1 = 2$, we have that $q_1^{a_1+1} \times \log q_1 > 1$ since $a_1 \geq 20$ due to the lemma 2.8. Since the both sides of the inequality is greater that 1 for all primes q_i which divides n, then we can multiply the inequalities to obtain

$$(\log n \times \log\log n)^{\pi(q_m)} > n \times N_m \times \prod_{i=1}^{m} \log q_i$$

where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order m. If we apply the logarithm to the both sides of the inequality, then we would have

$$\pi(q_m) \times (\log\log n + \log\log\log n) > \log n + \log N_m + \sum_{i=1}^{m} \log\log q_i$$

which is equivalent to

$$\pi(q_m) \times (\log\log n + \log\log\log n) > \log n + \theta(q_m) + \sum_{i=1}^{m} \log\log q_i.$$

If we apply the lemma 2.4, then we would have

$$1.25506 \times \frac{q_m}{\log q_m} \times (\log\log n + \log\log\log n) > \log n + \theta(q_m) + \sum_{i=1}^{m} \log\log q_i.$$

Let's introduce the lemma 2.2 in this inequality and thus

$$1.25506 \times \frac{q_m}{\log q_m} \times (\log\log n + \log\log\log n) > \log n + (1 - \frac{1}{\log q_m}) \times q_m + \sum_{i=1}^{m} \log\log q_i.$$

In addition, we can transform this into

$$1.25506 \times \frac{q_m}{\log q_m} \times (\log\log n + \log\log\log n) > q_m + (1 - \frac{1}{\log q_m}) \times q_m + \sum_{i=1}^{m} \log\log q_i$$

because of the lemma 2.6. If we divide the both sides by q_m, then

$$1.25506 \times \frac{1}{\log q_m} \times (\log\log n + \log\log\log n) > 1 + 1 - \frac{1}{\log q_m} + \sum_{i=1}^{m} \frac{\log\log q_i}{q_m}.$$

According to the lemma 3.3, we know that

$$-\frac{1}{\log q_m} + \sum_{i=1}^{m} \frac{\log\log q_i}{q_m} = \alpha > 0.$$

Consequently, we would have that

$$1.25506 \times (\frac{\log\log n}{\log q_m} + \frac{\log\log\log n}{\log q_m}) > 2 + \alpha.$$

If we use the lemma 3.2, then

$$1.25506 \times (1 + \frac{1}{2 \times \log^2 q_m} + \frac{\log\log q_m}{\log q_m} + \frac{1}{2 \times \log^3 q_m}) > 2 + \alpha.$$

We know that

$$1.25506 \times (1 + \frac{1}{2 \times \log^2 q_m} + \frac{\log\log q_m}{\log q_m} + \frac{1}{2 \times \log^3 q_m})$$
$$\leq 1.25506 \times (1 + \frac{1}{2 \times \log^2 953} + \frac{\log\log 953}{\log 953} + \frac{1}{2 \times \log^3 953})$$

and we have that

$$1.25506 \times (1 + \frac{1}{2 \times \log^2 953} + \frac{\log\log 953}{\log 953} + \frac{1}{2 \times \log^3 953}) \approx 1.62266460495.$$

Consequently, we have that

$$2 > 1.25506 \times (1 + \frac{1}{2 \times \log^2 q_m} + \frac{\log\log q_m}{\log q_m} + \frac{1}{2 \times \log^3 q_m}) > 2 + \alpha > 2$$

and

$$2 > 2$$

is a contradiction. To sum up, we obtain a contradiction just assuming that $n > 5040$ is the smallest integer such that $\mathsf{Robins}(n)$ does not hold.

Theorem 4.2 $\mathsf{Robins}(n)$ *holds for all* $n > 5040$.

Proof Due to the theorem 4.1, we can assure there is not any natural number $n > 5040$ such that $\mathsf{Robins}(n)$ does not hold.

Theorem 4.3 *The Riemann Hypothesis is true.*

Proof This is a direct consequence of theorems 1.1 and 4.2

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