

# Proof of the Riemann Hypothesis

Frank Vega

the date of receipt and acceptance should be inserted later

**Abstract** The Riemann hypothesis has been considered the most important unsolved problem in mathematics. Robin criterion states that the Riemann hypothesis is true if and only if the inequality  $\sigma(n) < e^\gamma \times n \times \log \log n$  holds for all natural numbers  $n > 5040$ , where  $\sigma(n)$  is the sum-of-divisors function of  $n$  and  $\gamma \approx 0.57721$  is the Euler-Mascheroni constant. We show that the Robin inequality is true for all natural numbers  $n > 5040$  which are not divisible by the prime 3. Moreover, we prove that the Robin inequality is true for all natural numbers  $n > 5040$  which are divisible by the prime 3. Consequently, the Robin inequality is true for all natural numbers  $n > 5040$  and thus, the Riemann hypothesis is true.

**Keywords** Riemann hypothesis · Robin inequality · sum-of-divisors function · prime numbers · Riemann zeta function

**Mathematics Subject Classification (2010)** MSC 11M26 · MSC 11A41 · MSC 11A25

## 1 Introduction

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part  $\frac{1}{2}$  [3]. The Riemann hypothesis belongs to the David Hilbert's list of 23 unsolved problems [3]. Besides, it is one of the Clay Mathematics Institute's Millennium Prize Problems [3]. As usual  $\sigma(n)$  is the sum-of-divisors function of  $n$  [4]:

$$\sum_{d|n} d$$

---

F. Vega  
CopSonic, 1471 Route de Saint-Nauphary 82000 Montauban, France  
ORCID: 0000-0001-8210-4126  
E-mail: vega.frank@gmail.com

where  $d \mid n$  means the integer  $d$  divides  $n$  and  $d \nmid n$  means the integer  $d$  does not divide  $n$ . Define  $f(n)$  to be  $\frac{\sigma(n)}{n}$ . Say Robins( $n$ ) holds provided

$$f(n) < e^\gamma \times \log \log n.$$

The constant  $\gamma \approx 0.57721$  is the Euler-Mascheroni constant and  $\log$  is the natural logarithm. The importance of this property is:

**Theorem 1.1** Robins( $n$ ) holds for all natural numbers  $n > 5040$  if and only if the Riemann hypothesis is true [9].

It is known that Robins( $n$ ) holds for many classes of numbers  $n$ . Robins( $n$ ) holds for all natural numbers  $n > 5040$  that are not divisible by 2 [4]. In addition, we show that Robins( $n$ ) holds for all natural numbers  $n > 5040$  that are not divisible by 3. Furthermore, we prove that Robins( $n$ ) holds for all natural numbers  $n > 5040$  that are divisible by 3. Putting all together yields the proof that the Riemann hypothesis is true.

## 2 A Central Lemma

These are known results:

**Lemma 2.1** [4]. For  $n > 1$ :

$$f(n) < \prod_{q \mid n} \frac{q}{q-1}. \quad (2.1)$$

**Lemma 2.2** [5].

$$\prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \prod_{k=1}^{\infty} \frac{1}{1 - \frac{1}{q_k^2}} = \zeta(2) = \frac{\pi^2}{6}. \quad (2.2)$$

The following is a key lemma. It gives an upper bound on  $f(n)$  that holds for all natural numbers  $n$ . The bound is too weak to prove Robins( $n$ ) directly, but is critical because it holds for all natural numbers  $n$ . Further the bound only uses the primes that divide  $n$  and not how many times they divide  $n$ .

**Lemma 2.3** Let  $n > 1$  and let all its prime divisors be  $q_1 < \dots < q_m$ . Then,

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^m \frac{q_i + 1}{q_i}.$$

*Proof* We use that lemma 2.1:

$$f(n) < \prod_{i=1}^m \frac{q_i}{q_i - 1}.$$

Now for  $q > 1$ ,

$$\frac{1}{1 - \frac{1}{q^2}} = \frac{q^2}{q^2 - 1}.$$

So

$$\begin{aligned} \frac{1}{1 - \frac{1}{q^2}} \times \frac{q+1}{q} &= \frac{q^2}{q^2 - 1} \times \frac{q+1}{q} \\ &= \frac{q}{q-1}. \end{aligned}$$

Then by lemma 2.2,

$$\prod_{i=1}^m \frac{1}{1 - \frac{1}{q_i^2}} < \zeta(2) = \frac{\pi^2}{6}.$$

Putting this together yields the proof:

$$\begin{aligned} f(n) &< \prod_{i=1}^m \frac{q_i}{q_i - 1} \\ &\leq \prod_{i=1}^m \frac{1}{1 - \frac{1}{q_i^2}} \times \frac{q_i + 1}{q_i} \\ &< \frac{\pi^2}{6} \times \prod_{i=1}^m \frac{q_i + 1}{q_i}. \end{aligned}$$

### 3 A Basic Case

In basic number theory, for a given prime number  $p$ , the  $p$ -adic order of a natural number  $n$  is the highest exponent  $v_p \geq 1$  such that  $p^{v_p}$  divides  $n$ . This is a known result:

**Lemma 3.1** *In general, we know that Robins( $n$ ) holds for a natural number  $n > 5040$  that satisfies either  $v_2(n) \leq 19$ ,  $v_3(n) \leq 12$  or  $v_7(n) \leq 6$ , where  $v_p(n)$  is the  $p$ -adic order of  $n$  [6].*

We can easily prove that Robins( $n$ ) is true for certain kind of numbers:

**Lemma 3.2** *Robins( $n$ ) holds for  $n > 5040$  when  $q \leq 7$ , where  $q$  is the largest prime divisor of  $n$ .*

*Proof* Let  $n > 5040$  and let all its prime divisors be  $q_1 < \dots < q_m \leq 5$ , then we need to prove

$$f(n) < e^\gamma \times \log \log n$$

that is true when

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} \leq e^\gamma \times \log \log n$$

according to the lemma 2.1. For  $q_1 < \dots < q_m \leq 5$ ,

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5}{1 \times 2 \times 4} = 3.75 < e^\gamma \times \log \log(5040) \approx 3.81.$$

However, we know for  $n > 5040$

$$e^\gamma \times \log \log(5040) < e^\gamma \times \log \log n$$

and therefore, the proof is complete when  $q_1 < \dots < q_m \leq 5$ . The remaining case is for  $n > 5040$  when all its prime divisors are  $q_1 < \dots < q_m \leq 7$ . Robins( $n$ ) holds for  $n > 5040$  when  $v_7(n) \leq 6$  according to the lemma 3.1 [6]. Hence, it is enough to prove this for those natural numbers  $n > 5040$  when  $7^7 \mid n$ . For  $q_1 < \dots < q_m \leq 7$ ,

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5 \times 7}{1 \times 2 \times 4 \times 6} = 4.375 < e^\gamma \times \log \log(7^7) \approx 4.65.$$

However, for  $n > 5040$  and  $7^7 \mid n$ :

$$e^\gamma \times \log \log(7^7) \leq e^\gamma \times \log \log n$$

and as a consequence, the proof is complete when  $q_1 < \dots < q_m \leq 7$ .

#### 4 A Better Bound

This is a known result:

**Lemma 4.1** [10]. For  $x > 1$ :

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + B + \frac{1}{\log^2 x} \quad (4.1)$$

where

$$B = 0.2614972128 \dots$$

denotes the (Meissel-)Mertens constant [7].

We show a better result:

**Lemma 4.2** For  $x \geq 11$ , we have

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + \gamma - 0.12.$$

*Proof* Let's define  $H = \gamma - B$  [7]. The lemma 4.1 is the same as

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + \gamma - \left(H - \frac{1}{\log^2 x}\right).$$

For  $x \geq 11$ ,

$$\left(H - \frac{1}{\log^2 x}\right) > \left(0.31 - \frac{1}{\log^2 11}\right) > 0.12$$

and thus,

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + \gamma - \left(H - \frac{1}{\log^2 x}\right) < \log \log x + \gamma - 0.12.$$

## 5 On a Square Free Number

We know the following results:

**Lemma 5.1** [4]. For  $0 < a < b$ :

$$\frac{\log b - \log a}{b - a} = \frac{1}{(b - a)} \int_a^b \frac{dt}{t} > \frac{1}{b}. \quad (5.1)$$

**Lemma 5.2** [4]. For  $q > 0$ :

$$\log(q + 1) - \log q = \int_q^{q+1} \frac{dt}{t} < \frac{1}{q}. \quad (5.2)$$

We recall that an integer  $n$  is said to be square free if for every prime divisor  $q$  of  $n$  we have  $q^2 \nmid n$  [4].

**Lemma 5.3** Robins( $n$ ) holds for all natural numbers  $n > 5040$  that are square free [4].

**Lemma 5.4** For a square free number

$$n = q_1 \times \cdots \times q_m$$

such that  $q_1 < q_2 < \cdots < q_m$  are odd prime numbers,  $q_m \geq 11$  and  $3 \nmid n$ , then:

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(n) \leq e^\gamma \times n \times \log \log(2^{19} \times n).$$

*Proof* By induction with respect to  $\omega(n)$ , that is the number of distinct prime factors of  $n$  [4]. Put  $\omega(n) = m$  [4]. We need to prove the assertion for those integers with  $m = 1$ . From a square free number  $n$ , we obtain

$$\sigma(n) = (q_1 + 1) \times (q_2 + 1) \times \cdots \times (q_m + 1) \quad (5.3)$$

when  $n = q_1 \times q_2 \times \cdots \times q_m$  [4]. In this way, for every prime number  $q_i \geq 11$ , then we need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \left(1 + \frac{1}{q_i}\right) \leq e^\gamma \times \log \log(2^{19} \times q_i). \quad (5.4)$$

For  $q_i = 11$ , we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \left(1 + \frac{1}{11}\right) \leq e^\gamma \times \log \log(2^{19} \times 11)$$

is actually true. For another prime number  $q_i > 11$ , we have

$$\left(1 + \frac{1}{q_i}\right) < \left(1 + \frac{1}{11}\right)$$

and

$$\log \log(2^{19} \times 11) < \log \log(2^{19} \times q_i)$$

which clearly implies that the inequality (5.4) is true for every prime number  $q_i \geq 11$ . Now, suppose it is true for  $m-1$ , with  $m \geq 2$  and let us consider the assertion for those square free  $n$  with  $\omega(n) = m$  [4]. So let  $n = q_1 \times \cdots \times q_m$  be a square free number and assume that  $q_1 < \cdots < q_m$  for  $q_m \geq 11$ .

*Case 1:*  $q_m \geq \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = \log(2^{19} \times n)$ .

By the induction hypothesis we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1 + 1) \times \cdots \times (q_{m-1} + 1) \leq e^\gamma \times q_1 \times \cdots \times q_{m-1} \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

and hence

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1 + 1) \times \cdots \times (q_{m-1} + 1) \times (q_m + 1) \leq$$

$$e^\gamma \times q_1 \times \cdots \times q_{m-1} \times (q_m + 1) \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

when we multiply the both sides of the inequality by  $(q_m + 1)$ . We want to show

$$e^\gamma \times q_1 \times \cdots \times q_{m-1} \times (q_m + 1) \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1}) \leq$$

$$e^\gamma \times q_1 \times \cdots \times q_{m-1} \times q_m \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = e^\gamma \times n \times \log \log(2^{19} \times n).$$

Indeed the previous inequality is equivalent with

$$q_m \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) \geq (q_m + 1) \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

or alternatively

$$\frac{q_m \times (\log \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1}))}{\log q_m} \geq \frac{\log \log(2^{19} \times q_1 \times \cdots \times q_{m-1})}{\log q_m}.$$

We can apply the inequality in lemma 5.1 just using  $b = \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m)$  and  $a = \log(2^{19} \times q_1 \times \cdots \times q_{m-1})$ . Certainly, we have

$$\begin{aligned} \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log(2^{19} \times q_1 \times \cdots \times q_{m-1}) &= \\ \log \frac{2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m}{2^{19} \times q_1 \times \cdots \times q_{m-1}} &= \log q_m. \end{aligned}$$

In this way, we obtain

$$\frac{q_m \times (\log \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1}))}{\log q_m} > \frac{q_m}{\log(2^{19} \times q_1 \times \cdots \times q_m)}.$$

Using this result we infer that the original inequality is certainly satisfied if the next inequality is satisfied

$$\frac{q_m}{\log(2^{19} \times q_1 \times \cdots \times q_m)} \geq \frac{\log \log(2^{19} \times q_1 \times \cdots \times q_{m-1})}{\log q_m}$$

which is trivially true for  $q_m \geq \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m)$  [4].

Case 2:  $q_m < \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = \log(2^{19} \times n)$ .

We need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \frac{\sigma(n)}{n} \leq e^\gamma \times \log \log(2^{19} \times n).$$

We know  $\frac{3}{2} < 1.503 < \frac{4}{2.66}$ . Nevertheless, we could have

$$\frac{3}{2} \times \frac{\sigma(n)}{n} \times \frac{\pi^2}{6} < \frac{4 \times \sigma(n)}{3 \times n} \times \frac{\pi^2}{2 \times 2.66}$$

and therefore, we only need to prove

$$\frac{\sigma(3 \times n)}{3 \times n} \times \frac{\pi^2}{5.32} \leq e^\gamma \times \log \log(2^{19} \times n)$$

where this is possible because of  $3 \nmid n$ . If we apply the logarithm to the both sides of the inequality, then we obtain

$$\log\left(\frac{\pi^2}{5.32}\right) + (\log(3+1) - \log 3) + \sum_{i=1}^m (\log(q_i + 1) - \log q_i) \leq \gamma + \log \log \log(2^{19} \times n).$$

In addition, note that  $\log\left(\frac{\pi^2}{5.32}\right) < \frac{1}{2} + 0.12$ . However, we know

$$\gamma + \log \log q_m < \gamma + \log \log \log(2^{19} \times n)$$

since  $q_m < \log(2^{19} \times n)$ . We use that lemma 5.2 for each term  $\log(q+1) - \log q$  and thus,

$$0.12 + \frac{1}{2} + \frac{1}{3} + \frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq 0.12 + \sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \log \log q_m$$

where  $q_m \geq 11$ . Hence, it is enough to prove

$$\sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \log \log q_m - 0.12$$

but this is true according to the lemma 4.2 for  $q_m \geq 11$ . In this way, we finally show the lemma is indeed satisfied.

## 6 Main Insight

The next result is a main insight.

**Lemma 6.1** *Let  $n > 5040$  and let all its prime divisors be  $q_1 < \cdots < q_m$ . When  $q_m \geq 11$ ,  $3 \nmid n$  and  $2^{20} \mid n$ , then*

$$\frac{\pi^2}{6} \times \prod_{i=1}^m \frac{q_i + 1}{q_i} \leq e^\gamma \times \log \log n.$$

*Proof* We need to prove that

$$\frac{\pi^2}{6} \times \prod_{i=1}^m \frac{q_i + 1}{q_i} \leq e^\gamma \times \log \log n.$$

Using the formula (5.3) for the square free numbers, then we obtain that is equivalent to

$$\frac{\pi^2}{6} \times \frac{\sigma(n')}{n'} \leq e^\gamma \times \log \log n$$

where  $n' = q_1 \times \dots \times q_m$  is the square free kernel of the natural number  $n$  [4]. We know that  $2^{20} \mid n$  and thus,

$$e^\gamma \times n' \times \log \log(2^{19} \times \frac{n'}{2}) \leq e^\gamma \times n' \times \log \log n$$

because of  $2^{19} \times \frac{n'}{2} \leq n$  where  $2^{20} \mid n$  and  $2 \mid n'$ . So,

$$\frac{\pi^2}{6} \times \sigma(n') \leq e^\gamma \times n' \times \log \log(2^{19} \times \frac{n'}{2}).$$

According to the formula (5.3) for the square free numbers and  $2 \mid n'$ , then,

$$\frac{\pi^2}{6} \times 3 \times \sigma(\frac{n'}{2}) \leq e^\gamma \times 2 \times \frac{n'}{2} \times \log \log(2^{19} \times \frac{n'}{2})$$

which is the same as

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(\frac{n'}{2}) \leq e^\gamma \times \frac{n'}{2} \times \log \log(2^{19} \times \frac{n'}{2})$$

where this is true according to the lemma 5.4 when  $3 \nmid \frac{n'}{2}$  and  $q_m \geq 11$ . To sum up, the proof is complete.

## 7 Proof of the Riemann Hypothesis

Let  $q_1 = 2, q_2 = 3, \dots, q_m$  denote the first  $m$  consecutive primes, then an integer of the form  $\prod_{i=1}^m q_i^{a_i}$  with  $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$  is called an Hardy-Ramanujan integer [4]. A natural number  $n$  is called superabundant precisely when, for all natural numbers  $m < n$

$$f(m) < f(n).$$

**Lemma 7.1** *If  $n$  is superabundant, then  $n$  is an Hardy-Ramanujan integer [2].*

**Lemma 7.2** *The smallest counterexample of the Robin inequality greater than 5040 must be a superabundant number [1].*

This is an important lemma that we use:



**Lemma 7.3** Let  $x \geq 11$ . For  $y > x$  we have [8]:

$$\frac{\log \log y}{\log \log x} < \frac{\sqrt{y}}{\sqrt{x}}.$$

**Theorem 7.4** The Riemann hypothesis is true.

*Proof* Let  $\prod_{i=1}^m q_i^{a_i}$  be the representation of  $n$  as a product of primes  $q_1 < \dots < q_m$  with natural numbers as exponents  $a_1, \dots, a_m$ . In this way, we assume that  $n > 5040$  could be the smallest integer such that Robins( $n$ ) does not hold. According to the lemmas 7.1 and 7.2, the primes  $q_1 < \dots < q_m$  must be the first  $m$  consecutive primes and  $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$  since  $n > 5040$  should be an Hardy-Ramanujan integer. We know that  $n > 5040$  complies that Robins( $n$ ) holds when  $v_2(n) \leq 19$  or  $q_m \leq 7$  according to the lemmas 3.1 and 3.2. Therefore, the natural number  $n > 5040$  complies with  $q_m \geq 11$  and  $2^{20} \mid n$ . So,

$$\frac{\pi^2}{6} \times \frac{3}{4} \times \prod_{i=1}^m \frac{q_i + 1}{q_i} \leq e^\gamma \times \log \log \frac{n}{3^{v_3(n)}}$$

because of the lemma 6.1. This is equivalent to

$$\frac{\pi^2}{8} \times \prod_{i=1}^m \frac{q_i + 1}{q_i} \leq e^\gamma \times \log \log \frac{n}{3^{v_3(n)}}.$$

If we divide the two sides of the previous inequality by  $e^\gamma \times \log \log n$ , then

$$\frac{\frac{\pi^2}{8} \times \prod_{i=1}^m \frac{q_i + 1}{q_i}}{e^\gamma \times \log \log n} \leq \frac{\log \log \frac{n}{3^{v_3(n)}}}{\log \log n}.$$

We use that lemma 7.3 to show that

$$\frac{\log \log \frac{n}{3^{v_3(n)}}}{\log \log n} > \frac{1}{\sqrt{3^{v_3(n)}}}.$$

We know that Robins( $n$ ) holds for a natural number  $n > 5040$  when  $v_3(n) \leq 12$ . Consequently, we obtain that

$$\frac{\frac{\pi^2}{8} \times \sqrt{3^{12}} \times \prod_{i=1}^m \frac{q_i + 1}{q_i}}{e^\gamma \times \log \log n} \leq \frac{1}{\sqrt{3^{v_3(n) - 12}}}.$$

We have that

$$\frac{\pi^2}{8} \times \sqrt{3^{12}} \geq \frac{\pi^2}{6}.$$

We use that theorem 2.2 to show that

$$\frac{\pi^2}{6} \times \prod_{i=1}^m \frac{q_i + 1}{q_i} > \left( \prod_{i=1}^m \frac{q_i^2}{q_i^2 - 1} \right) \times \prod_{i=1}^m \frac{q_i + 1}{q_i}.$$

Besides,

$$\left( \prod_{i=1}^m \frac{q_i^2}{q_i^2 - 1} \right) \times \prod_{i=1}^m \frac{q_i + 1}{q_i} = \prod_{i=1}^m \frac{q_i}{q_i - 1}$$

because of

$$\frac{q}{q-1} = \frac{q^2}{q^2-1} \times \frac{q+1}{q}.$$

Consequently, we obtain that

$$\frac{\prod_{i=1}^m \frac{q_i}{q_i-1}}{e^\gamma \times \log \log n} < \frac{\frac{\pi^2}{8} \times \sqrt{3^{12}} \times \prod_{i=1}^m \frac{q_i+1}{q_i}}{e^\gamma \times \log \log n}$$

and thus,

$$\frac{f(n)}{e^\gamma \times \log \log n} < 1$$

according to the lemma 2.1 and  $\frac{1}{\sqrt{3^{v_3(n)-12}}} < 1$ . That is the same as

$$f(n) < e^\gamma \times \log \log n.$$

However, this is a contradiction, since  $\text{Robins}(n)$  does not hold under our initial assumption. Finally, we can see that the Riemann hypothesis is true because of the theorem 1.1.

## Acknowledgments

The author would like to thank Richard J. Lipton and Craig Helfgott for helpful comments and his mother, maternal brother and his friend Sonia for their support.

## References

1. Akbary, A., Friggstad, Z.: Superabundant numbers and the Riemann hypothesis. *The American Mathematical Monthly* **116**(3), 273–275 (2009). DOI doi:10.4169/193009709X470128
2. Alaoglu, L., Erdős, P.: On highly composite and similar numbers. *Transactions of the American Mathematical Society* **56**(3), 448–469 (1944). DOI doi:10.2307/1990319
3. Borwein, P.B., Choi, S., Rooney, B., Weirathmueller, A.: *The Riemann Hypothesis: A Resource for the Afficionado and Virtuoso Alike*, vol. 27. Springer Science & Business Media (2008)
4. Choie, Y., Lichiardopol, N., Moree, P., Solé, P.: On Robin's criterion for the Riemann hypothesis. *Journal de Théorie des Nombres de Bordeaux* **19**(2), 357–372 (2007). DOI doi:10.5802/jtnb.591
5. Edwards, H.M.: *Riemann's Zeta Function*. Dover Publications (2001)
6. Hertlein, A.: Robin's Inequality for New Families of Integers. *Integers* **18** (2018)
7. Mertens, F.: Ein Beitrag zur analytischen Zahlentheorie. *J. reine angew. Math.* **1874**(78), 46–62 (1874). DOI 10.1515/crll.1874.78.46. URL <https://doi.org/10.1515/crll.1874.78.46>
8. Nazardonyavi, S., Yakubovich, S.: Superabundant numbers, their subsequences and the Riemann hypothesis (2013). At <https://arxiv.org/pdf/1211.2147.pdf>
9. Robin, G.: Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann. *J. Math. pures appl* **63**(2), 187–213 (1984)
10. Rosser, J.B., Schoenfeld, L.: Approximate Formulas for Some Functions of Prime Numbers. *Illinois Journal of Mathematics* **6**(1), 64–94 (1962). DOI doi:10.1215/ijm/1255631807