Theory of Geometric Shape Conversion

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Abstract: In this paper we have proposed a new theory about geometric shape conversion. Through a particular process where the number of steps of the process tends to infinity, keeping one or more shapes of two or more shapes the same throughout the whole process and by gradually changing its size, shifting another shape toward another specific new size or new shape is the main topic of discussion of the theory. Again, if the shape that changes its size does not shift toward another specific new size or new shape then it is considered as error and this theory also discussed the properties of that error.

Keywords: Geometric shapes, shape conversion, new theory, geometric limit, shape conversion error.

1. Introduction

Geometry is the most important sector of mathematics. In this theory we will discuss about geometric shape conversion where the shapes can be one-dimensional or two-dimensional or three-dimensional. Through a particular process where the number of steps of the process tends to infinity, keeping one or more shapes of two or more shapes the same throughout the whole process and by gradually changing its size, shifting another shape toward another specific new size or new shape is the main topic of discussion of the theory. Again, if the shape that changes its size does not shift toward another specific new size or new shape then it is considered as error and this theory also discussed the properties of that error. Here every shape is important and necessary for that particular process. In other words, through a particular process where the number of steps is infinite, using two or more shapes to gradually shift one shape of those two or more shapes toward a specific new size or new shape is the basic concept of the theory. For a better understanding of the main concept of theory the following definitions are important.

Definition 1.1 (Base shape): The shape that always remains unchanged or same is called base shape.

Definition 1.2 (Converting shape): The shape that gradually changes its size and shifts toward another specific new size or new shape is called Converting shape.

Definition 1.3 (Final shape): The converting shape gradually changes its size and shifts toward another specific new size or new shape and that specific size or shape is called final shape.

Definition 1.4 (Converting process): Through a particular process whose number of steps tends to infinity, a converting shape gradually changes its size that particular process is called converting process.

All the base shapes are used in every step of the converting process. Now, by using the definitions we can write that by using one or more base shapes, a converting shape and a converting process, shifting the converting shape toward a final shape is the main concept of the theory. As an example, if we take a circle as a base shape and an isosceles triangle as a converting shape, then by using a converting process we can get an equilateral triangle as the final shape. There are many such examples in the paper. Again, if the converting shape does not shift toward a final shape then it will be considered as error.

This theory reveals a new aspect of geometry and demonstrates a different relation between geometry and limit. The number of steps of the converting process cannot be finite. By using a process where the number of steps is infinite, shifting one shape towards another specific size or shape is the uniqueness of the theory.

2. Main concepts and definitions

For a better understanding of the theory the following concepts and definitions are important.

Base shape: A base shape is an important part of a converting process. The size of a base shape remains the same in the whole converting process. All the base shapes are necessary for each step of the converting process. Base shapes help the converting shape to gradually change its size and shift toward a final shape. All the base shapes must be used in each step of the converting process and without the base shapes the converting process is incomplete.

Definition 2.1 (friend shape): Two same types of shapes are called friend shape of each other.

As an example, any two triangles will be called friend shape of each other. Again, a triangle and a quadrilateral are not same type of shapes. So, a triangle and a quadrilateral can never be a friend shape of each other.

Converting shape: In the introduction section we have given the definition of the converting shape. By the help of the base shapes and converting process, a converting shape gradually changes its size. Generally a converting shape shift toward a final shape but in some cases there is no final shape.

Concept of converting process: In the introduction section we have given the definition of converting process. According to the definition we know that the number of steps of the converting process tends to infinity. In other words, the number of steps of the converting process is infinite. The converting process describes how a converting shape gradually changes its size. There are two types of converting process. They are as follows:

1. Type-1 converting process
2. Type-2 converting process

Definition 2.2 (Main position): The position of one or more base shapes and a converting shape at the beginning of the converting process is called main position.

Type-1 converting process: In the first step of this converting process, by using one or more base shapes and a converting shape, a process will be applied on the converting shape so that we get a specific friend shape of the converting shape.

In the second step, by using those one or more base shapes and the specific friend shape of the converting shape obtained from the first step, the same process of first step which has been applied on the converting shape will be applied on the friend shape of the converting shape obtained from the first step so that we get another specific friend shape of the converting shape.

Here we have used a word specific friend shape which indicates that in every step of the converting process there can be more than one friend shapes of the converting shape. But one of them should be considered as main friend shape of the converting shape.

Definition 2.3 (nth main friend shape of a converting shape): The specific friend shape of a converting shape which will be obtained in the n th step of the converting process will be called n th main friend shape of the converting shape.

n th step of type-1 converting process: In the n th step of the converting process, by using one or more base shapes and the main friend shape of the converting shape obtained from the $(n - 1)$ th step, the same process of second step will be applied on the $(n - 1)$ th main friend shape of the converting shape so that we get n th main friend shape of the converting shape where $n > 2$.

Type-2 converting process and the nth step: In the first step of this converting process, by using one or more base shapes and a converting shape, a process will be applied on the converting shape so that we get a specific friend shape of the converting shape.

In the second step, by using those one or more base shapes and the specific friend shape of the converting shape obtained in the first step, a different process from the process which has been applied on the converting shape in the first step will be applied on the friend shape of the converting shape obtained from the first step so that we get another specific friend shape of the converting shape.

The main aim is to get a friend shape of the converting shape in each step of the converting process. Let n is the number of steps. When $n > 1$, in the n th step a process will be applied on the specific friend shape of the converting shape obtained from the $(n - 1)$ th step so that a new specific friend shape of the converting shape is obtained. In this type of converting process, a same process will not be applied in every step to get a new specific friend shape of the converting shape. But all the processes applied in the steps of this converting process must follow a particular pattern. By using that pattern the procedure of n th step will be easily described.

Note: Here the specific friend shape of the converting shape is the n th main friend shape of the converting shape.

Definition 2.3 (nth new size of a converting shape): The n th main friend shape of a converting shape obtained in the n th step of the converting process is called n th new size of that converting shape.

The concept of gradually shifting one shape toward another one: In every step of the converting process, a friend shape of the converting shape will be obtained. That friend shape will be considered as the new size of the converting shape. In this way the converting shape will gradually change its size in every step of the converting process and obtain a new size. When the number of steps of the converting process tends to infinity, if then the new size of the converting shape approaches to a specific size or shape then that specific size or shape will be considered as final shape otherwise it will be considered as error. We must get a friend shape of the converting shape in every step of the converting process.

Now it is clear from the above concepts and definitions that in every step of the converting process a process will be applied so that we get a friend shape of the converting shape. The obtained friend shape of the converting shape in a step will be considered as the new size of the converting shape. When the number of steps of the converting process will tend to infinity, a new specific size of the converting shape or a new specific different shape will be obtained. To make these concepts clear let us give an example. Let, a circle is the base shape and an isosceles triangle is the converting shape. The three vertices of the isosceles will be on the circumference of the circle and this is the main position. In the first step we will increase the three medians of the isosceles triangle and the three increased medians will intersect at three points of the circumference of the circle. Connecting those three points we will get a new isosceles triangle which will be considered as 1st main friend shape of the converting shape. In the second step we will apply the same process on the 1st main friend shape of the converting shape. This time we will increase the three medians of the 1st main friend shape of the converting shape and the three increased medians will intersect at three points of the circumference of the circle. Connecting those three points we will get a new isosceles triangle which will be considered as 2nd main friend shape of the converting shape. Now we can write the n th step of the converting process and it will be, In the n th step of the converting process, by using the base shape and the main friend shape of the converting shape obtained from the $(n - 1)$ th step, the same process of second step will be applied on the $(n - 1)$ th main friend shape of the converting shape so that we get n th main friend shape of the converting shape where $n > 2$. In the n th step, the n th main friend shape of converting shape will be an equilateral triangle when $n \rightarrow +\infty$. So, in this case the equilateral triangle is the final shape. The proof of this example has been shown in this paper.

3. Main topics of discussion of the theory

There are four main topics of discussion of the theory. They are as follows:

1. Shape conversion
2. Secondary shape conversion
3. Shape conversion error

4. Secondary shape conversion error

3.1 Shape conversion

Definition 3.1.1 (Shape conversion): By using a converting process, one or more base shapes, a converting shape, and a main position, shifting the converting shape toward a final shape is called shape conversion.

Basically shape conversion means the conversion of the converting shape toward a final shape. There are four types of shape conversion. They are as follows:

1. One-dimensional shape conversion
2. Two-dimensional shape conversion
3. Three-dimensional shape conversion
4. Mixed-dimensional shape conversion

Definition 3.1.2 (One-dimensional shape conversion): The shape conversion in which, all the base shapes, the converting shape and the final shape are one-dimensional that shape conversion is called one-dimensional shape conversion.

Definition 3.1.3 (Two-dimensional shape conversion): The shape conversion in which, all the base shapes, the converting shape and the final shape are two-dimensional that shape conversion is called two-dimensional shape conversion.

Definition 3.1.4 (Three-dimensional shape conversion): The shape conversion in which, all the base shapes, the converting shape and the final shape are three-dimensional that shape conversion is called three-dimensional shape conversion.

Definition 3.1.5 (Mixed-dimensional shape conversion): The shape conversion in which, all the base shapes and the converting shape are same dimensional but the final shape is different dimensional that shape conversion is called mixed-dimensional shape conversion.

The properties of a shape conversion are as follows:

1. There is only one converting shape in a shape conversion but the number of base shape can be more than one.
2. By the help of base shapes the converting shape gradually changes its size. So in every step of the converting process all the base shapes must be used.
3. In the first step the converting shape and in another step the main friend shape of the converting shape must be used. In a step of the converting process, a process must be applied on the main friend shape of the converting shape obtained from the previous step to get a new main friend shape of the converting shape and in case of the first step a process must be applied on the converting shape. In case of applying the process all the base shapes must be used.
4. By using the same base shapes, converting process and main position if a friend shape of the converting shape is used as converting shape then the same final shape will be obtained. In this case the friend shape of the converting shape cannot be congruent to the converting shape and there will be such infinitely many friend shapes of the converting shape.

Steps of shape conversion

There are 8 steps of shape conversion. They are as follows

1. Base shape or base shapes

2. Converting shape
3. Main position
4. First few steps of converting process
5. nth step of the converting process
6. Final shape
7. Figure
8. Proof

1. Base shape or base shapes

In this step we have to mention the names and properties of the base shapes. If there is only one base shape then it will be called base shape. But if there are more than one base shape then the base shapes will be called base shape 1, base shape 2 etc.

2. Converting shape

In this step we have to mention the name and properties of the converting shape. The properties of the converting shape must support the main position.

3. Main position

The position of one or more base shapes and converting shape at the beginning of the converting process will be mentioned in this step.

4. First few steps of converting process

The first few steps of the converting process (more than one) will be described in this step so that we can get a clear idea about the procedure of nth step of the converting process. In case of naming the sides of 1st main friend shape of the converting shape, we have to use 1 as subscript below the letters which will be used to name the sides of that converting shape. Again, In case of naming the sides of 2nd main friend shape of a converting shape, we have to use 2 as subscript below the letters which will be used to name the sides of that converting shape. In this pattern we have to continue naming the sides of the main friend shape of the converting shape in every step. But if some common letters are used to name any base shape and converting shape then we do not need to use the subscripts below those letters.

5. nth step of the converting process

The general statement of the processes applied in every step of the converting process will be mentioned and described in this step.

6. Final shape

In this step we have to mention the name, properties and position of the final shape. In case of naming the sides of the final shape we have to use ∞ as subscript below the letters which will be used to name the sides of that converting shape. But if some common letters are used to name any base shape and converting shape then we do not need to use ∞ as the subscript below those letters in case of naming the final shape.

7. Figure

The figure of a shape conversion is divided into two parts. They are as follows:

- i. Converting process visualization
- ii. Final shape visualization

i. Converting process visualization

In this step, we have to present at least first three steps of the converting process by using figures so that we get a clear idea. We have to present every step of the converting process by using different figures and by using arrow symbol we have to show that the converting process starts from the step one and goes up to infinity.

ii. Final shape visualization

In this step, we have to present the base shapes and the final shape by using figure.

8. Proof

In this step, we have to present a proof of the shape conversion by using the figures of the Converting process visualization step. But in this step we can use different figures with additional properties for the convenience of proof.

According to the above definitions and concepts here we will present five shape conversions. To prove those shape conversions easily we have to prove two formulas. The formulas are as follows:

Note: Here $f^2(x)$ means $f(f(x))$, $f^3(x)$ means $f(f(f(x)))$ etc.

Formula 1: Let, $x > a$ and $f(x) > b \geq a$. If $f(x) < x$, then $\lim_{n \rightarrow +\infty} f^n(x) = b$.

Proof: Here, $f(x) < x$ and $f(x) > b \geq a$, so $f(f(x)) < f(x) \Rightarrow f^2(x) < f(x)$

Similarly, $f(f(f(x))) < f(f(x)) \Rightarrow f^3(x) < f^2(x)$

So we get,

$$f(x) > f^2(x) > f^3(x) > \dots > f^n(x) > b$$

Now, if we increase the value of n the value of $f^n(x)$ will decrease and get closer to b . If n approaches to $+\infty$ $f^n(x)$ will approach to b . So,

$$\lim_{n \rightarrow +\infty} f^n(x) = b$$

Formula 2: Let, $f(x) > a$ when $x < a$ and $f(x) < a$ when $x > a$. If $f(x) > f(f(f(x))) > a$ and $f(f(x)) < f(f(f(f(x)))) < a$ when $x < a$ and $f(x) < f(f(f(x))) < a$ and $f(f(x)) > f(f(f(f(x)))) > a$ when $x > a$ then $\lim_{n \rightarrow +\infty} f^{2n-1}(x) = \lim_{n \rightarrow +\infty} f^{2n}(x) = a$.

Proof: Let, when $x < a$, $f(x) > f(f(f(x))) > a \Rightarrow f(x) > f^3(x) > a$. Again, $f(f(x)) < a$ when $x < a$. So $f(f(f(x))) > a$ as $f(x) > a$ when $x < a$.

As $f(x) > f(f(f(x))) > a$ when $x < a$ so,

$$\begin{aligned} f(f(f(x))) &> f(f(f(f(f(x)))) > a \\ &\Rightarrow f^3(x) > f^5(x) > a \end{aligned}$$

Similarly, $f^5(x) > f^7(x) > a$. So we get,

$$f(x) > f^3(x) > f^5(x) > f^7(x) > \dots > f^{2n-1}(x) > a \text{ when } x < a$$

Again,

When $x < a$, $f(f(x)) < f(f(f(f(x)))) < a \Rightarrow f^2(x) < f^4(x) < a$. Again, $f(f(f(x))) > a$ when $x < a$. So $f(f(f(f(x)))) < a$ as $f(x) < a$ when $x > a$.

As, $f(f(x)) < f(f(f(f(x)))) < a$ when $x < a$ so,

$$f\left(f\left(f\left(f\left(f(x)\right)\right)\right)\right) < f\left(f\left(f\left(f\left(f\left(f\left(f(x)\right)\right)\right)\right)\right)\right) < a$$

$$\Rightarrow f^4(x) < f^6(x) < a$$

Similarly, $f^6(x) < f^8(x) < a$. So we get,

$$f^2(x) < f^4(x) < f^6(x) < f^8(x) < \dots < f^{2n}(x) < a \text{ when } x < a$$

Now, if we increase the value of n the value of $f^{2n-1}(x)$ will decrease and the value of $f^{2n}(x)$ will increase and get closer to a when $x < a$. If n approaches to $+\infty$ $f^{2n-1}(x)$ and $f^{2n}(x)$ will approach to a . So,

$$\lim_{n \rightarrow +\infty} f^{2n-1}(x) = \lim_{n \rightarrow +\infty} f^{2n}(x) = a$$

By using the similar process it can be shown that

$$f(x) < f^3(x) < f^5(x) < f^7(x) < \dots < f^{2n-1}(x) < a \text{ when } x > a$$

And,

$$f^2(x) > f^4(x) > f^6(x) > f^8(x) > \dots > f^{2n}(x) > a \text{ when } x > a$$

Now, if we increase the value of n the value of $f^{2n-1}(x)$ will increase and the value of $f^{2n}(x)$ will decrease and get closer to a when $x > a$. If n approaches to $+\infty$ $f^{2n-1}(x)$ and $f^{2n}(x)$ will approach to a . So,

$$\lim_{n \rightarrow +\infty} f^{2n-1}(x) = \lim_{n \rightarrow +\infty} f^{2n}(x) = a$$

I think the proofs of these formulas are not so much important. The statements are the proofs itself.

Shape conversion 1

i. Base shape: A circle of radius r .

ii. Converting shape: An isosceles triangle where the length of the base is a and the length of the legs is b . The properties of the isosceles triangle must follow the main position.

iii. Main position: The three vertices of the converting shape will be on the circumference of the base shape.

iv. First two steps of converting process: In the first step we will increase the three medians of the converting shape and the three increased medians will intersect at three points of the circumference of the base shape. Connecting those three points we will get a new isosceles triangle which will be considered as 1st main friend shape of the converting shape. In the second step we will apply the same process on the 1st main friend shape of the converting shape. This time we will increase the three medians of the 1st main friend shape of the converting shape and the three increased medians will intersect at three points of the circumference of the base shape. Connecting those three points we will get another new isosceles triangle which will be considered as 2nd main friend shape of the converting shape.

v. nth step of the converting process: In the n th step of the converting process, by using the base shape and the main friend shape of the converting shape obtained in the $(n-1)$ th step, the same process of second step will be applied on the $(n-1)$ th main friend shape of the converting shape so that we get n th main friend shape of the converting shape where $n > 2$.

vi. Final shape: An equilateral triangle. The length of one side of the equilateral triangle will be $\sqrt{3}r$ and the three vertices of the final shape will be on the circumference of the base shape.

vii. Figure

(a) Converting process visualization

Figure 3.1.1: Here $AB_1CA_1BC_1$ is the base shape. ABC is the converting shape. $A_1B_1C_1$ is the 1st main friend shape of the converting shape. In the second step $A_2B_2C_2$ is the 2nd main friend shape of the converting shape. In the third step $A_3B_3C_3$ is the third main friend shape of the converting shape. Here, in every step the name of the base shape has changed but the size is always same.

(b) Final shape visualization

Figure 3.1.2: Here $A_\infty B_\infty C_\infty$ is the final shape.

viii. Proof

Before we prove the shape conversion, it is necessary to generate the formulas to calculate the length of the sides of the second triangle obtained by extending the medians of the first isosceles triangle inscribed in a circle. In other words, If we extend the three medians of any inscribed isosceles triangle in a circle, the extended medians will intersect the circumference of the circle at three points. Connecting those three points of intersection we will get a new triangle. Now we have to generate the formulas to calculate the length of the sides of this triangle. In this case we will use a method. We will consider the base of the first isosceles triangle as the main side and one of the legs as the secondary side. Then we will represent the length of secondary side by using the length of main side. Let α is the ratio of secondary side and main side. In this case we need some important theorems. They are as follows:

Theorem 3.1 (Apollonius's theorem): The sum of the squares of any two sides of any triangle equals to twice its square on half of the third side, along with the twice of its square on the median bisecting the third side.

Theorem 3.2: Two angles at the circumference subtended by the same arc are equal.

Theorem 3.3: The centroid of a triangle divides each median in the ratio 2:1.

Figure 3.1.3: ABC is the first isosceles triangle inscribed in the circle $AFBDCE$. After extending the medians of the triangle ABC , they intersect at three points D, E and F of the circumference of the circle. Connecting D, E and E, F and F, D , the second triangle DEF has been obtained.

Here the main side of the triangle ABC is BC . Let the secondary side is AB . So, $\alpha = \frac{AB}{BC} \Rightarrow AB = \alpha BC$. Again, $AC = \alpha BC$.

By using the Apollonius's theorem we can write from the figure 3.1.3,

$$\begin{aligned} AB^2 + AC^2 &= 2(AH^2 + BH^2) \\ \Rightarrow AH^2 &= \frac{(\alpha BC)^2 + (\alpha BC)^2}{2} - \left(\frac{BC}{2}\right)^2 \\ AH &= \frac{BC\sqrt{4\alpha^2 - 1}}{2} \end{aligned}$$

According to theorem 4.4 we can write,

$$\begin{aligned} AG:GH &= 2:1 \\ \Rightarrow AH:GH &= 3:1 \\ \Rightarrow GH &= \frac{BC\sqrt{4\alpha^2 - 1}}{2} \times \frac{1}{3} \\ GH &= \frac{BC\sqrt{4\alpha^2 - 1}}{6} \end{aligned}$$

Look at the figure 3.1.3, we can see the two angles $\angle DBC$ and $\angle DAC$ are subtended by the same arc CD . According to theorem 3.2, $\angle DBC = \angle DAC$

Again, the two angles $\angle ADB$ and $\angle ACB$ are subtended by the same arc AB . That's why $\angle ADB = \angle ACB$

Now in case of triangle BHD and triangle AHC , $\angle DBH = \angle HAC$ and $\angle HDB = \angle ACH$. So, $\triangle BHD$ and $\triangle AHC$ are similar. That's why,

$$\begin{aligned} \frac{DH}{HC} &= \frac{BH}{AH} \\ \Rightarrow DH &= \frac{BC}{2} \times \frac{BC}{2} \times \frac{2}{BC\sqrt{4\alpha^2 - 1}} \end{aligned}$$

$$DH = \frac{BC}{2\sqrt{4\alpha^2 - 1}}$$

Now,

$$GD = GH + DH$$

$$\begin{aligned} &= \frac{BC\sqrt{4\alpha^2 - 1}}{6} + \frac{BC}{2\sqrt{4\alpha^2 - 1}} \\ &= \frac{BC(2\alpha^2 + 1)}{3\sqrt{4\alpha^2 - 1}} \end{aligned}$$

Again, according to the Apollonius's theorem we can write from the figure 3.1.3,

$$\begin{aligned} AC^2 + BC^2 &= 2(CJ^2 + AJ^2) \\ \Rightarrow CJ^2 &= \frac{(\alpha BC)^2 + BC^2}{2} - \left(\frac{\alpha BC}{2}\right)^2 \\ CJ &= \frac{BC\sqrt{\alpha^2 + 2}}{2} \end{aligned}$$

According to theorem 3.3,

$$\begin{aligned} CG:GJ &= 2:1 \\ \Rightarrow CG:CJ &= 2:3 \\ \Rightarrow CG &= \frac{BC\sqrt{\alpha^2 + 2}}{2} \times \frac{2}{3} \\ CG &= \frac{BC\sqrt{\alpha^2 + 2}}{3} \end{aligned}$$

Again, we can see the two angles $\angle DFC$ and $\angle DAC$ are subtended by the same arc CD . According to theorem 3.2, $\angle DFC = \angle DAC$. Similarly, the two angles $\angle ACF$ and $\angle ADF$ are subtended by the same arc AF . That's why $\angle ACF = \angle ADF$.

Now is case of $\triangle FGD$ and $\triangle AGC$, $\angle DFG = \angle GAC$ and $\angle FDG = \angle GCA$. So, $\triangle FGD$ and $\triangle AGC$ are similar. That's why,

$$\begin{aligned} \frac{DF}{AC} &= \frac{GD}{CG} \\ \Rightarrow DF &= \frac{BC(2\alpha^2 + 1)}{3\sqrt{4\alpha^2 - 1}} \times \alpha BC \times \frac{3}{BC\sqrt{\alpha^2 + 2}} \\ DF &= \frac{\alpha BC(2\alpha^2 + 1)}{\sqrt{(4\alpha^2 - 1)(\alpha^2 + 2)}} \end{aligned}$$

By using the similar process we can show that,

$$DE = \frac{\alpha BC(2\alpha^2 + 1)}{\sqrt{(4\alpha^2 - 1)(\alpha^2 + 2)}}$$

According to theorem 3.3,

$$JG:CG = 1:2$$

$$\Rightarrow JG = \frac{CG}{2}$$

$$JG = \frac{BC\sqrt{\alpha^2 + 2}}{6}$$

Now in case of, $\triangle AFJ$ and $\triangle BCJ$, $\angle FAJ = \angle BCJ$ and $\angle AFJ = \angle JBC$ (Theorem 3.2). So, $\triangle AFJ$ and $\triangle BCJ$ are similar. That's why,

$$\frac{FJ}{BJ} = \frac{AJ}{CJ}$$

$$\Rightarrow FJ = \frac{\alpha BC}{2} \times \frac{\alpha BC}{2} \times \frac{2}{BC\sqrt{\alpha^2 + 2}}$$

$$FJ = \frac{\alpha^2 BC}{2\sqrt{\alpha^2 + 2}}$$

Now,

$$FG = FJ + JG$$

$$= \frac{\alpha^2 BC}{2\sqrt{\alpha^2 + 2}} + \frac{BC\sqrt{\alpha^2 + 2}}{6}$$

$$= \frac{BC(2\alpha^2 + 1)}{3\sqrt{\alpha^2 + 2}}$$

Again, by using the Apollonius's theorem we can write,

$$BG = \frac{BC\sqrt{\alpha^2 + 2}}{3}$$

In case of $\triangle EFG$ and $\triangle BCG$, $\angle EFG = \angle CBG$ and $\angle FEG = \angle BCG$ (Theorem 3.2). So, $\triangle EFG$ and $\triangle BCG$ are similar. That's why,

$$\frac{EF}{BC} = \frac{FG}{BG}$$

$$\Rightarrow EF = \frac{BC(2\alpha^2 + 1)}{3\sqrt{\alpha^2 + 2}} \times BC \times \frac{3}{BC\sqrt{\alpha^2 + 2}}$$

$$EF = \frac{BC(2\alpha^2 + 1)}{\alpha^2 + 2}$$

Here, $DF = DE$ that's why triangle DEF is an isosceles triangle. So it is proved that, If the medians of an isosceles triangle inscribed in a circle are extended, they will intersect at three points of the circumference of the circle. The triangle obtained by connecting the three points of intersection will also be an isosceles triangle. Depending on these calculations now we will prove the shape conversion 1.

Look at the figure 3.1.1, in the first step of the converting process, ABC is the converting shape and the length of base of the converting shape is a and length of the legs is b . According to the above process, the ratio of the secondary side and the main side is

$$\alpha = \frac{b}{a}$$

If $b > a$ then $\alpha > 1$ and if $b < a$ then $0 < \alpha < 1$. According to the above calculations the triangle $A_1B_1C_1$ is an isosceles triangle. Again,

$$A_1B_1 = A_1C_1 = \frac{\alpha BC(2\alpha^2 + 1)}{\sqrt{(4\alpha^2 - 1)(\alpha^2 + 2)}} \text{ and } B_1C_1 = \frac{BC(2\alpha^2 + 1)}{\alpha^2 + 2}$$

In this case, the ratio of the secondary side and the main side is

$$\alpha_1 = \frac{\alpha\sqrt{\alpha^2 + 2}}{\sqrt{4\alpha^2 - 1}}$$

In other way we can write,

$$\alpha_1 = f(\alpha) = \frac{\alpha\sqrt{\alpha^2 + 2}}{\sqrt{4\alpha^2 - 1}}$$

Similarly in the second step of the converting process, the ratio of the secondary side and the main side of the 2nd main friend shape of the converting shape will be,

$$\alpha_2 = f(f(\alpha)) = f^2(\alpha) = \frac{\alpha_1\sqrt{\alpha_1^2 + 2}}{\sqrt{4\alpha_1^2 - 1}}$$

Similarly in the nth step of the converting process, the ratio of the secondary side and the main side of the nth main friend shape of the converting shape will be,

$$\alpha_n = f^n(\alpha) = \frac{\alpha_{n-1}\sqrt{(\alpha_{n-1})^2 + 2}}{\sqrt{\{4(\alpha_{n-1})^2 - 1\}}}$$

Let $b > a$. So, $\alpha > 1$. When $\alpha > 1$ in that case,

$$\alpha > \frac{\alpha\sqrt{\alpha^2 + 2}}{\sqrt{4\alpha^2 - 1}} > 1$$

$$\Rightarrow \alpha > f(\alpha) > 1$$

As, $\alpha > 1$, $f(\alpha) > 1$ and $\alpha > f(\alpha)$ that's why here we can apply the formula 1. So, by using the formula 1 we can write,

$$\lim_{n \rightarrow +\infty} f^n(\alpha) = 1$$

Again, let $b < a$. So, $\alpha > 0$ and $\alpha < 1$. When $0 < \alpha < 1$ in that case,

$$\frac{\alpha\sqrt{\alpha^2 + 2}}{\sqrt{4\alpha^2 - 1}} > 1$$

$$\Rightarrow f(\alpha) > 1$$

As, $\alpha > f(\alpha) > 1$ when $\alpha > 1$ so similarly we can write,

$$f(\alpha) > f(f(\alpha)) > 1 \text{ when } f(\alpha) > 1$$

So in this case applying the formula 1 we can write,

$$\lim_{n \rightarrow +\infty} f^n(\alpha) = 1$$

So, in the n th step of the converting process, the ratio of the secondary side and the main side of the n th main friend shape of the converting shape will be 1 when $n \rightarrow +\infty$. That's why, all the sides of the n th main friend shape of the converting shape will be of equal length when $n \rightarrow +\infty$. So, the final shape will be an equilateral triangle. As the length of one side of an inscribed equilateral triangle in a circle is $\sqrt{3}r$ where the radius of that circle is r , so in this case the length of one side of the final shape will be $\sqrt{3}r$ because the radius of the base shape is r . Again, the converting process clarifies that the three vertices of the final shape will be on the circumference of the base shape.

Here we can see the converting shape is an isosceles triangle inscribed in the base shape and the properties of the final shape do not depend on the properties of the converting shape. Again, there are infinitely many isosceles triangles inscribed in that base shape which are not congruent to converting shape. So any of them can be used as converting shape and for that the same final shape will be obtained. Thus this is an example of shape conversion.

Shape conversion 2

i. Base shape: A right triangle whose base is b and height is h .

ii. Converting shape: A rectangle where the length of one vertical side is p and the length of one horizontal side is q . The properties of the converting shape must support the main position.

iii. Main position: The converting shape will be inscribed in the base shape so that one vertex of the converting shape coincides with the right-angle vertex of the base shape.

iv. First two steps of converting process: In the first step, at first we will connect the midpoints of two vertical sides of the converting shape by a line segment and extend the line segment up to hypotenuse. Then we will draw a perpendicular bisector of this extended line segment and extend both sides of that perpendicular bisector up to hypotenuse and base of the base shape. Again, from that point of the hypotenuse where the perpendicular bisector of the first line segment will intersect we will draw a perpendicular line on the base of base shape. Considering this perpendicular line on the base of base shape and that perpendicular bisector of the first line segment as two sides of a rectangle we will draw a rectangle so that one vertex of that rectangle coincides with the right-angle vertex of the base shape. This new rectangle will be considered as 1st main friend shape of the converting shape. In the second step we will apply the same method on the 1st main friend shape of the converting shape so that we get second main friend shape of the converting shape.

v. n th step of the converting process In the n th step of the converting process, by using the base shape and the main friend shape of the converting shape obtained in the $(n - 1)$ th step, the same process of second step will be applied on the $(n - 1)$ th main friend shape of the converting shape so that we get n th main friend shape of the converting shape where $n > 2$.

vi. Final shape: A rectangle whose length of one vertical side will be $\frac{2bp}{3(b-q)}$ and length of one horizontal side will be $\frac{b}{3}$. So the area will be $\frac{2b^2p}{9(b-q)}$. The final shape will be inscribed in the base shape so that one vertex of the converting shape coincides with the right-angle vertex of the base shape.

vii. Figure

(a) Converting process visualization

Figure 3.1.4: Here XYZ is the base shape. $ABCY$ is the converting shape. $A_1B_1C_1Y$ is the 1st main friend shape of the converting shape. In the second step $A_2B_2C_2Y$ is the 2nd main friend shape of the converting shape. In the third step $A_3B_3C_3Y$ is the third main friend shape of the converting shape.

(b) Final shape visualization

Figure 3.1.5: Here $A_\infty B_\infty C_\infty Y$ is the final shape.

viii. Proof

Look at the figure of first step of the converting process.

Figure 3.1.6: Here XYZ is the base shape. $ABCY$ is the converting shape. $A_1B_1C_1Y$ is the 1st main friend shape of the converting shape.

Let us connect R, F where $FR \perp YZ$. Here, $BC = p, YC = q, YZ = b$. Let, $\frac{b}{p} = n \Rightarrow b = np$. Again, $CZ = b - q$.
Let, $\frac{b-q}{p} = m \Rightarrow mp = b - q$.

Here we can see, $\triangle BCZ$ and $\triangle RFZ$ are similar. That's why,

$$\frac{FZ}{RF} = \frac{CZ}{BC}$$

$$\Rightarrow FZ = \frac{mp}{p} \times \frac{p}{2} = \frac{mp}{2}$$

Now,

$$C_1Z = YZ - \frac{YZ - FZ}{2}$$

$$= np - \frac{np}{2} + \frac{mp}{4}$$

$$= \frac{2np + mp}{4}$$

Again, $\triangle B_1C_1Z$ and $\triangle BCZ$ are similar. That's why,

$$\frac{B_1C_1}{C_1Z} = \frac{BC}{CZ}$$

$$\Rightarrow B_1C_1 = \frac{2np + mp}{4} \times \frac{p}{mp} = \frac{2np + mp}{4m}$$

By using the similar process in the second step of the converting process we get the length of one vertical side of the second main friend shape of the converting shape. So,

$$B_2C_2 = \frac{10np + mp}{16m} = \frac{2np(1 + 4) + mp}{4^2m}$$

Similarly,

$$B_3C_3 = \frac{42np + mp}{64m} = \frac{2np(1 + 4 + 4^2) + mp}{4^3m}$$

$$B_4C_4 = \frac{170np + mp}{256m} = \frac{2np(1 + 4 + 4^2 + 4^3) + mp}{4^4m}$$

Now, here get a beautiful pattern. According to that pattern we can write that the length of one vertical side of the n th main friend shape of the converting shape will be

$$B_nC_n = \frac{2np(1 + 4 + 4^2 + 4^3 + \dots + 4^x) + mp}{4^{x+1}m}$$

So, the length of one vertical side of the final shape will be

$$\lim_{x \rightarrow +\infty} \frac{2np(1 + 4 + 4^2 + 4^3 + \dots + 4^x) + mp}{4^{x+1}m}$$

$$= \lim_{x \rightarrow +\infty} \frac{2np(1 + 4 + 4^2 + 4^3 + \dots + 4^x)}{4^{x+1}m} + \lim_{x \rightarrow +\infty} \frac{mp}{4^{x+1}m}$$

Here, the solution of the limit is

$$\lim_{x \rightarrow +\infty} \frac{(1 + 4 + 4^2 + 4^3 + \dots + 4^x)}{4^{x+1}} = \frac{1}{3}$$

So we can write,

$$\begin{aligned} \lim_{x \rightarrow +\infty} \frac{2np(1 + 4 + 4^2 + 4^3 + \dots + 4^x)}{4^{x+1}m} + \lim_{x \rightarrow +\infty} \frac{mp}{4^{x+1}m} \\ = \frac{2np}{3m} + 0 \\ = \frac{2bp}{3(b-q)} \end{aligned}$$

Again, as every main friend shape of the converting shape is a rectangle so $\Delta B_{\infty}C_{\infty}Z$ and ΔBCZ are similar. Now,

$$\begin{aligned} YC_{\infty} &= YZ - C_{\infty}Z \\ &= b - \frac{2b}{3} = \frac{b}{3} \end{aligned}$$

So, the area of the final shape will be

$$\frac{2b^2p}{9(b-q)}$$

Now we will present the shape conversion 3. To prove shape conversion3 we need to understand some important definitions, lemmas and theorems. They are as follows:

Definition 3.1.6 (Apex and secondary vertices of an isosceles triangle): The vertex opposite the base is called apex and the other two vertices is called secondary vertices of an isosceles triangle.

Definition 3.1.7 (Main sides and secondary sides of a rectangle): If we consider any two parallel sides of a rectangle as the main sides then the other two parallel sides will be called secondary sides of that rectangle.

Definition 3.1.8 (Base median of an isosceles triangle): The median to the base of an isosceles triangle is called base median of that isosceles triangle.

Definition 3.1.9 (Special point): A point on any side of two equal sides of an isosceles triangle at a distance of x from the apex is called a special point where x is greater than one-third of the length of that side.

Definition 3.1.10 (General line segment): The line segment drawn from a secondary vertex to a special point where the special point lies on the opposite side of the secondary vertex is called a general line segment.

Lemma 3.1.1: Suppose, the apex of an isosceles triangle lies on a line and the base lies on other line which is parallel to the first line. If two equal general line segments on two equal sides of the isosceles triangle are taken and

extended they will intersect the first line at two points. The distance between those two points will be greater than the base of the isosceles triangle.

Figure 3.1.7: ABC is an isosceles triangle. BD and CE are equal general line segments on AC and AB respectively.

Here, $\angle AEG = \angle BEC$ and $\angle AGE = \angle BCE$. So, $\triangle AEG$ and $\triangle BEC$ are similar. So,

$$\frac{AE}{BE} = \frac{AG}{BC}$$

According to the definition of general line segment, $AE > \frac{1}{3}AB$. So, $BE < \frac{2}{3}AB$. That's why, $\frac{AE}{BE} > \frac{1}{2}$. So,

$$\frac{AG}{BC} > \frac{1}{2} \Rightarrow AG > \frac{BC}{2}$$

Similarly we can show,

$$\frac{AF}{BC} > \frac{1}{2} \Rightarrow AF > \frac{BC}{2}$$

So we get,

$$AG + AF > BC$$

$$\Rightarrow GF > BC$$

Thus it is proved that the distance between G and F is greater than the base of the triangle ABC .

Theorem 3.1.1: Suppose, the apex of an isosceles triangle lies on the midpoint of a main side of a rectangle and the secondary vertices lie on the secondary sides of that rectangle. If two equal general line segments on two equal sides of the isosceles triangle are taken and extended and at the same time the base median is also extended, they will intersect the perimeter of the rectangle at three points. The triangle obtained by connecting those three points will also be an isosceles triangle and the secondary vertices of the obtained isosceles triangle will lie on the secondary sides of that rectangle, the apex of that triangle will lie on the midpoint of the other main side of that rectangle and the base of that isosceles triangle will be equal to the base of the first isosceles triangle.

Figure 3.1.8: $PQRS$ is a rectangle. PQ and SR are the main sides and PS and QR are the secondary sides of the rectangle. ABC is the first isosceles triangle and DEF is the second obtained isosceles triangle. BM and CN are equal general line segments on AC and AB respectively.

Here, $AB = AC$, $AP = AQ$ and $\angle APB = \angle AQC = 90^\circ$. So, $\triangle APB$ and $\triangle AQC$ are congruent. That's why, $BP = CQ$. So, $PQ = SR = BC$ and $PQ \parallel SR \parallel BC$.

According to the lemma 3.1.1, $XY > BC$ and $AY = AX$. Again, $PQ = SR = BC$. So, $XY > PQ$. That means the two points X and Y will always be on the extended part of PQ . So, if we connect C, Y the line segment CY will obviously intersect the secondary side PS of the rectangle at the point F . Similarly, if we connect B, X the line segment BX will obviously intersect the secondary side QR of the rectangle at the point E .

Again, the base median AO of the triangle ABC will intersect the main side RS at the point D . Connecting the points D, E and F the second triangle will be obtained. Now, we have to prove $\triangle DEF$ is also an isosceles triangle and $FE = BC$.

Here, BM and CN are equal general line segments on AC and AB respectively. That's why, $AM = AN$, $BN = CM$ and BC is the common side. So, $\triangle BMC$ and $\triangle BNC$ are congruent. That's why, $\angle NCB = \angle MBC$ or $\angle FCB = \angle EBC$.

Again, $\angle FBC = \angle ECB = 90^\circ$. So, $\triangle FBC$ and $\triangle ECB$ are congruent. That's why, $BF = CE$. So, $FE = BC$ and $FE \parallel BC$. Now, AD is the perpendicular bisector of BC . So, AD will also be the perpendicular bisector of EF . That's why, $ED = FD$. So, $\triangle DEF$ is also an isosceles triangle. Again, $\triangle FSD$ and $\triangle ERD$ are congruent. So, $SD = DR$. That's why, D is the midpoint of SR . Thus the theorem is proved.

Here we can see the converting shape is rectangle inscribed in the base shape and the properties of the final shape do not depend on the properties of the converting shape. Again, there are infinitely many rectangles inscribed in that base shape which are not congruent to converting shape. So any of them can be used as converting shape and for that the same final shape will be obtained. Thus this is an example of shape conversion.

Shape conversion 3

i. Base shape: A rectangle where the length of a main side is $2x$ and the length of a secondary side is $2y$.

ii. Converting shape: An isosceles triangle whose base is $2x$ and height is q .

iii. Main position: The apex of the converting shape will lie on the midpoint of a main side of the base shape and the secondary vertices will lie on the secondary sides of base shape.

iv. First two steps of converting process: In the first step, we will increase the three medians of the converting shape. The three extended medians will intersect the perimeter of the base shape at three points. According to theorem 3.1.1, connecting those three points we will get a new isosceles triangle. This new isosceles triangle will be considered as 1st main friend shape of the converting shape. In the second step we will apply the same method on the 1st main friend shape of the converting shape so that we get second main friend shape of the converting shape.

v. nth step of the converting process: In the n th step of the converting process, by using the base shape and the main friend shape of the converting shape obtained in the $(n - 1)$ th step, the same process of second step will be applied on the $(n - 1)$ th main friend shape of the converting shape so that we get n th main friend shape of the converting shape where $n > 2$.

vi. Final shape: An isosceles triangle whose base will be $2x$ and height will be $\frac{3y}{2}$ that means the area will be $\frac{3x}{2}$.

vii. Figure

(a) Converting process visualization

Figure 3.1.9: Here $PQRS$ is the base shape. PQ and SR are the main sides and PS and QR are the secondary sides of the base shape. ABC is the converting shape. $A_1B_1C_1$ is the 1st main friend shape of the converting shape. In the second step $A_2B_2C_2$ is the 2nd main friend shape of the converting shape. In the third step $A_3B_3C_3$ is the third main friend shape of the converting shape.

(b) Final shape visualization

Figure 3.1.10: Here, $A_\infty B_\infty C_\infty$ is the final shape.

viii. Proof

Look at the figure of first step of the converting process.

Figure 3.1.11: Here $PQRS$ is the base shape. PQ and SR are the main sides and PS and QR are the secondary sides of the base shape. ABC is the converting shape. $A_1B_1C_1$ is the 1st main friend shape of the converting shape.

Let us connect M, N where $NM \perp BC$. Here, $PQ = SR = 2x$ and $PS = QR = 2y$. Again, CM is the median on the side AB . So, M is the midpoint of AB and $AM > \frac{1}{3}AB$. That's why, according to the definition of special point M is a special point on AB . Again, the two medians on the equal sides will also be considered as two equal general line segments.

Now, by using the theorem 3.1.1 and its proof we get, the triangle $A_1B_1C_1$ is an isosceles triangle because it is obtained by increasing the medians of the triangle ABC , $BC = B_1C_1 = 2x$, $PQ \parallel SR \parallel BC$ and the point B_1 and C_1 will lie on the secondary side of the rectangle $PQRS$. Again, $AD = q$, A_1D_1 is the height of $A_1B_1C_1$.

Here, $\triangle ABD$ and $\triangle MBN$ are similar. So,

$$\frac{BN}{BD} = \frac{BM}{AB}$$

$$\Rightarrow BN = \frac{BD}{2} = \frac{x}{2}$$

So,

$$CN = BC - BN = \frac{3x}{2}$$

Again,

$$\frac{MN}{AD} = \frac{BM}{AB}$$

$$\Rightarrow MN = \frac{AD}{2} = \frac{q}{2}$$

Again, $\triangle NCM$ and $\triangle BCC_1$ are similar. So,

$$\frac{BC_1}{MN} = \frac{BC}{CN}$$

$$\Rightarrow BC_1 = 2x \times \frac{2}{3x} \times \frac{q}{2} = \frac{2q}{3}$$

Now,

$$A_1D_1 = BC_1 + A_1D$$

$$= \frac{2q}{3} + (2y - q)$$

$$= \frac{6y - q}{3}$$

Here, for any value of q where $0 < q < 2y$ the converting shape will support the main position. So in other way we can write,

$$A_1D_1 = f(q) = \frac{6y - q}{3} \text{ where } 0 < q < 2y$$

Let, A_2D_2 is the height of the triangle $A_2B_2C_2$. By using the theorem 3.1.1 and its proof we get, the triangle $A_2B_2C_2$ is an isosceles triangle because it is obtained by increasing the medians of the triangle $A_1B_1C_1$, $B_2C_2 = B_1C_1 = 2x$ and the point B_2 and C_2 will lie on the secondary side of the rectangle $PQRS$.

Now, by using the above method we get,

$$A_2D_2 = \frac{6y - \left(\frac{6y - q}{3}\right)}{3}$$

In other way we can write,

$$A_2D_2 = f(f(q)) = f^2(q) = \frac{6y - \left(\frac{6y - q}{3}\right)}{3} = \frac{12y + q}{9}$$

According to the theorem 3.1.1, all the friend shapes of the converting shape will be isosceles triangle and the length of the base will be $2x$. So, the length of the n th main friend shape of the converting shape will be $2x$ when $n \rightarrow +\infty$.

Again, let A_3D_3 is the height of $\Delta A_3B_3C_3$ and A_4D_4 is the height of $\Delta A_4B_4C_4$. By applying the above method we get,

$$A_3D_3 = \frac{6y - \left(\frac{12y+q}{9}\right)}{3}$$

In other way we can write,

$$A_3D_3 = f\left(f\left(f(q)\right)\right) = f^3(q) = \frac{6y - \left(\frac{12y+q}{9}\right)}{3} = \frac{42y - q}{27}$$

Again,

$$A_4D_4 = \frac{6y - \left(\frac{42y-q}{27}\right)}{3}$$

In other way we can write,

$$A_4D_4 = f\left(f\left(f\left(f(q)\right)\right)\right) = f^4(q) = \frac{6y - \left(\frac{42y-q}{27}\right)}{3} = \frac{120y + q}{81}$$

Similarly, the height of $\Delta A_nB_nC_n$ will be

$$A_nD_n = f^n(q)$$

Now we can see, here

$$\begin{aligned} \frac{6y - q}{3} &> \frac{3y}{2} \text{ when } q < \frac{3y}{2} \text{ and } \frac{6y - q}{3} < \frac{3y}{2} \text{ when } q > \frac{3y}{2} \\ \Rightarrow f(q) &> \frac{3y}{2} \text{ when } q < \frac{3y}{2} \text{ and } f(q) < \frac{3y}{2} \text{ when } q > \frac{3y}{2} \end{aligned}$$

Again,

$$\begin{aligned} \frac{42y - q}{27} &> \frac{3y}{2} \text{ when } q < \frac{3y}{2} \text{ and } \frac{42y - q}{27} < \frac{3y}{2} \text{ when } q > \frac{3y}{2} \\ \Rightarrow f^3(q) &> \frac{3y}{2} \text{ when } q < \frac{3y}{2} \text{ and } f^3(q) < \frac{3y}{2} \text{ when } q > \frac{3y}{2} \end{aligned}$$

Again,

$$\begin{aligned} \frac{12y + q}{9} &> \frac{3y}{2} \text{ when } q > \frac{3y}{2} \text{ and } \frac{12y + q}{9} < \frac{3y}{2} \text{ when } q < \frac{3y}{2} \\ \Rightarrow f^2(q) &> \frac{3y}{2} \text{ when } q > \frac{3y}{2} \text{ and } f^2(q) < \frac{3y}{2} \text{ when } q < \frac{3y}{2} \end{aligned}$$

Again,

$$\begin{aligned} \frac{120y + q}{81} &> \frac{3y}{2} \text{ when } q > \frac{3y}{2} \text{ and } \frac{120y + q}{81} < \frac{3y}{2} \text{ when } q < \frac{3y}{2} \\ \Rightarrow f^4(q) &> \frac{3y}{2} \text{ when } q > \frac{3y}{2} \text{ and } f^4(q) < \frac{3y}{2} \text{ when } q < \frac{3y}{2} \end{aligned}$$

Again,

$$\frac{6y - q}{3} > \frac{42y - q}{27}$$

$$\Rightarrow q < \frac{3y}{2}$$

So we can write,

$$\frac{6y - q}{3} > \frac{42y - q}{27} > \frac{3y}{2} \text{ when } q < \frac{3y}{2}$$

$$\Rightarrow f(q) > f^3(q) > \frac{3y}{2} \text{ when } q < \frac{3y}{2}$$

Again,

$$\frac{120y + q}{81} > \frac{12y + q}{9}$$

$$\Rightarrow q < \frac{3y}{2}$$

So we can write,

$$\frac{3y}{2} > \frac{120y + q}{81} > \frac{12y + q}{9} \text{ when } q < \frac{3y}{2}$$

$$\Rightarrow \frac{3y}{2} > f^4(q) > f^2(q) \text{ when } q < \frac{3y}{2}$$

Again,

$$\frac{6y - q}{3} < \frac{42y - q}{27}$$

$$\Rightarrow q > \frac{3y}{2}$$

So we can write,

$$\frac{6y - q}{3} < \frac{42y - q}{27} < \frac{3y}{2} \text{ when } q > \frac{3y}{2}$$

$$\Rightarrow f(q) < f^3(q) < \frac{3y}{2} \text{ when } q > \frac{3y}{2}$$

Again,

$$\frac{12y + q}{9} > \frac{120y + q}{81}$$

$$\Rightarrow q > \frac{3y}{2}$$

So we can write,

$$\frac{12y + q}{9} > \frac{120y + q}{81} > \frac{3y}{2} \text{ when } q > \frac{3y}{2}$$

$$\Rightarrow f^2(q) > f^4(q) > \frac{3y}{2} \text{ when } q > \frac{3y}{2}$$

Now, here we can apply the formula 2. By applying the formula we can write,

$$\begin{aligned} \lim_{n \rightarrow +\infty} f^{2n-1}(q) &= \lim_{n \rightarrow +\infty} f^{2n}(q) = \frac{3y}{2} \\ \Rightarrow \lim_{n \rightarrow +\infty} f^n(q) &= \frac{3y}{2} \end{aligned}$$

So, the height of nth main friend shape of the converting shape will be

$$A_n D_n = \frac{3y}{2} \text{ when } n \rightarrow +\infty$$

So, the final shape is an isosceles triangle whose base is $2x$ and height is $\frac{3y}{2}$. That's why, the area will be $\frac{3x}{2}$.

Here we can see the converting shape is an isosceles triangle inscribed in the base shape and the properties of the final shape do not depend on the properties of the converting shape. Again, there are infinitely many isosceles triangles inscribed in that base shape where the height of every isosceles triangle is greater than 0 and less than $2y$ and the base is $2x$. So any of them can be used as converting shape and for that the same final shape will be obtained. Thus this is an example of shape conversion.

Shape conversion 4

i. Base shape: A line segment whose length is $2a$.

ii. Converting shape: A line segment whose length is b .

iii. Main position: One endpoint of the converting shape will lie on a endpoint of the base shape and other endpoint of the converting shape will lie on the base shape.

iv. First two steps of converting process: In the first step, we have to take a point at a distance of $\frac{1}{m}$ of the converting shape from the endpoint which lies on one endpoint of the base shape. Then we have to draw a line segment from that point to the other endpoint of the base shape. This new line segment will be the first main friend shape of the converting shape. In the second step, we have to take a point at a distance of $\frac{1}{m}$ of the 1st main friend shape of the converting shape from the endpoint which lies on one endpoint of the base shape. Then we have to draw a line segment from that point to the other endpoint of the base shape. This new line segment will be the 2nd main friend shape of the converting shape.

v. nth step of the converting process: In the nth step of the converting process, by using one or more base shapes and the main friend shape of the converting shape obtained in the $(n - 1)$ th step, the same process of second step will be applied on the $(n - 1)$ th main friend shape of the converting shape so that we get nth main friend shape of the converting shape where $n > 2$.

vi. Final shape: A line segment whose length will be $\frac{2am}{m+1}$ and one endpoint of the final shape will lie on a endpoint of the base shape and other endpoint of the final shape will lie on the base shape.

vii. Figure

(a) Converting process visualization

Figure 3.1.12: Here AB is the base shape. AC is the converting shape. BC_1 is the 1st main friend shape of the converting shape. In the second step AC_2 is the 2nd main friend shape of the converting shape. In the third step BC_3 is the third main friend shape of the converting shape.

(b) Final shape visualization

Figure 3.1.13: Here, AC_∞ or BC_∞ is the final shape.

Note: In the figure we have used some small line segments to indicate the endpoints of the converting shapes, its main friend shapes and final shape.

viii. Proof

Look at the figure of first step of the converting process. Here $AB = 2a$ and $AC = b$. Now, the length of 1st main friend shape of the converting shape will be

$$BC_1 = 2a - \frac{b}{m} = \frac{2am - b}{m}$$

Here, for any value of b where $0 < b < 2a$ the converting shape will support the main position. So in other way we can write,

$$BC_1 = f(b) = \frac{2am - b}{m} \text{ where } 0 < b < 2a$$

Now, by using the above method we get the length of the 2nd main friend shape of the converting shape will be,

$$AC_2 = \frac{2am - \left(\frac{2am - b}{m}\right)}{m} = \frac{2am^2 - 2am + b}{m^2}$$

In other way we can write, the length of the 2nd main friend shape of the converting shape will be,

$$AC_2 = f(f(b)) = f^2(b) = \frac{2am - \left(\frac{2am-b}{m}\right)}{m} = \frac{2am^2 - 2am + b}{m^2}$$

Similarly, the length of the 3rd main friend shape of the converting shape will be,

$$BC_3 = f(f(f(b))) = f^3(b) = \frac{2am - \left(\frac{2am^2 - 2am + b}{m^2}\right)}{m} = \frac{2am^3 - 2am^2 + 2am - b}{m^3}$$

Again, the length of the 4th main friend shape of the converting shape will be,

$$AC_4 = f(f(f(f(b)))) = f^4(b) = \frac{2am - \left(\frac{2am^3 - 2am^2 + 2am - b}{m^3}\right)}{m} = \frac{2am^4 - 2am^3 + 2am^2 - 2am + b}{m^4}$$

Now we can see, here

$$\begin{aligned} \frac{2am - b}{m} &> \frac{3y}{2} \text{ when } b < \frac{2am}{m+1} \text{ and } \frac{2am - b}{m} < \frac{2am}{m+1} \text{ when } b > \frac{2am}{m+1} \\ \Rightarrow f(b) &> \frac{2am}{m+1} \text{ when } b < \frac{2am}{m+1} \text{ and } f(b) < \frac{2am}{m+1} \text{ when } b > \frac{2am}{m+1} \end{aligned}$$

Again,

$$\begin{aligned} \frac{2am^3 - 2am^2 + 2am - b}{m^3} &> \frac{2am}{m+1} \text{ when } b < \frac{2am}{m+1} \\ \text{and } \frac{2am^3 - 2am^2 + 2am - b}{m^3} &< \frac{2am}{m+1} \text{ when } b > \frac{2am}{m+1} \\ \Rightarrow f^3(b) &> \frac{2am}{m+1} \text{ when } b < \frac{2am}{m+1} \text{ and } f^3(b) < \frac{2am}{m+1} \text{ when } b > \frac{2am}{m+1} \end{aligned}$$

Again,

$$\begin{aligned} \frac{2am^2 - 2am + b}{m^2} &> \frac{2am}{m+1} \text{ when } b > \frac{2am}{m+1} \text{ and } \frac{2am^2 - 2am + b}{m^2} < \frac{2am}{m+1} \text{ when } b < \frac{2am}{m+1} \\ \Rightarrow f^2(b) &> \frac{2am}{m+1} \text{ when } b > \frac{2am}{m+1} \text{ and } f^2(b) < \frac{2am}{m+1} \text{ when } b < \frac{2am}{m+1} \end{aligned}$$

Again,

$$\begin{aligned} \frac{2am^4 - 2am^3 + 2am^2 - 2am + b}{m^4} &> \frac{2am}{m+1} \text{ when } b > \frac{2am}{m+1} \\ \text{and } \frac{2am^4 - 2am^3 + 2am^2 - 2am + b}{m^4} &< \frac{2am}{m+1} \text{ when } b < \frac{2am}{m+1} \\ \Rightarrow f^4(b) &> \frac{2am}{m+1} \text{ when } b > \frac{2am}{m+1} \text{ and } f^4(b) < \frac{2am}{m+1} \text{ when } b < \frac{2am}{m+1} \end{aligned}$$

Again,

$$\frac{2am - b}{m} > \frac{2am^3 - 2am^2 + 2am - b}{m^3}$$

$$\Rightarrow b < \frac{2am}{m+1}$$

So we can write,

$$\frac{2am - b}{m} > \frac{2am^3 - 2am^2 + 2am - b}{m^3} > \frac{2am}{m+1} \text{ when } b < \frac{2am}{m+1}$$

$$\Rightarrow f(b) > f^3(b) > \frac{2am}{m+1} \text{ when } b < \frac{2am}{m+1}$$

Again,

$$\frac{2am^4 - 2am^3 + 2am^2 - 2am + b}{m^4} > \frac{2am^2 - 2am + b}{m^2}$$

$$\Rightarrow b < \frac{2am}{m+1}$$

So we can write,

$$\frac{2am}{m+1} > \frac{2am^4 - 2am^3 + 2am^2 - 2am + b}{m^4} > \frac{2am^2 - 2am + b}{m^2} \text{ when } b < \frac{2am}{m+1}$$

$$\Rightarrow \frac{2am}{m+1} > f^4(b) > f^2(b) \text{ when } b < \frac{2am}{m+1}$$

Again,

$$\frac{2am - b}{m} < \frac{2am^3 - 2am^2 + 2am - b}{m^3}$$

$$\Rightarrow b > \frac{2am}{m+1}$$

So we can write,

$$\frac{2am - b}{m} < \frac{2am^3 - 2am^2 + 2am - b}{m^3} < \frac{2am}{m+1} \text{ when } b > \frac{2am}{m+1}$$

$$\Rightarrow f(b) < f^3(b) < \frac{2am}{m+1} \text{ when } b > \frac{2am}{m+1}$$

Again,

$$\frac{2am^2 - 2am + b}{m^2} > \frac{2am^4 - 2am^3 + 2am^2 - 2am + b}{m^4}$$

$$\Rightarrow b > \frac{2am}{m+1}$$

So we can write,

$$\frac{2am^2 - 2am + b}{m^2} > \frac{2am^4 - 2am^3 + 2am^2 - 2am + b}{m^4} > \frac{2am}{m+1} \text{ when } b > \frac{2am}{m+1}$$

$$\Rightarrow f^2(b) > f^4(b) > \frac{2am}{m+1} \text{ when } b > \frac{2am}{m+1}$$

Now, here we can apply the formula 2. By applying the formula we can write,

$$\begin{aligned} \lim_{n \rightarrow +\infty} f^{2n-1}(b) &= \lim_{n \rightarrow +\infty} f^{2n}(b) = \frac{2am}{m+1} \\ \Rightarrow \lim_{n \rightarrow +\infty} f^n(b) &= \frac{2am}{m+1} \end{aligned}$$

So, the length of nth main friend shape of the converting shape will be

$$= \frac{2am}{m+1} \text{ when } n \rightarrow +\infty$$

So, the length of the final shape is $\frac{2a}{m+1}$.

Here we can see the converting shape is a line segment on the base shape and the properties of the final shape do not depend on the properties of the converting shape. Again, there are infinitely many line segments whose length is greater than 0 and less than $2a$. So any of them can be used as converting shape and for that the same final shape will be obtained. Thus this is an example of shape conversion.

To present shape conversion 5 we need to understand some important definitions. They are as follows:

Definition 3.1.11 (Main vertex of a parallelogram): In a parallelogram, the vertex which is created by the two adjacent sides where the adjacent angle is an acute angle that vertex is called main vertex of that parallelogram.

Definition 3.1.12 (Main side and secondary side of a parallelogram): In a parallelogram, the two adjacent sides which create a main vertex if one of them is called main side then the other side will be called secondary side.

To clearly understand the definitions let us give an example.

Figure 3.1.14: Here ABCD is a parallelogram.

Here, $\angle ADC = 60^\circ$. So the vertex D will be a main vertex. If we consider AD as main side then CD will be the secondary side. Again, if we consider CD as main side then AD will be the secondary side. Similarly, B will also be a main vertex because, $\angle ABC = 60^\circ$.

Note: In a parallelogram, there are two main vertices.

Shape conversion 5

i. Base shape: A rectangle where the length of a main side is $2x$ and the length of a secondary side is $2y$.

ii. Converting shape: A parallelogram whose main side is p , secondary is q and the vertical distance between main side and its opposite side is d .

iii. Main position: One main side of the converting shape will lie on a main side of the base shape so that the main vertex lies on one endpoint of that main side of the base shape. The converting shape will be totally inside the base shape.

iv. First two step of converting process: Let, the main side of the converting shape which lies on a main side of the base shape is main side-1 of the converting shape, the main vertex which lies on one endpoint of that main side of the base shape is main vertex-1 of the converting shape and the secondary side which helps to create the main vertex-1 is secondary side-1.

Now in the first step, we will draw a line segment through the main vertex-1 of the converting shape so that the line segment divides the angle between secondary side-1 of the converting shape and secondary side of the base shape into two equal parts and then we will increase the diagonal opposite the main vertex-1 of the converting shape so that the increased part of the diagonal intersects that line segment at a point. Considering the distance between that point of intersection and the main vertex-1 of the converting shape as the secondary side and the main side of the converting shape as the main side we will draw a new parallelogram. This new parallelogram will be the 1st main friend shape of the converting shape. In the second step we will apply the same process on the 1st main friend shape of the converting shape to get the 2nd main friend shape of the converting shape.

v. nth step of the converting process: In the n th step of the converting process, by using one or more base shapes and the main friend shape of the converting shape obtained in the $(n - 1)$ th step, the same process of second step will be applied on the $(n - 1)$ th main friend shape of the converting shape so that we get n th main friend shape of the converting shape where $n > 2$.

vi. Final shape: A rectangle whose one side will be p and other side will be $\frac{pd}{p - \sqrt{q^2 - d^2}}$. So the area will be $\frac{p^2 d}{p - \sqrt{q^2 - d^2}}$.

vii. Figure

(a) Converting process visualization

Figure 3.1.15: Here PQRS is the base shape. PQ and SR are the main sides and PS and QR are the secondary sides of the base shape. ABCS is the converting shape. A_1B_1CS is the 1st main friend shape of the converting shape. In the second step A_2B_2CS is the 2nd main friend shape of the converting shape. In the third step A_3B_3CS is the third main friend shape of the converting shape.

(b) Final shape visualization

Figure 3.1.16: Here, $A_\infty B_\infty CS$ is the final shape.

viii. Proof

Look at the figure of first step of the converting process. Here, $\angle PSC = 90^\circ$, so $\angle PSA = 90^\circ - \angle ASC$. Now we get,

$$\angle A_1 SC = \angle ASC + \frac{90^\circ - \angle ASC}{2}$$

Similarly,

$$\angle A_2 SC = \angle ASC + \frac{90^\circ - \angle ASC}{2} + \frac{90^\circ - \angle ASC}{4} = \angle ASC + \frac{90^\circ - \angle ASC}{2} + \frac{90^\circ - \angle ASC}{2^2}$$

Again,

$$\begin{aligned} \angle A_3 SC &= \angle ASC + \frac{90^\circ - \angle ASC}{2} + \frac{90^\circ - \angle ASC}{4} + \frac{90^\circ - \angle ASC}{8} \\ &= \angle ASC + \frac{90^\circ - \angle ASC}{2} + \frac{90^\circ - \angle ASC}{2^2} + \frac{90^\circ - \angle ASC}{2^3} \end{aligned}$$

Now, here we can see a pattern. According to the pattern in case of n th main friend shape $A_n B_n CS$ of the converting shape we can write,

$$\angle A_n SC = \angle ASC + \frac{90^\circ - \angle ASC}{2} + \frac{90^\circ - \angle ASC}{2^2} + \frac{90^\circ - \angle ASC}{2^3} + \dots + \frac{90^\circ - \angle ASC}{2^n}$$

When $n \rightarrow +\infty$ we can write,

$$\begin{aligned} \lim_{n \rightarrow \infty} \angle A_n SC &= \angle ASC + \frac{90^\circ - \angle ASC}{2} + \frac{90^\circ - \angle ASC}{2^2} + \frac{90^\circ - \angle ASC}{2^3} + \dots \\ &= \angle ASC + (90^\circ - \angle ASC) \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots \right) \\ &= \angle ASC + 90^\circ - \angle ASC = 90^\circ \end{aligned}$$

In other words, we can write the angle $\angle A_\infty SC$ in the final shape will be 90° . So, the side $A_\infty S$ will be on the side PS .

Again, $\angle ACS = \angle A_1 CS = \angle A_2 CS = \angle A_3 CS = \dots = \angle A_\infty CS$

Now given that, $SC = p$, $AS = q$ and the vertical distance between AB and SC is d . Look at the figure bellow,

Figure 3.1.17: Here, $ABCS$ is the converting shape and $A_{\infty}B_{\infty}CS$ is the final shape.

Here, $AD \perp SC$ so $AD = d$. Again, $\triangle ADC$ and $\triangle A_{\infty}SC$ are similar. So we can write,

$$\frac{AD}{DC} = \frac{A_{\infty}S}{SC}$$

$$\Rightarrow A_{\infty}S = \frac{dp}{p - \sqrt{q^2 - d^2}}$$

As, every main friend shape of the converting shape is a parallelogram so the opposite sides are parallel and equal. That's why, $A_{\infty}S = B_{\infty}C$ and $A_{\infty}B_{\infty} = SC$.

So, the side SC of the final shape is p and the side $A_{\infty}S$ is $\frac{dp}{p - \sqrt{q^2 - d^2}}$. As, $\angle A_{\infty}SC = 90^\circ$ so the final shape is a rectangle. That's why, the area will be $\frac{p^2 d}{p - \sqrt{q^2 - d^2}}$.

One important thing to notice is, in every step we divided the angle between secondary side of base shape and main friend shape of the converting shape into two equal parts. But here we could apply a different converting process. In the first step we could divide the angle into two equal parts, in the second step we could divide the angle into three equal parts, in the third step we could divide the angle into four equal parts etc. In that case the converting process would be type-2 converting process.

In case of every example, by the help of base shapes the converting shape has gradually changed its size and in every step of the converting process the base shape has been used. In the first step the converting shape and in another step the main friend shape of the converting shape have been used. In a step of the converting process, a process has been applied on the main friend shape of the converting shape obtained from the previous step to get a new main friend shape of the converting shape and in case of the first step a process has been applied on the converting shape. In case of applying the process the base shape has been used. Again, by using the same base shapes, converting process and main position, infinitely many friend shapes of the converting shape which are not congruent can be used as converting shape and for that the same final shape will be obtained. So, these are perfect example of shape conversion. Here we have demonstrated five shape conversions. The first three shape conversions and the fifth one are two-dimensional shape conversion and the fourth shape conversion is a one-dimensional shape conversion. Again, in case of these shape conversions, type-1 converting process has been applied. Thus we can set a lot of examples of shape conversion. Here we have shown some important shape conversions.

Types of shape conversion depending on the properties of final shape

There are three types of shape conversion depending on the properties of final shape. They are as follows:

1. Congruent shape based shape conversion
2. Friend shape based shape conversion
3. Different shape based shape conversion

Definition 3.1.13 (Congruent shape based shape conversion): In which shape conversion, the converting shape, main friend shapes of the converting shape and the final shape are congruent that shape conversion is called congruent shape based shape conversion.

Definition 3.1.14 (Friend shape based shape conversion): In which shape conversion, the final shape is a friend shape of the converting shape that shape conversion is called friend shape based shape conversion.

Definition 3.1.15 (Different shape based shape conversion): In which shape conversion, the type of final shape is different from the type of the converting shape that shape conversion is called different shape based shape conversion.

Note: Almost all the shape conversions are conditionally congruent shape based shape conversions.

According to the definition, the above shape conversions are friend shape based shape conversion. In case of the first shape conversion if we take an equilateral triangle as converting shape then by applying the converting process we will get the same equilateral triangle in every step and so the final shape will also be equilateral. So in this condition shape conversion 1 will be a congruent shape based shape conversion. Again, in case of other shape conversions if we take the final shapes as the converting shapes then those shape conversions will be also be congruent shape based shape conversions. But in case of shape conversion 5 it is not possible because if we take the final shape as converting shape then the converting process cannot be applied. So, almost all the shape conversions are conditionally congruent shape based shape conversions. Again, we can show some examples of different shape based shape conversions such as we can show one side of a quadrilateral is gradually vanishing and the quadrilateral is converting to a triangle. The mixed-dimensional shape conversion will also be a different shape based shape conversion.

3.2 Secondary shape conversion

Definition 3.2.1 (Secondary shape conversion): The shape conversion which is dependent on other shape conversion by the following two ways will be called secondary shape conversion.

- i. If a shape conversion is presented by indirectly using another shape conversion, then the shape conversion will be considered as a secondary shape conversion.
- ii. By using the same base shape, converting process and same type of converting shape if we just change the main position of a shape conversion and get the same final shape then the shape conversion will be considered as a secondary shape conversion.

Note: 1. Any type of shape conversion can be dependent on other any type of shape conversion such as a two-dimensional shape conversion can be dependent on a one-dimensional shape conversion.

2. A secondary shape conversion is dependent on a shape conversion. Similarly, we can show a shape conversion is dependent on a secondary shape conversion. But which will be discovered first will be considered as shape conversion.

Steps of secondary shape conversion

All the steps are same as the steps of the shape conversion.

Types of secondary shape conversion

As there are four types of shape conversions depending on dimensions and a secondary shape conversion is dependent on other shape conversion so the secondary shape conversion will also be of four types depending on dimensions. They are one-dimensional, two-dimensional, three-dimensional and mixed-dimensional secondary shape conversion. Again, similarly the secondary shape conversion will also be of three types depending on the

properties of final shape. They are congruent shape based secondary shape conversion, friend shape based secondary shape conversion and different shape based secondary shape conversion.

Now, here we will present 2 secondary shape conversions. Before we present the first secondary shape conversion, we have to prove an important theorem.

Theorem 3.2.1: Suppose, the apex of an isosceles triangle lies on the midpoint of a main side of a rectangle and the base lies on the other main side of that rectangle. Again, the base is greater than half of the main side and less than or equal to the main side. If the three medians of the isosceles triangle extended, they will intersect the perimeter of the rectangle at three points. The triangle obtained by connecting those three points will also be an isosceles triangle and the secondary vertices of the obtained isosceles triangle will lie on the secondary sides of that rectangle, the apex of that isosceles triangle will lie on the midpoint of the other main side of that rectangle and the base of that triangle will be equal to the base of the first isosceles triangle.

Figure 3.2.1: $PQRS$ is a rectangle. PQ and SR are the main sides and PS and QR are the secondary sides of the rectangle. ABC is the first isosceles triangle and DEF is the second obtained isosceles triangle.

Here, CN , AD , BM are the medians on the AB , BC , AC respectively. Again, $\frac{1}{2}SR < BC < SR$. Now, in case of $\triangle ANY$ and $\triangle BNC$, $AN = BN$, $\angle ANY = \angle BNC$ and $\angle NAY = \angle NBC$. So, $\triangle ANY$ and $\triangle BNC$ are congruent. That's why, $BC = AY$. Similarly, we can show $BC = AX$.

Again, A is the midpoint of PQ . So, $AP = \frac{1}{2}PQ$. As $BC > \frac{1}{2}SR$ and $BC = AY$, so AY is always greater than AP . That's why, the point Y will always lie on the increased part of QP . So, the line segment CY will intersect PS at a point. Let the point is F . Similarly, the line segment BX will intersect QR at the point E . Now, connecting the points D, E and F the triangle DEF will be obtained. So, the secondary vertices of the obtained triangle will lie on the secondary sides of the rectangle $PQRS$.

Now, we have shown $BC = AY$ and $BC = AX$. So, $AY = AX$. Again, $AP = AQ$. So, $PY = QX$. Now, in case of $\triangle FPY$ and $\triangle EQX$, $PY = QX$, $\angle FPY = \angle EQX = 90^\circ$ and $\angle FYP = \angle EXQ$. So, $\triangle FPY$ and $\triangle EQX$ are congruent. That's why, $FP = EQ$. So, $EF \parallel PQ \parallel SR$ and $EF = PQ = SR$.

Here, A is the midpoint of PQ and $AD \perp BC$ or, $AD \perp SR$. So, D is the midpoint of SR . Again, as $EF \parallel PQ \parallel SR$, so AD is the perpendicular bisector of EF . That's why, $DE = DF$. That's why, the triangle DEF is an isosceles triangle.

Secondary shape conversion 1

i. Base shape: A rectangle where the length of a main side is $2x$ and the length of a secondary side is $2y$.

ii. Converting shape: An isosceles triangle whose base is p and height is $2y$.

iii. Main position: The apex of the converting shape will lie on the midpoint of a main side of the base shape and the base will lie on the other main side of base shape.

iv. First two step of converting process: In the first step, we will increase the three medians of the converting shape. The three extended medians will intersect the perimeter of the base shape at three points. According to

theorem 3.2.1, connecting those three points we will get a new isosceles triangle. This new isosceles triangle will be considered as 1st main friend shape of the converting shape. In the second step we will apply the same method on the 1st main friend shape of the converting shape so that we get second main friend shape of the converting shape.

v. nth step of the converting process: In the n th step of the converting process, by using the base shape and the main friend shape of the converting shape obtained in the $(n - 1)$ th step, the same process of second step will be applied on the $(n - 1)$ th main friend shape of the converting shape so that we get n th main friend shape of the converting shape where $n > 2$.

vi. Final shape: An isosceles triangle whose base will be $2x$ and height will be $\frac{3y}{2}$ that means the area will be $\frac{3xy}{2}$.

vii. Figure

(a) Converting process visualization

Figure 3.2.2: Here $PQRS$ is the base shape. PQ and SR are the main sides and PS and QR are the secondary sides of the base shape. ABC is the converting shape. $A_1B_1C_1$ is the 1st main friend shape of the converting shape. In the second step $A_2B_2C_2$ is the 2nd main friend shape of the converting shape. In the third step $A_3B_3C_3$ is the third main friend shape of the converting shape.

(b) Final shape visualization

Figure 3.2.3: Here $A_\infty B_\infty C_\infty$ is the final shape.

viii. Proof

Look at the figure of first step of the converting process. According to the theorem 3.2.1, if the base BC of the converting shape is greater than $\frac{1}{2}SR$ and less than or equal to SR then $A_1B_1C_1$ will also be an isosceles triangle and the secondary vertices of $A_1B_1C_1$ will lie on the secondary sides of the base shape, the apex of $A_1B_1C_1$ will lie on the midpoint of the other main side of the base shape and the base of $A_1B_1C_1$ will be equal to the base of the converting shape.

Now, let BC is so smaller than $\frac{1}{2}SR$ and B_1C_1 is also smaller than $\frac{1}{2}SR$. Look at the figure below,

Figure 3.2.4: Here PQRS is the base shape. PQ and SR are the main sides and PS and QR are the secondary sides of the base shape. ABC is the converting shape. $A_1B_1C_1$ is the 1st main friend shape of the converting shape.

Now, in case of $\triangle BMC$ and $\triangle AMC_1$, $AM = BM$, $\angle AMC_1 = \angle BMC$ and $\angle MAC_1 = \angle MBC$. So, $\triangle BMC$ and $\triangle AMC_1$ are congruent. That's why, $BC = AC_1$. Similarly, we can show $BC = AB_1$. Again, $AA_1 \perp BC$ and $PQ \parallel SR$. So, $A_1A \perp PQ$ or $A_1A \perp B_1C_1$. As, $BC = AC_1$ and $BC = AB_1$ so $AB_1 = AC_1$. That's why, A is the midpoint of B_1C_1 and we have shown $A_1A \perp B_1C_1$. So, $\triangle A_1B_1C_1$ is also an isosceles triangle and $B_1C_1 = 2BC$.

Again, let B_2C_2 and B_3C_3 are also smaller than $\frac{1}{2}SR$. By using the above process we get $\triangle A_2B_2C_2$ and $\triangle A_3B_3C_3$ are isosceles and,

$$B_2C_2 = 2B_1C_1 = 2^2BC \text{ and } B_3C_3 = 2B_2C_2 = 2^3BC$$

Again, let the base of the nth main friend shape of the converting shape is also less than $\frac{1}{2}SR$. By using the above process we get the nth main friend shape of the converting shape will be an isosceles triangle and the base will be,

$$B_nC_n = 2^nBC$$

But,

$$\lim_{n \rightarrow +\infty} 2^nBC = \infty$$

So, for a finite value of n , 2^nBC will be greater than $\frac{1}{2}SR$ and less than or equal to SR . Then for that value of n the base of the $(n+1)$ th main friend shape of the converting shape will be greater than $\frac{1}{2}SR$ and less than or equal to SR . So according to the theorem 3.2.1, the $(n+2)$ th main friend shape of the converting shape will also be an isosceles triangle and the secondary vertices of $(n+2)$ th main friend shape will lie on the secondary sides of the base shape, the apex will lie on the midpoint of the other main side of the base shape and the base will be equal to the base of the converting shape.

Now, the position of $(n+2)$ th main friend shape of the converting shape and base shape is completely same to the main position of the shape conversion 3. Again in the shape conversion 3, the converting shape is also an isosceles triangle, base shape is a rectangle and the converting process is same. So, the final shape will also be same to the final shape of the shape conversion 3. By using the proof of the shape conversion 1 we get, the final shape will be an isosceles triangle whose base will be $2x$ and height will be $\frac{3y}{2}$. So the area will be $\frac{3xy}{2}$.

Here type-1 converting process has been applied and this is a two-dimensional secondary shape conversion. We can everything is same to the shape conversion 3 except the main position. But after changing the main position the final shape is same. That's why, this is a secondary shape conversion.

Secondary shape conversion 2

i. Base shape: A rectangle where the length of a main side is $2x$ and the length of a secondary side is $2y$.

ii. Converting shape

Converting shape: An isosceles triangle whose base is $2x$ and height is q .

iii. Main position: The apex of the converting shape will lie on the midpoint of a main side of the base shape and the secondary vertices will lie on the secondary sides of base shape.

iv. First two step of converting process: In the first step, we have to take a point at a distance of $\frac{1}{m}$ of the height of converting shape from its apex. Then through that point we have to draw a line segment parallel to the main sides of the base shape so that the line segment intersects the secondary sides of the base shape at two points. Now a new triangle will be obtained by connecting these two points of intersection with the midpoint of that main side of the base shape where the apex of the converting shape is not lied. This new triangle will be the 1st main friend shape of the converting shape. In the second step, we have to take a point at a distance of $\frac{1}{m}$ of the height of 1st main friend shape of the converting shape from its apex. Then through that point we have to draw a line segment parallel to the main sides of the base shape so that the line segment intersects the secondary sides of the base shape at two points. Now a new triangle will be obtained by connecting these two points of intersection with the midpoint of that main side of the base shape where the apex of 1st main friend shape of the converting shape is not lied. This new triangle will be the 2nd main friend shape of the converting shape.

v. nth step of the converting process: In the nth step of the converting process, by using the base shape and the main friend shape of the converting shape obtained in the $(n - 1)$ th step, the same process of second step will be applied on the $(n - 1)$ th main friend shape of the converting shape so that we get nth main friend shape of the converting shape where $n > 2$.

vi. Final shape: An isosceles triangle whose base will be $2x$ and height will be $\frac{2ym}{m+1}$. So, the area will be $\frac{2xym}{m+1}$.

vii. Figure

(a) Converting process visualization

Figure 3.2.5: Here PQRS is the base shape. PQ and SR are the main sides and PS and QR are the secondary sides of the base shape. ABC is the converting shape. $A_1B_1C_1$ is the 1st main friend shape of the converting shape. In the second step $A_2B_2C_2$ is the 2nd main friend shape of the converting shape. In the third step $A_3B_3C_3$ is the third main friend shape of the converting shape.

(b) Final shape visualization

Figure 3.2.6: Here $A_{\infty}B_{\infty}C_{\infty}$ is the final shape.

viii. Proof

Look at the figure of first step of the converting process. Here $B_1C_1 \parallel PQ \parallel SR$ so $B_1C_1 = PQ = SR = 2x$. Again, A and A_1 are the midpoints of PQ and SR respectively. So, if we connect A, A_1 then AA_1 will be the perpendicular bisector of B_1C_1 . That's why, A_1B_1 will be equal to A_1C_1 . So, the triangle $A_1B_1C_1$ will be an isosceles triangle whose base will be $2x$. Similarly, $\Delta A_2B_2C_2$ and $\Delta A_3B_3C_3$ will also be isosceles triangle and their base will be $2x$. So, it is proved that all the main friend shapes of the converting shape will be isosceles triangle and their base will be $2x$. That's why, n th main friend shape of the converting shape will be an isosceles triangle and its base will be $2x$ when $n \rightarrow +\infty$. Now we have to calculate the height of the final shape.

Now, if we make a new shape conversion by considering the vertical distance between the midpoints of the main sides of the base shape as new base shape, the height of the converting shape as new converting shape, the height of the 1st main friend shape of the converting shape as 1st main friend shape of the new converting shape etc. then by comparing the converting processes we get the new shape conversion is completely same to the shape conversion 4. So, by using the shape conversion 4 we get the height of the final shape will be $\frac{2ym}{m+1}$.

Here, $\frac{2ym}{m+1} < 2y$ so the secondary vertices of the final shape will lie on the secondary sides of base shape. At last the area of the final shape will be $\frac{2xy}{m+1}$. Thus this is an example of type-2 secondary shape conversion and here type-1 converting process has been applied. This is also a two-dimensional shape conversion. Here we have indirectly used shape conversion 4. That's why, this is a secondary shape conversion.

3.3 Shape conversion error

Definition 3.3.1 (Shape conversion error): By using a converting process, one or more base shapes, a converting shape and a main position if it is not possible to shift the converting shape toward a specific shape then it will be called a shape conversion error.

The properties of a shape conversion are as follows:

1. There is only one converting shape in a shape conversion error but the number of base shape can be more than one.
2. By the help of base shapes the converting shape gradually changes its size. So in every step of the converting process all the base shapes must be used.
3. In the first step the converting shape and in another step the main friend shape of the converting shape must be used. In a step of the converting process, a process must be applied on the main friend shape of the converting shape obtained from the previous step to get a new main friend shape of the converting shape and in case of the first step a process must be applied on the converting shape. In case of applying the process all the base shapes must be used.

4. By using the same base shapes, converting process and main position if a friend shape of the converting shape is used as converting shape then the same thing will be happened, in other words no final shape will be obtained. In this case the friend shape of the converting shape cannot be congruent to the converting shape and there will be such infinitely many friend shapes of the converting shape.

Steps of shape conversion error

There are 7 steps of shape conversion error. They are as follows

1. Base shape or base shapes
2. Converting shape
3. Main position
4. First two step of converting process
5. nth step of the converting process
6. Figure
7. Proof

Almost all the steps are same as the steps of the shape conversion. As there is no final shape so the step to describe the properties of the final shapes is not needed. Thus the step final shape visualization will also be canceled. Now, here we will present 1 shape conversion error.

Shape conversion error 1

i. Base shapes

Base shape 1: A rectangle whose length of one main side is x and length of one secondary side is y .

Base shape 2: A rectangle whose length of one main side is p where $\frac{x}{2} < p < x$ and length of one secondary side is q .

ii. Converting shape: An isosceles triangle whose base is x and height is h .

iii. Main position: Let a main side of the base shape 1 is main side 1 and another main side is main side 2. Similarly, a main side of the base shape 2 is main side 1 and another main side is main side 2. The midpoint of main side 1 of the base shape 1 will lie on the midpoint of main side 1 of the base shape 2. The apex of the converting shape will lie on the midpoint of a main side 1 of the base shape 1 and the secondary vertices will lie on the secondary sides.

iv. First four steps of converting process: In the first step, we have to draw two line segments from the midpoints of the legs of the converting shape perpendicular to the base. Then we have to increase those line segments so that they intersect a two points of the main side 2 of the base shape 2. Considering the distance between these two points of intersection as base and the midpoint of the main side 1 of base shape 2 as apex, we have to draw an isosceles triangle which will be the 1st main friend shape of the converting shape. In the second step, we have to draw two line segments from the midpoints of the legs of 1st main friend shape of the converting shape perpendicular to the base. Then we have to increase those line segments so that they intersect a two points of the main side 2 of the base shape 1. Considering four times the distance between these two points of intersection as base and the midpoint of the main side 1 of base shape 1 as apex, we have to draw an isosceles triangle which will be the 2nd main friend shape of the converting shape. In the 3rd step the same procedure of the first step will be applied on the 2nd main friend shape of the converting shape and in the 4th step the same procedure of the second step will be applied on the 3rd main friend shape of the converting shape

v. nth step of the converting process: In the nth step of the converting process, by using base shape and the $(n - 1)$ th main friend shape of converting shape obtained in the $(n - 1)$ th step, the same process of second step will be applied on the $(n - 1)$ th main friend shape of converting shape so that we get nth main friend shape of

converting shape where $n > 3$ and n is even. Again the n th step of the converting process, by using base shape and the $(n - 1)$ th main friend shape of converting shape obtained in the $(n - 1)$ th step, the same process of third step will be applied on the $(n - 1)$ th main friend shape of converting shape so that we get n th main friend shape of converting shape where $n > 3$ and n is odd.

vi. Figure

Converting process visualization

Figure 3.3.1: Here $ABCD$ and $PQRS$ are the first and second base shapes respectively. AB is the main side 1 of the base shape 1 and SR is the main side 1 of the base shape 2. XYZ is the converting shape. XY_1Z_1 is the 1st main friend shape of the converting shape. In the second step XCD is the 2nd main friend shape of the converting shape. In the third step XY_3Z_3 is the third main friend shape of the converting shape.

vii. Proof

Let the height of the converting shape is XX' . Now look at the figure bellow,

Figure 3.3.2: Here XYZ is the converting shape and XX' is the height.

Here, E, F are the midpoints of XZ and XY . Again, $NF \perp YZ$ and $ME \perp YZ$. Now we can see, $\triangle FNY$ and $\triangle XX'Y$ are similar. So,

$$\frac{FY}{XY} = \frac{NY}{X'Y}$$

$$\Rightarrow NY = \frac{X'Y}{2}$$

Similarly,

$$NY = \frac{X'Z}{2}$$

That's why,

$$MN = \frac{YZ}{2} = \frac{x}{2}$$

Now look at the first step of the converting process. Here, $YZ = x = CD$ because the length of one main side of the base shape 1 is x . So, $YZ \parallel CD \parallel AB \parallel PQ$. That's why, MNZ_1Y_1 is a rectangle and we can write, $MN = Y_1Z_1 = \frac{x}{2}$.

So, the base of 1st main friend shape of the converting shape is $\frac{x}{2}$ and the height is q because the length of one secondary side of the base shape 2 is q .

Again, by using the above method we get in the second step,

$$N_1M_1 = Y_2Z_2 = \frac{Y_1Z_1}{2} = \frac{x}{4}$$

So, the base of the 2nd main friend shape of the converting shape is, $4Y_2Z_2 = x = CD$ and the height is y because the length of one secondary side of the base shape 1 is y .

Again, by using the above method we get in the third step,

$$N_2M_2 = Y_3Z_3 = \frac{CD}{2} = \frac{x}{2}$$

So, the base of 3rd main friend shape of the converting shape is $\frac{x}{2}$ and the height is q because the length of one secondary side of the base shape 2 is q .

Again, by using the above method we get in the 4th step,

$$N_3M_3 = Y_4Z_4 = \frac{Y_3Z_3}{2} = \frac{x}{4}$$

So, the base of the 2nd main friend shape of the converting shape is, $4Y_4Z_4 = x = CD$ and the height is y because the length of one secondary side of the base shape 1 is y .

Now we can see, in the odd steps we will get a main friend shape of the converting shape whose base and height will be $\frac{x}{2}$ and q respectively. Again, in the even steps we will get a main friend shape of the converting shape whose base and height will be x and y respectively. So it is not possible to specify the properties of n th main friend shape of the converting shape when $n \rightarrow +\infty$. That's why, no final shape will be obtained.

Again, we can see the height of the converting shape does not matter. If we take the height greater than 0 and less than y then we will get the same 1st main friend shape of the converting shape and no final shape will be obtained.. So there are infinitely many friend shapes of the converting shape which can be used as converting shape and they are not congruent. So it is a perfect example of shape conversion error. Again, in this case we have used type-2 converting process because in every step a same process is not applied.

Types of shape conversion error

There are three types of shape conversion errors. They are as follows:

1. One-dimensional shape conversion error
2. Two-dimensional shape conversion error
3. Three-dimensional shape conversion error

Definition 3.3.1 (One-dimensional shape conversion error): The shape conversion error in which, all the base shapes and the converting shape are one-dimensional that shape conversion error is called one-dimensional shape conversion error.

Definition 3.3.2 (Two-dimensional shape conversion error): The shape conversion error in which, all the base shapes and the converting shape are two-dimensional that shape conversion error is called two-dimensional shape conversion error.

Definition 3.3.3 (Three-dimensional shape conversion error): The shape conversion error in which, all the base shapes and the converting shape are three-dimensional that shape conversion error is called three-dimensional shape conversion error.

So, the above shape conversion error is a two-dimensional shape conversion error.

3.4 Secondary shape conversion error

Definition 4.2.1 (Secondary shape conversion error): The shape conversion error which is dependent on other shape conversion error by the following two ways will be called secondary shape conversion.

- i. If a shape conversion error is presented by indirectly using another shape conversion error, then the shape conversion error will be considered as a secondary shape conversion error.
- ii. By using the same base shape, converting process and same type of converting shape if we just change the main position of a shape conversion error and do not get a final shape then the shape conversion error will be considered as a secondary shape conversion error.

Note: 1. Any type of shape conversion error can be dependent on other any kind of shape conversion error such as a two-dimensional shape conversion error be dependent on a one-dimensional shape conversion error.

2. A secondary shape conversion error is dependent on a shape conversion error. Similarly, we can show a shape conversion error is dependent on a secondary shape conversion error. But which will be discovered first will be considered as shape conversion error.

Steps of secondary shape conversion error

All the steps are same as the steps of the shape conversion error.

Types of secondary shape conversion

As there are three types of shape conversion errors depending on dimensions and a secondary shape conversion error is dependent on other shape conversion error so the secondary shape conversion error will also be of three types depending on dimensions. They are one-dimensional, two-dimensional and three-dimensional secondary shape conversion error.

4. Discussion and conclusion

Geometry is one of the most important branches of mathematics. This theory demonstrates a new kind of relation between geometry and limit. Through a converting process, one or more base shapes, one converting shape and a main position, shifting the converting shape toward a final shape is the main concept of the theory. In case of the converting process, in the first step the converting shape and in another step the main friend shape of the converting shape must be used. In a step of the converting process, a process must be applied on the main friend shape of the converting shape obtained from the previous step to get a new main friend shape of the converting shape and in case of the first step a process must be applied on the converting shape. In case of applying the process all the base shapes must be used. The main friend shape obtained in a step is considered as the new size of the converting shape. So, it is the idea that the converting shape will obtain a new size in every step and gradually shift toward a final size. We cannot apply such a process in a step to get a main friend shape of the converting shape where any base shape or converting shape is not used. Whatever the process is it is not important, it is important that a process must be applied on the converting shape and main friend shape of the converting shape.

In this paper, we have presented some important shape conversions, secondary shape conversions and a shape conversion error. These examples are enough to understand the main idea and establish the theory. We didn't set any example of third-dimensional, mixed-dimensional shape conversion and secondary shape conversion error. Our main aim is to establish the concept of the theory and according to the concept it is clear that it is obviously possible to demonstrate a third-dimensional or mixed-dimensional shape conversion or secondary shape conversion error. By using the two-dimensional shape conversions we can present three-dimensional shape conversion such as by using shape conversion 5 we can show that if we consider a rectangular solid as base shape and a parallelepiped as the converting shape then by using the similar main position and converting process of shape conversion 5 we will get a final shape which will be a rectangular solid. Again, in case of shape conversion 5 if we apply the converting process differently then we will get a mixed-dimensional shape conversion. In other words, (look at the shape conversion 5) if we divide the angle between that secondary side of the converting shape and other main side of the base shape the converting shape (parallelogram) will gradually shift toward a straight line segment. So the final shape will be one dimensional. That's why it will be a mixed-dimensional shape conversion. Similarly, we can easily present a secondary shape conversion error.

We know, in case of theorem, corollary, lemma etc. there is a statement. Now, for a shape conversion and secondary shape conversion the first six steps will be considered as statement and for a shape conversion error and secondary shape conversion error the first five steps will be considered as statement. Again in case of some shape conversions and secondary shape conversions, we have used two figures to visualize the final shape. The reason is we can calculate the properties of the final shape but cannot clarify where will be the actual position of the final shape. So, in this kind of situation more than one figure are allowed.

In the converting process, every step is dependent on the previous step because we need to use the main friend shape of the converting shape obtained from the previous step to get the next main friend shape of the converting shape. We know in case of the type-1 converting process, a same process is applied in every step. So, a key factor of this converting process is

$$\lim_{n \rightarrow +\infty} f^n(x)$$

If the limit exists then it will be called shape conversion or secondary shape conversion otherwise it will be a shape conversion error or secondary shape conversion error. I believe geometry is the heart of mathematics. Without geometry mathematics is not so much interesting. This theory shows a new side of geometry. There is a huge possibility of this theory. If we properly understand the main concept of the theory then it is very clear that we can set a lot of examples of shape conversion, secondary shape conversion, shape conversion error and secondary shape conversion error to enrich the theory. So we can conclude that this theory describes a new relation between geometry and limit and demonstrate a new way to convert geometric shapes. I also believe that every topic is important to enrich the world of mathematics. Thus we cannot avoid the importance of this theory.

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