The Nicolas criterion for the Riemann Hypothesis

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Abstract

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. For every prime number p_n, we define the sequence $X_n = \prod_{q \leq p_n} \frac{q}{q-1} - e^{\gamma} \times \log \theta(p_n)$, where $\theta(x)$ is the Chebyshev function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. The Nicolas criterion states that the Riemann hypothesis is true if and only if $X_n > 0$ holds for all primes $p_n > 2$. For every prime number $p_k > 2$, $X_k > 0$ is called the Nicolas inequality. We prove that the Nicolas inequality holds for all primes $p_n > 2$. In this way, we demonstrate that the Riemann hypothesis is true.

Keywords: Riemann hypothesis, Nicolas inequality, prime numbers, Chebyshev function, Monotonicity
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1. Introduction

The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$ [1]. The Riemann hypothesis belongs to the David Hilbert's list of 23 unsolved problems [1]. Besides, it is one of the Clay Mathematics Institute's Millennium Prize Problems [1]. In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

with the sum extending over all prime numbers p that are less than or equal to x [2]. For every prime p_n, we define the sequence

$$X_n = \prod_{q \leq p_n} \frac{q}{q-1} - e^{\gamma} \times \log \theta(p_n).$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and log is the natural logarithm. The importance of this property is:

Theorem 1.1. *[3], [4]. $X_n > 0$ holds for all primes $p_n > 2$ if and only if the Riemann hypothesis is true. Moreover, the Riemann hypothesis is false if and only if there are infinitely many prime numbers q_i for which $X_i \leq 0$ and infinitely many other prime numbers r_j for which $X_j > 0$.*

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We use the following properties of the Chebyshev function:

Theorem 1.2. *[5].*

$$\lim_{x\to\infty} \frac{\theta(x)}{x} = 1.$$

Theorem 1.3. *[6]. For $x \geq 41$:*

$$\theta(x) > (1 - \frac{1}{\log(x)}) \times x.$$

Besides, we use the following result:

Theorem 1.4. *[6]. For $x \geq 286$:*

$$\prod_{q \leq x} \frac{q}{q-1} < e^{\gamma} \times (\log x + \frac{1}{2 \times \log(x)}).$$

We also use the Mertens' theorem which states:

Theorem 1.5. *[7].*

$$\lim_{x\to\infty} (\frac{1}{\log x} \times \prod_{q \leq x} \frac{q}{q-1}) = e^{\gamma}.$$

In mathematics, a subsequence is a sequence that can be derived from another sequence by deleting some or no elements without changing the order of the remaining elements. Let Z_i be the infinite and biggest subsequence contained in X_n such that it is strictly decreasing. This time every index i does not correspond to a prime number p_i, but for a new arrangement of enumerating the elements in the sequence Z_i. Besides, we show that $\lim_{i\to\infty} Z_i = 0$ after of applying the sandwich theorem. This implies that $\limsup_{i\to\infty} Z_i = 0$. However, under the assumption that the Nicolas inequality fails for a prime big enough, then $\limsup_{i\to\infty} Z_i < 0$. By contraposition, we show that the Riemann hypothesis is actually true.

2. Results

For every prime number $p_n > 2$, we define the sequence $Y_n = \frac{e^{\frac{1}{2\times\log(p_n)}}}{(1-\frac{1}{\log(p_n)})}$.

Theorem 2.1. *For every prime number $p_n > 2$, the sequence Y_n is strictly decreasing.*

Proof. For every real value $x \geq 3$, we state the function

$$k(x) = \frac{e^{\frac{1}{2\times\log(x)}}}{(1 - \frac{1}{\log(x)})}$$

which is equivalent to

$$k(x) = g(x) \times h(u)$$

where $g(x) = e^{\frac{1}{2\times\log(x)}}$ and $h(u) = \frac{u}{u-1}$ for $u = \log(x)$. We know that $g(x)$ decreases as $x \geq 3$ increases, Moreover, we note that $h(u)$ decreases as $u > 1$ increases where $u = \log(x) > 1$ for $x \geq 3$. In conclusion, we can see that the function $k(x)$ is monotonically decreasing for every real value $x \geq 3$ and therefore, the sequence Y_n is monotonically decreasing as well. In addition, Y_n is essentially a strictly decreasing sequence, since there is not any natural number $n > 1$ such that $Y_n = Y_{n+1}$. □

We will prove another important result:

Theorem 2.2. *Let $q_1, q_2, \ldots, q_m$ denote the first m consecutive primes such that $q_1 < q_2 < \cdots < q_m$ and $q_m > 286$. Then*

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} < e^{\gamma} \times \log\left(Y_m \times \theta(q_m)\right).$$

Proof. From the theorem 1.3, we know that

$$\theta(q_m) > (1 - \frac{1}{\log(q_m)}) \times q_m.$$

In this way, we can show that

$$\begin{aligned} \log\left(Y_m \times \theta(q_m)\right) &> \log\left(Y_m \times (1 - \frac{1}{\log(q_m)}) \times q_m\right) \\ &= \log q_m + \log\left(Y_m \times (1 - \frac{1}{\log(q_m)})\right). \end{aligned}$$

We know that

$$\begin{aligned} \log\left(Y_m \times (1 - \frac{1}{\log(q_m)})\right) &= \log\left(\frac{e^{\frac{1}{2\times\log(q_m)}}}{(1 - \frac{1}{\log(q_m)})} \times (1 - \frac{1}{\log(q_m)})\right) \\ &= \log\left(e^{\frac{1}{2\times\log(q_m)}}\right) \\ &= \frac{1}{2 \times \log(q_m)}. \end{aligned}$$

Consequently, we obtain that

$$\log q_m + \log\left(Y_m \times (1 - \frac{1}{\log(q_m)})\right) \geq (\log q_m + \frac{1}{2 \times \log(q_m)}).$$

Due to the theorem 1.4, we prove that

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} < e^{\gamma} \times (\log q_m + \frac{1}{2 \times \log(q_m)}) < e^{\gamma} \times \log\left(Y_m \times \theta(q_m)\right)$$

when $q_m > 286$. □

Let's define the sequences:

$$V_n = e^{\gamma} \times \log Y_n$$

and

$$W_n = \prod_{q \leq p_n} \frac{q}{q - 1} - e^{\gamma} \times \log\left(Y_n \times \theta(p_n)\right).$$

We obtain a key theorem:

Theorem 2.3. *For all primes $p_n > 286$, we show that the inequalities $W_n < X_n < V_n$ are always satisfied.*

Proof. According to the theorem 2.2, we have that for all primes $p_n > 286$:

$$\prod_{q \leq p_n} \frac{q_i}{q_i - 1} < e^{\gamma} \times \log (Y_n \times \theta(p_n))$$

which is equivalent to

$$\prod_{q \leq p_n} \frac{q_i}{q_i - 1} - e^{\gamma} \times \log \theta(p_n) < e^{\gamma} \times \log Y_n$$

and thus,

$$X_n < e^{\gamma} \times \log Y_n = V_n.$$

Besides, we know that

$$e^{\gamma} \times \log (Y_n \times \theta(p_n)) > e^{\gamma} \times \log \theta(p_n)$$

and therefore,

$$W_n = \prod_{q \leq p_n} \frac{q}{q - 1} - e^{\gamma} \times \log (Y_n \times \theta(p_n)) < X_n.$$

□

Theorem 2.4.

$$\lim_{n \to \infty} V_n = 0.$$

Proof. We obtain that

$$\begin{aligned} \lim_{n \to \infty} (e^{\gamma} \times \log Y_n) &= \lim_{n \to \infty} (e^{\gamma} \times \log \frac{e^{\frac{1}{2 \times \log(p_n)}}}{(1 - \frac{1}{\log(p_n)})}) \\ &= e^{\gamma} \times \log 1 \\ &= 0 \end{aligned}$$

since $\lim_{n \to \infty} Y_n = 1$. □

Theorem 2.5.

$$\lim_{n \to \infty} W_n = 0.$$

Proof. We know by the theorem 1.5:

$$\lim_{n \to \infty} (\frac{1}{\log p_n} \times \prod_{q \leq p_n} \frac{q}{q - 1}) = e^{\gamma},$$

and we have by the theorem 1.2:

$$\lim_{n \to \infty} \frac{\theta(p_n)}{p_n} = 1.$$

Putting all this together yields the proof:

$$\lim_{n \to \infty} \left(\prod_{q \leq p_n} \frac{q}{q - 1} - e^{\gamma} \times \log (Y_n \times \theta(p_n)) \right) = \lim_{n \to \infty} (e^{\gamma} \times \log p_n - e^{\gamma} \times \log p_n) = 0$$

since $\lim_{n \to \infty} Y_n = 1$. □

Theorem 2.6. *The Riemann hypothesis is true.*

Proof. Let Z_i be the infinite and biggest subsequence contained in X_n such that it is strictly decreasing. Certainly, this infinite subsequence Z_i exists, since we know that $X_n < V_n$ and the sequence Y_n is strictly decreasing for every prime number $p_n > 2$. This time every index i does not correspond to a prime number p_i, but for a new arrangement of enumerating the elements in the sequence Z_i. In addition, we present the following statement:

$$\lim_{i\to\infty} Z_i = 0 \tag{1}$$

due to $W_j < Z_i < V_k$ and $\lim_{j\to\infty} W_j = \lim_{k\to\infty} V_k = 0$: This is the result of applying the sandwich theorem. We know that the Nicolas inequality holds for all primes $2 < p < 286$. By definition, the limit $\lim_{i\to\infty} Z_i$ exists if and only if

$$\lim_{i\to\infty} Z_i = \limsup_{i\to\infty} Z_i = \liminf_{i\to\infty} Z_i. \tag{2}$$

Suppose that $p > 286$ is the smallest prime number such that the Nicolas inequality is false where we know that Z_j is strictly decreasing (that is $Z_j > Z_{j+1}$). Under this assumption, there must exist some index m such that

$$Z_m \leq 0$$

and thus

$$Z_{m+1} < Z_m \leq 0.$$

This implies

$$\limsup_{m\to\infty} Z_m < 0$$

which is a contradiction with the limit superior in (2) and the value of (1). By contraposition, the Nicolas inequality would be satisfied for every prime p big enough. Consequently, there would be not infinitely many prime numbers for which the Nicolas inequality is unsatisfied. In this way, using the theorem 1.1, we can conclude that the Riemann hypothesis is true. □

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