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Short Note on the Riemann Hypothesis

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Abstract Robin criterion states that the Riemann hypothesis is true if and only if the inequality $\sigma(n) < e^{\gamma} \times n \times \log\log n$ holds for all natural numbers $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function of n and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. Let $q_1 = 2, q_2 = 3, \ldots, q_m$ denote the first m consecutive primes, then an integer of the form $\prod_{i=1}^{m} q_i^{a_i}$ with $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ is called an Hardy-Ramanujan integer. If the Riemann hypothesis is false, then there are infinitely many Hardy-Ramanujan integers $n > 5040$ such that Robin inequality does not hold and we prove that $n^{\left(1-\frac{0.6253}{\log q_m}\right)} < N_m$, where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order m and q_m is the largest prime divisor of n. In addition, we show that q_m will not have an upper bound by some positive value for these counterexamples and therefore, the value of q_m tends to infinity as n goes to infinity.

Keywords Riemann hypothesis · Robin inequality · sum-of-divisors function · prime numbers

Mathematics Subject Classification (2010) MSC 11M26 · MSC 11A41 · MSC 11A25

1 Introduction

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$ [4]. Let $N_m = 2 \times 3 \times 5 \times 7 \times 11 \times \cdots \times q_m$ denotes a primorial number of order m such that q_m is the m^{th} prime number [3]. As usual $\sigma(n)$ is the sum-of-divisors function of n [1]:

$$\sum_{d \mid n} d$$

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where $d \mid n$ means the integer d divides n and $d \nmid n$ means the integer d does not divide n. Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say $\mathsf{Robins}(n)$ holds provided

$$f(n) < e^{\gamma} \times \log\log n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and log is the natural logarithm. The importance of this property is:

Theorem 1.1 *If the Riemann hypothesis is false, then there are infinitely many natural numbers $n > 5040$ such that $\mathsf{Robins}(n)$ does not hold [4].*

We recall that an integer n is said to be square free if for every prime divisor q of n we have $q^2 \nmid n$ [1]. $\mathsf{Robins}(n)$ holds for all natural numbers $n > 5040$ that are square free [1]. Let $q_1 = 2, q_2 = 3, \ldots, q_m$ denote the first m consecutive primes, then an integer of the form $\prod_{i=1}^{m} q_i^{a_i}$ with $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ is called an Hardy-Ramanujan integer [1]. Based on the theorem 1.1, we know this result:

Theorem 1.2 *If the Riemann hypothesis is false, then there are infinitely many natural numbers $n > 5040$ which are an Hardy-Ramanujan integer and $\mathsf{Robins}(n)$ does not hold [1].*

We prove if the Riemann hypothesis is false, then there are infinitely many Hardy-Ramanujan integers $n > 5040$ such that $\mathsf{Robins}(n)$ does not hold and $n^{\left(1-\frac{0.6253}{\log q_m}\right)} < N_m$, where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order m and q_m is the largest prime divisor of n. Furthermore, we show that q_m will not have an upper bound by some positive value for these counterexamples and thus, the value of q_m tends to infinity as n goes to infinity.

2 Known Results

These are known results:

Theorem 2.1 *[1]. For $n > 1$:*

$$f(n) < \prod_{q \mid n} \frac{q}{q-1}.$$

Theorem 2.2 *Let $\frac{\pi^2}{6} \times \log\log n' \leq \log\log n$ for some $n > 5040$ such that n' is the square free kernel of the natural number n. Then $\mathsf{Robins}(n)$ holds [7].*

Theorem 2.3 $\mathsf{Robins}(n)$ *holds for all natural numbers $n > 5040$ when a prime $q \leq 1771559$ complies with $q \nmid n$ [7].*

Theorem 2.4 *[6]. For $q_m \geq 20000$, we have*

$$\log q_m < \log\log N_m + \frac{0.1253}{\log q_m}.$$

Theorem 2.5 *[5]. For $x \geq 286$:*

$$\prod_{q \leq x} \frac{q}{q-1} < e^{\gamma} \times (\log x + \frac{1}{2 \times \log(x)}).$$

Theorem 2.6 *[2]. For $x > -1$:*

$$\frac{x}{x+1} \leq \log(1+x).$$

3 Proof of Main Theorem

Theorem 3.1 *If the Riemann hypothesis is false, then there are infinitely many Hardy-Ramanujan integers $n > 5040$ such that $\mathsf{Robins}(n)$ does not hold and $n^{\left(1-\frac{0.6253}{\log q_m}\right)} < N_m$, where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order m and q_m is the largest prime divisor of n. In addition, q_m will not have an upper bound by some positive value for these counterexamples and therefore, the value of q_m tends to infinity as n goes to infinity.*

Proof Let $\prod_{i=1}^{m} q_i^{a_i}$ be the representation of some natural number $n > 5040$ as a product of primes $q_1 < \cdots < q_m$ with natural numbers as exponents $a_1, \ldots, a_m$. The primes $q_1 < \cdots < q_m$ must be the first m consecutive primes and $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ since the natural number $n > 5040$ could be an Hardy-Ramanujan integer. We assume that $\mathsf{Robins}(n)$ does not hold. Indeed, we know there are infinitely many Hardy-Ramanujan integers such as $n > 5040$ when the Riemann hypothesis is false according to the theorem 1.2. From the theorem 2.3, we know that necessarily $q_m \geq 1771559$. So,

$$e^{\gamma} \times \log\log n \leq f(n) < \prod_{q \leq q_m} \frac{q}{q-1} < e^{\gamma} \times (\log q_m + \frac{1}{2 \times \log(q_m)})$$

because of the theorems 2.1 and 2.5. Hence,

$$\log\log n < \log q_m + \frac{0.5}{\log(q_m)}.$$

From the theorem 2.4, we have that

$$\log\log n < \log\log N_m + \frac{0.1253}{\log q_m} + \frac{0.5}{\log(q_m)}.$$

That is the same as

$$\log\log n - \log\log N_m < \frac{0.6253}{\log q_m}.$$

Then,

$$\begin{aligned}\log\log n - \log\log N_m &= \log\left(\log N_m + \log(\frac{n}{N_m})\right) - \log\log N_m \\ &= \log\left(\log N_m \times (1 + \frac{\log(\frac{n}{N_m})}{\log N_m})\right) - \log\log N_m \\ &= \log\log N_m + \log(1 + \frac{\log(\frac{n}{N_m})}{\log N_m}) - \log\log N_m \\ &= \log(1 + \frac{\log(\frac{n}{N_m})}{\log N_m}).\end{aligned}$$

In addition, we know that

$$\log(1 + \frac{\log(\frac{n}{N_m})}{\log N_m}) \geq \frac{\log(\frac{n}{N_m})}{\log n}$$

using the theorem 2.6 since $\frac{\log(\frac{n}{N_m})}{\log N_m} > -1$. Certainly, we will have that

$$\log(1 + \frac{\log(\frac{n}{N_m})}{\log N_m}) \geq \frac{\frac{\log(\frac{n}{N_m})}{\log N_m}}{\frac{\log(\frac{n}{N_m})}{\log N_m} + 1} = \frac{\log(\frac{n}{N_m})}{\log(\frac{n}{N_m}) + \log N_m} = \frac{\log(\frac{n}{N_m})}{\log n}.$$

In this way, we have that

$$\frac{\log(\frac{n}{N_m})}{\log n} < \frac{0.6253}{\log q_m}$$

which is equivalent to

$$\log(\frac{n}{N_m}) < \log(n^{\frac{0.6253}{\log q_m}})$$

and thus

$$\frac{n}{N_m} < n^{\frac{0.6253}{\log q_m}}.$$

Finally, we obtain that

$$n^{\left(1 - \frac{0.6253}{\log q_m}\right)} < N_m.$$

Moreover, we know that q_m will not have an upper bound by some positive value for these counterexamples because of the theorem 2.2. Certainly, if there is a possible upper bound for q_m, then it cannot exist infinitely many Hardy-Ramanujan integers $n > 5040$ such that $\mathsf{Robins}(n)$ does not hold as a consequence of the theorem 2.2.

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